Cubical Pachner moves and cobordisms of immersions

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Simplicial Pachner moves

A *triangulation* of a manifold M is a simplicial complex homeomorphic to M. We require that the triangulation is a *PL manifold*, meaning the link of every vertex is a PL-sphere.

A *Pachner move* or *bistellar flip* on a n-dimensional triangulation T replaces a subcomplex B of T with B', where B and B' are complementary full-dimensional subcomplexes of a $(n+1)$-simplex.

Theorem (Pachner)

Any two triangulations of the same manifold are related through Pachner moves.

Cubical Pachner moves

A *cubical complex* is a finite collection of topological cubes glued together along unions of faces.

A *cubulation* of a manifold M is a cubical complex homeomorphic to M. We require that the cubulation is a PL manifold, meaning the boundary of the star of every vertex is a PL-sphere.

A *cubical Pachner move* or *cube flip* on a n-dimensional cubulation C replaces a subcomplex B of C with B', where B and B' are complementary full-dimensional subcomplexes of a $(n+1)$-cube, and B and B' are homeomorphic to balls.

Equivalence classes under flips

Question (Habegger)

Are any two cubulations of the same manifold related through cube flips?

Answer: No, consider a cubulation of S^1 with an even number of segments and one with an odd number of segments.

There are two classes of cubulations of S^1 under the equivalence relation of cube flips.

For S^2, there are at least two.

Immersions

Given manifolds N, M, a *immersion* is a map $f : N \to M$ such that for all $x \in N$, there is a neighborhood U of x such that f is a homeomorphism of U onto $f(U)$.

The *codimension* of an immersion $N \to M$ is $\dim(M) - \dim(N)$.

If C is a cubulation of M, we associate it with a codimension 1 immersion $f_c : N_c \to M$ as follows: View each n-dimensional cube as $[-1, 1]^n$, and immerse the n coordinate hyperplanes into each cube. Glue accordingly.

Cobordisms of immersions

Two codimension 1 immersions $f : N \to M$ and $f' : N' \to M$ are *cobordant* if there is a codimension 1 immersion $\Phi : X \to M \times [0, 1]$ where X is a manifold with boundary $N \sqcup N'$, Φ is transverse to the boundary, and $\Phi|_N = f \times \{0\}$, $\Phi_{N'} = f \times \{1\}$.

Observation: If two cubulations C, C' are related by cube flips, then their associated immersions f_C, $f_{C'}$ are cobordant.

Cube flips and cobordisms

Therefore, we have a well-defined map

$$\{\text{cubulations of } M \text{ mod cube flips}\} \to \{\text{codimension 1 immersions into } M \text{ mod cobordism}\}.$$

Theorem (Funar)

This map is a surjection.

Theorem (Adiprasito, L.)

This map is a bijection.

Immersions into spheres

The group of cobordisms of codimension 1 immersions into S^n is given by $\pi_n^s(\mathbf{RP}^\infty)$, the n-th stable homotopy group of $\mathbf{RP}^\infty$.

n	1	2	3	4	5	6	7	8	9
	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_8$	$\mathbf{Z}_2$	0	$\mathbf{Z}_2$	$\mathbf{Z}_{16} \oplus \mathbf{Z}_2$	$(\mathbf{Z}_2)^{\oplus 3}$	$(\mathbf{Z}_2)^{\oplus 4}$

Proof idea

By Funar, if two cubulations of M are cobordant, then there is a cubulation of $M \times [0, 1]$ which restricts to these two cubulations on the boundary.

We need to show that such a cubulation C of $M \times I$ exists such that C is *shellable* relative to $M \times \{0\}$.

Prove a cubical analogue of Morelli's theorem: Any cubulation of a ball becomes shellable after sufficiently many cubical stellar subdivisions.

Oriented matroids

To any rank r uniform oriented matroid $\mathcal{M}$ we can associate a unique cubulation of S^r whose associated immersion is a topological representation of $\mathcal{M}$.

Conjecture (Las Vergnas)

Any two such cubulations are related through "star-star" cubical Pachner moves.

As a consequence of our result, this is true if we allow all cubical Pachner moves.

Ball versions of this conjecture fail (Mnëv-Richter-Gebert, L.).

Thank you!