Tope graphs of COMs

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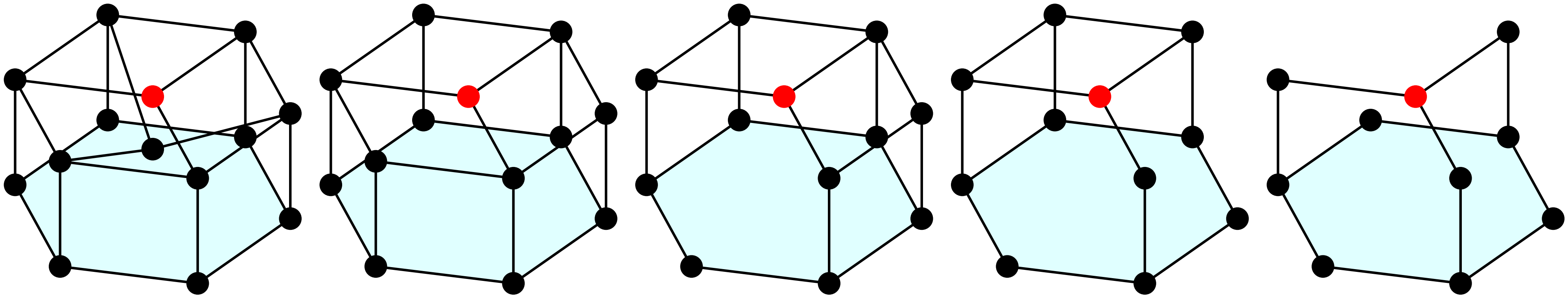
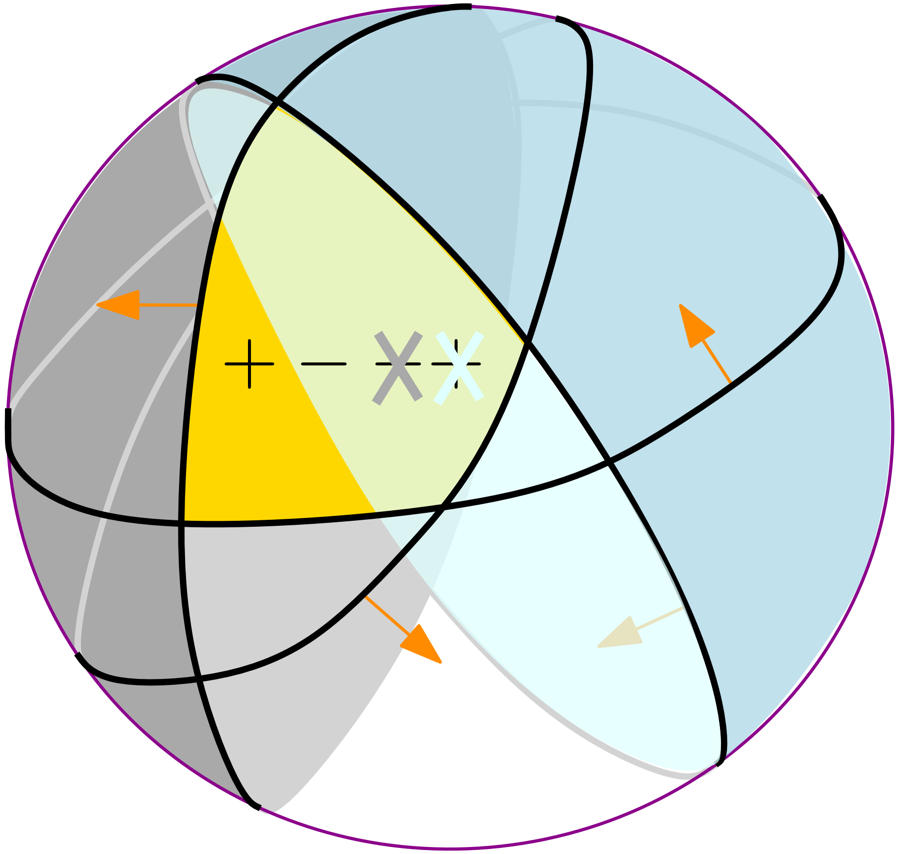
FMF, Univerza v Ljubljani



Combinatorial Geometries 2018: matroids, oriented matroids and applications

Complexes of oriented matroids



"Representative" example: arrangement of pseudospheres and pseudosemispheres

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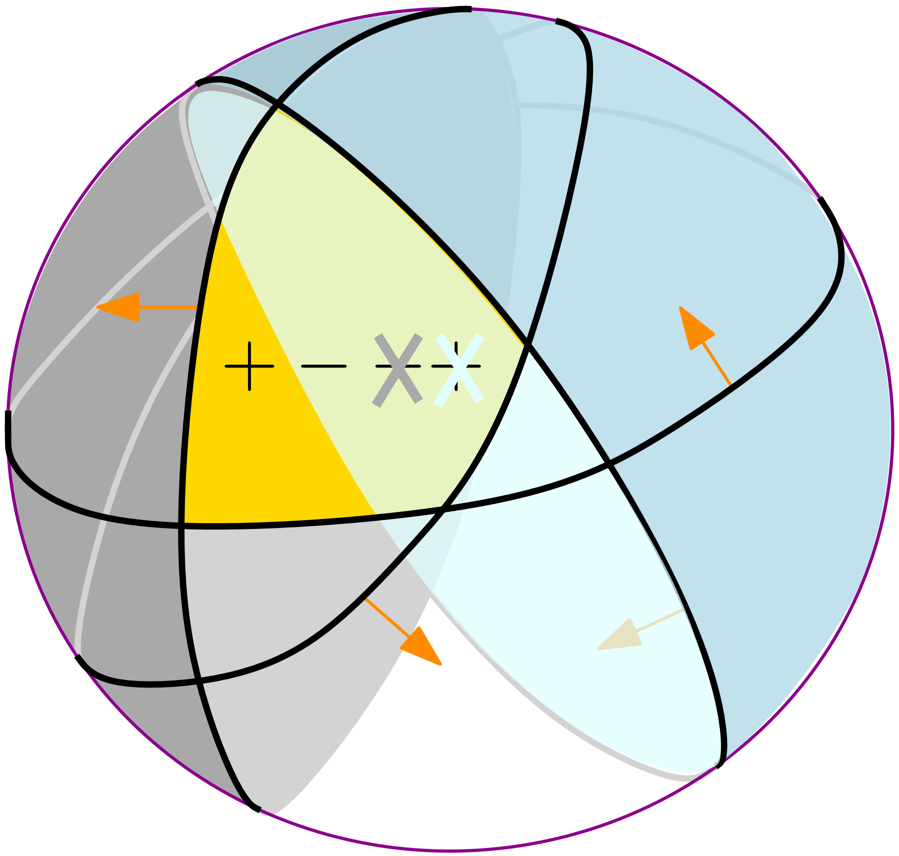

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topes $\mathcal{T}$ of $\mathcal{L}$= maximal cells

tope graph $G_{\mathcal{T}}$ = incidence graph

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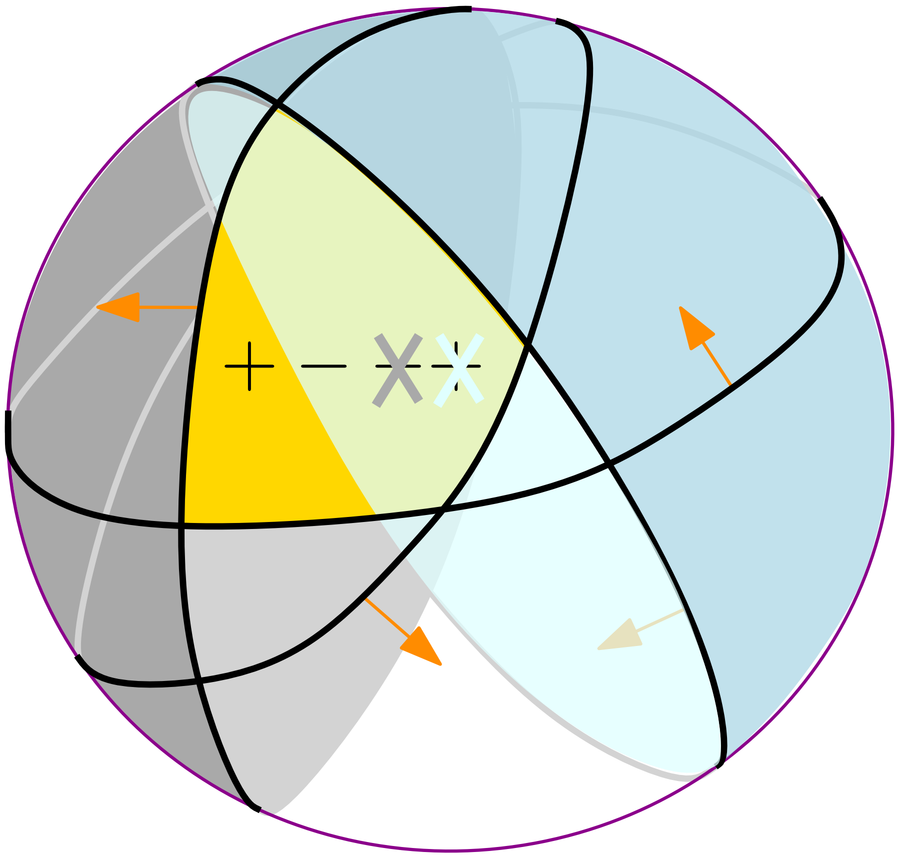

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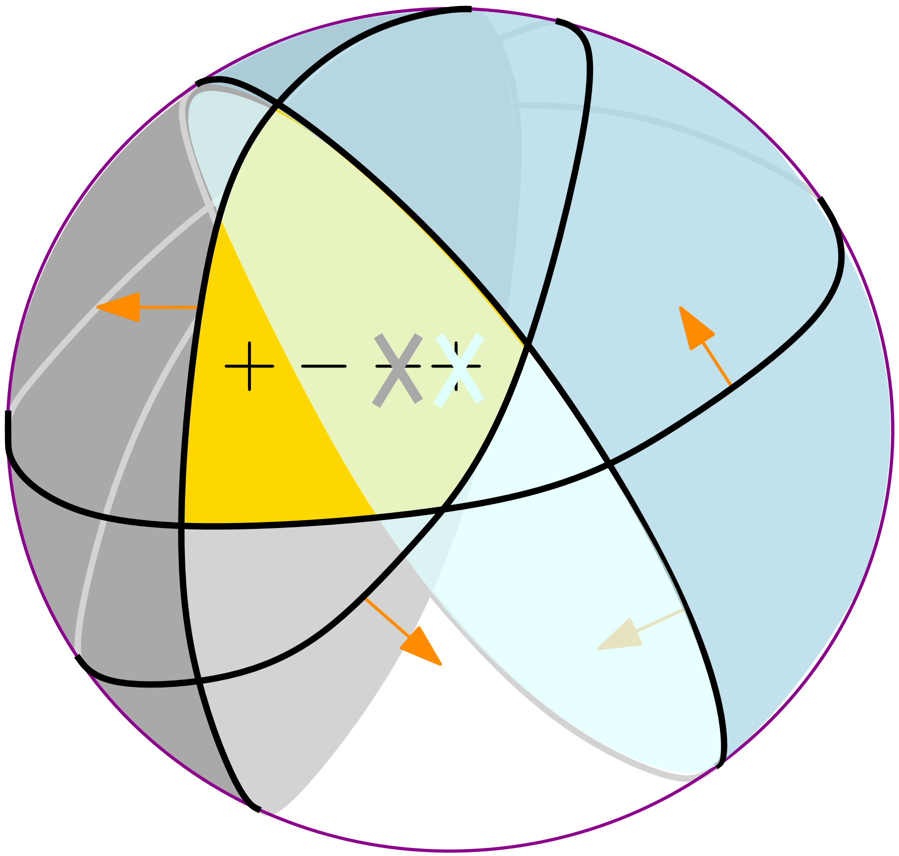

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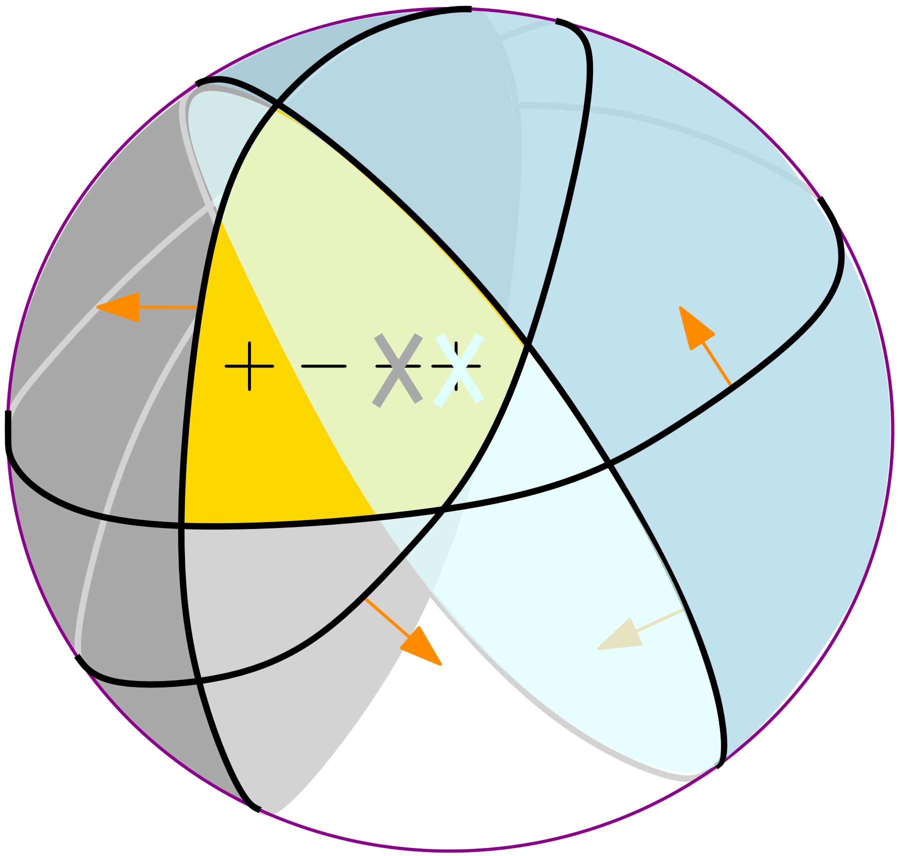

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$\rightsquigarrow$ special case: linear extension graphs of posets

A common generalization

∘ Covector axioms: $(E, \mathcal{L})$ **COM** iff

(FS) $\mathcal{L} \circ -\mathcal{L} \subseteq \mathcal{L}$

(SE) $\forall X, Y \in \mathcal{L}$ and $e \in S(X,Y) \exists Z \in \mathcal{L}$:
$Z_e = 0$ and $Z_f = X_f \circ Y_f$ for $f \notin S(X,Y)$.

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tope graphs are partial cubes and determine $\mathcal{L}$

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Partial cubes and partial cube minors

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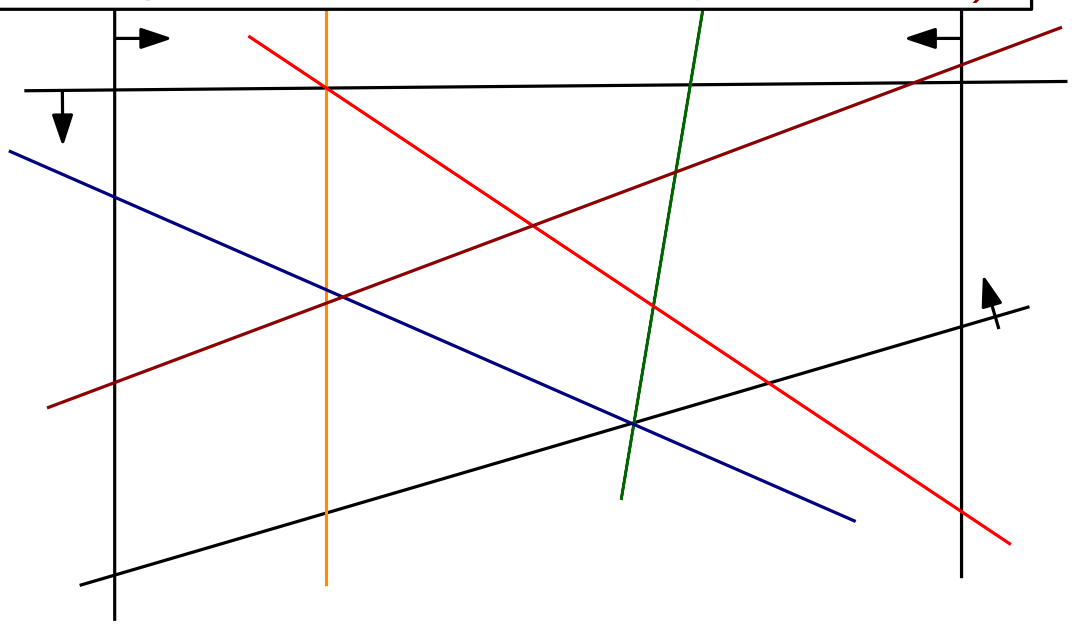

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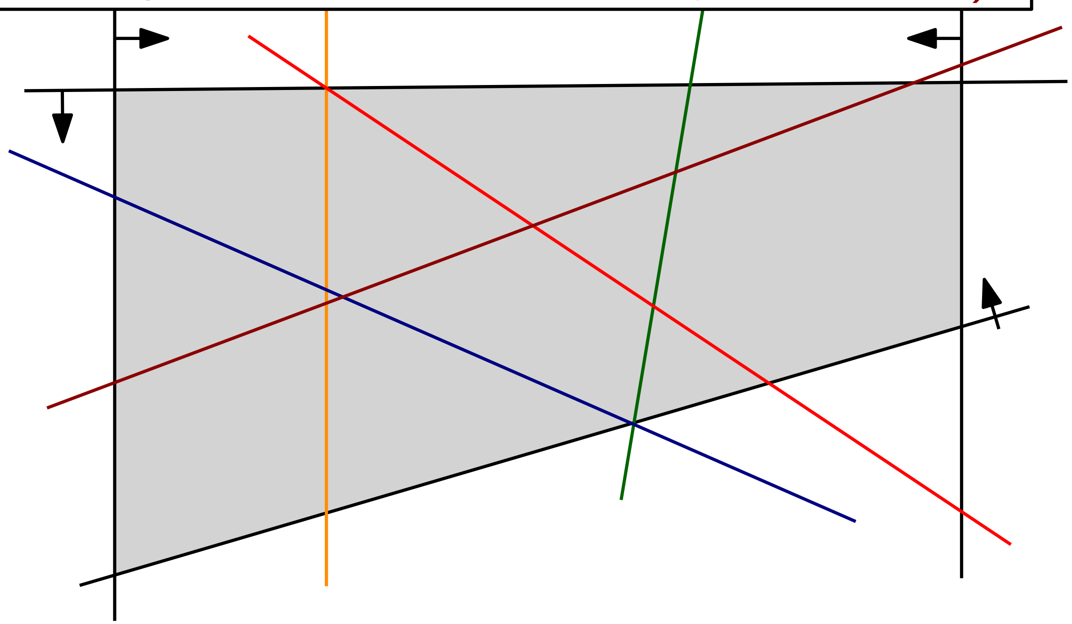

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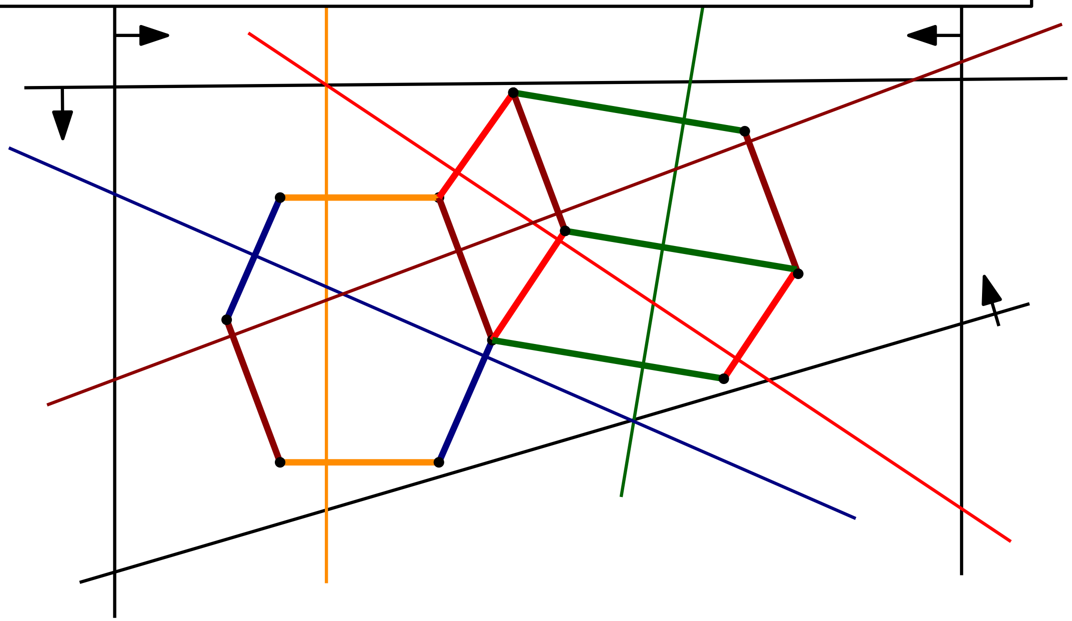

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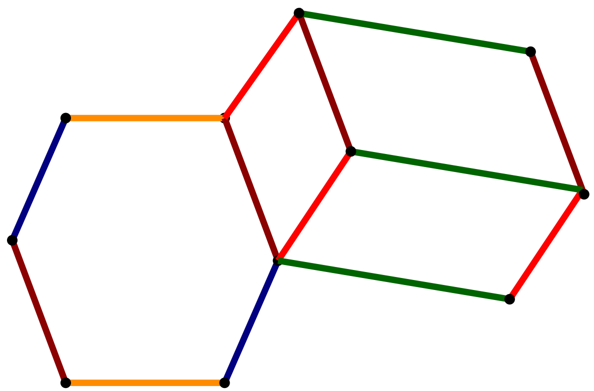

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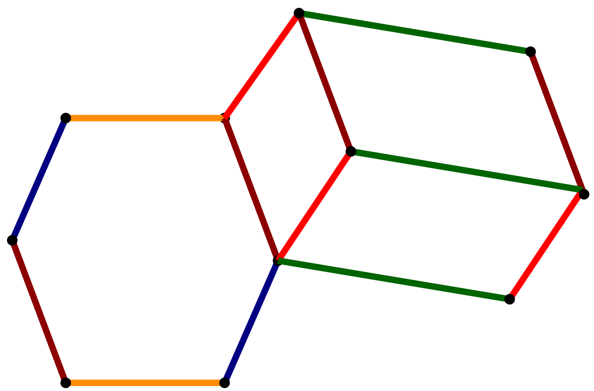

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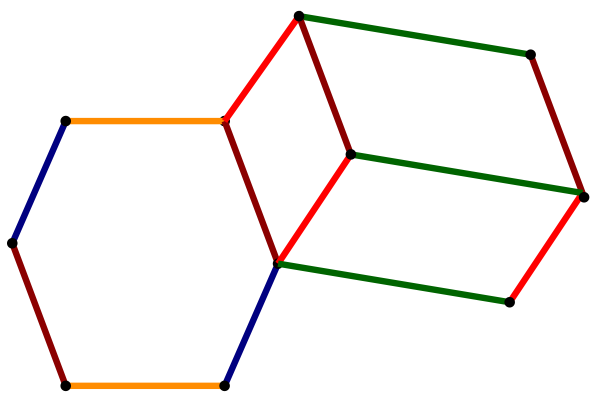

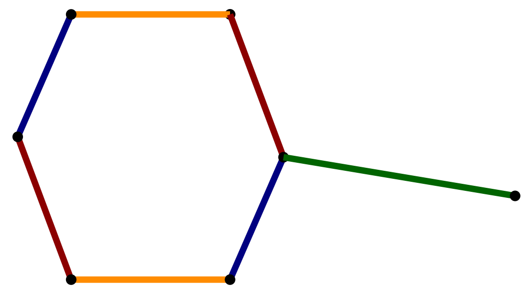

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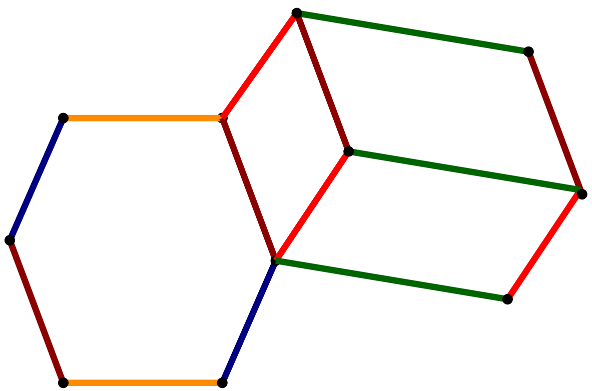

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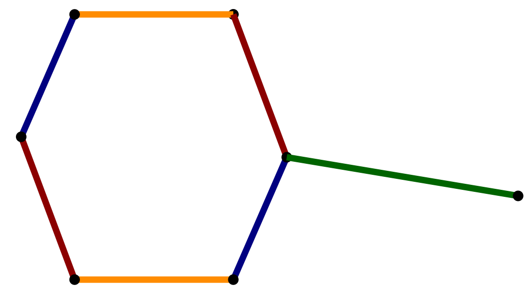

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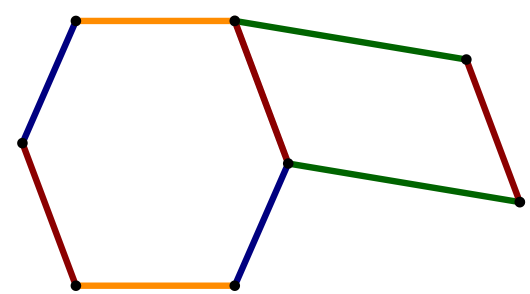

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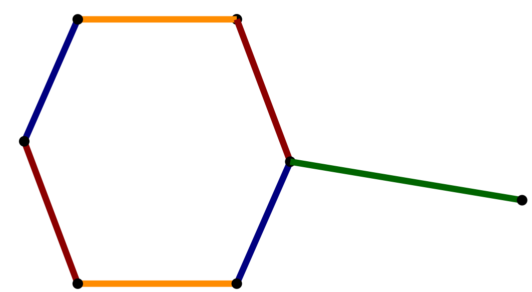

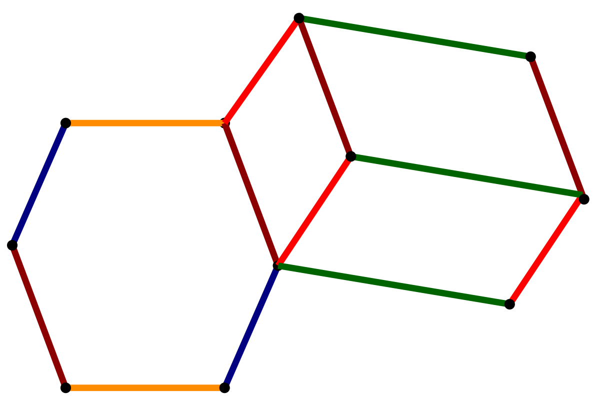

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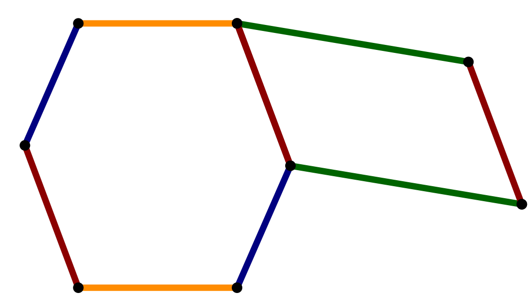

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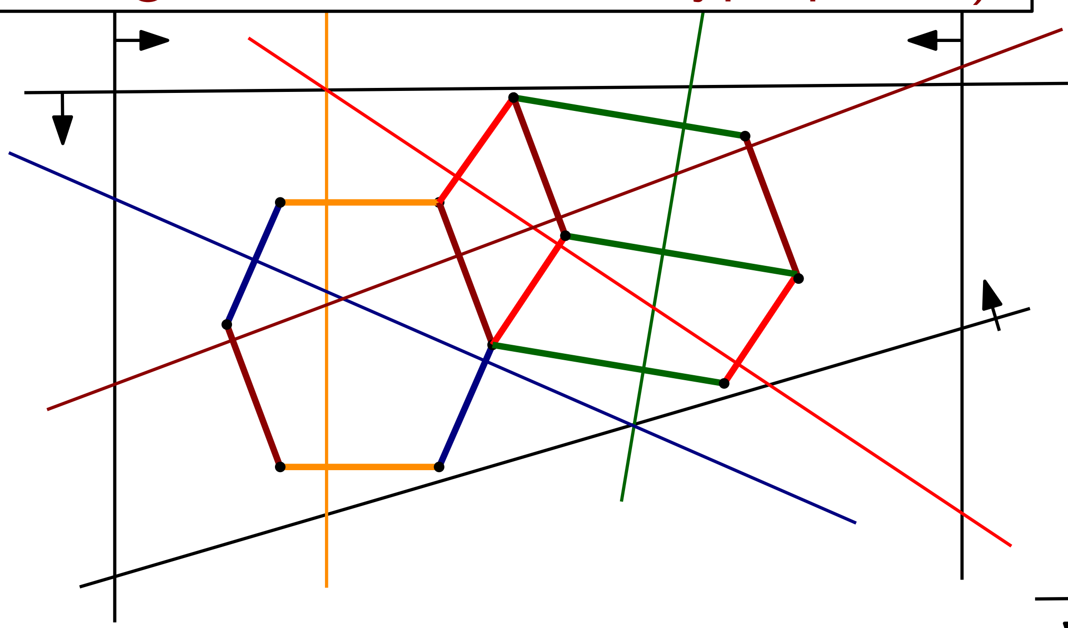

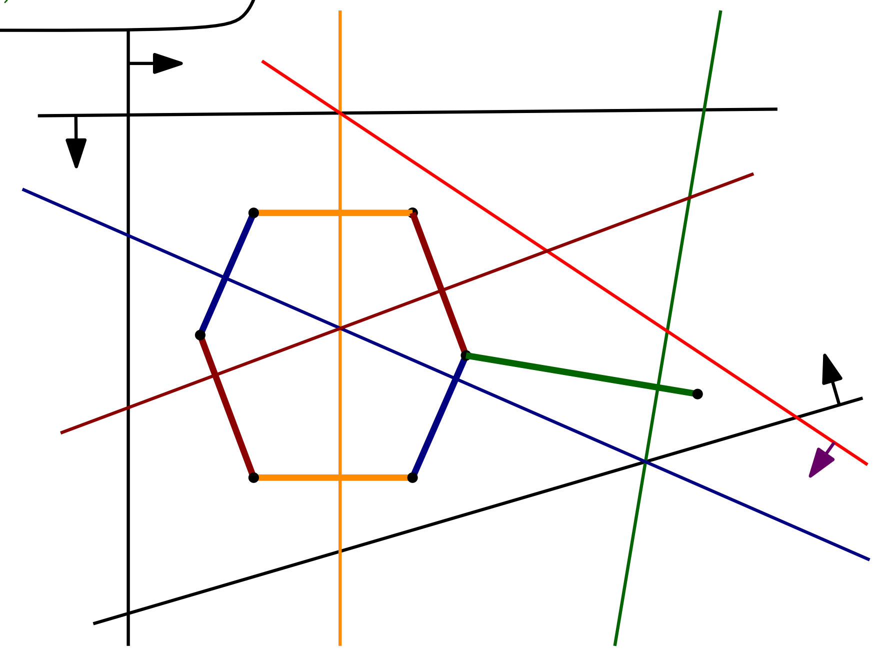

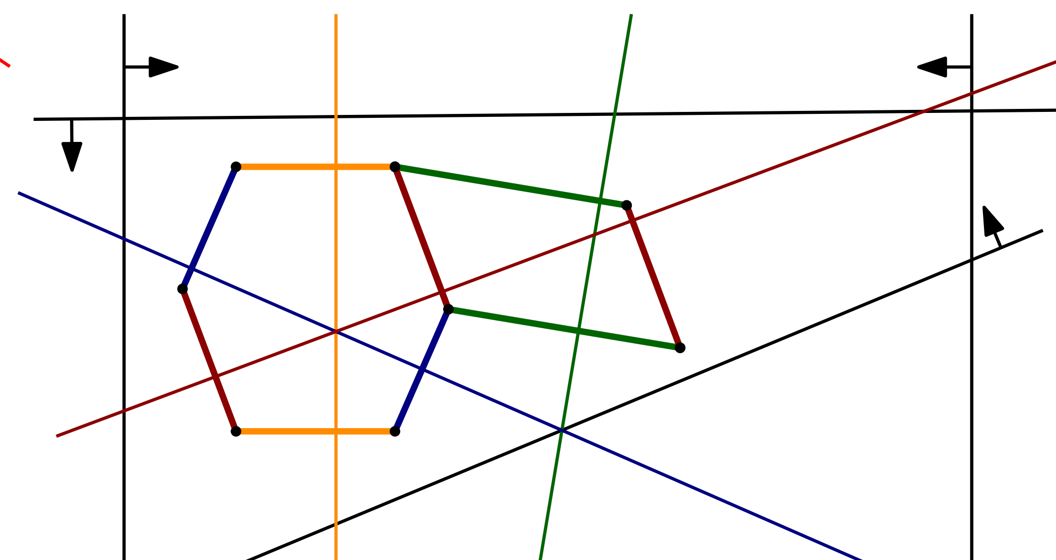

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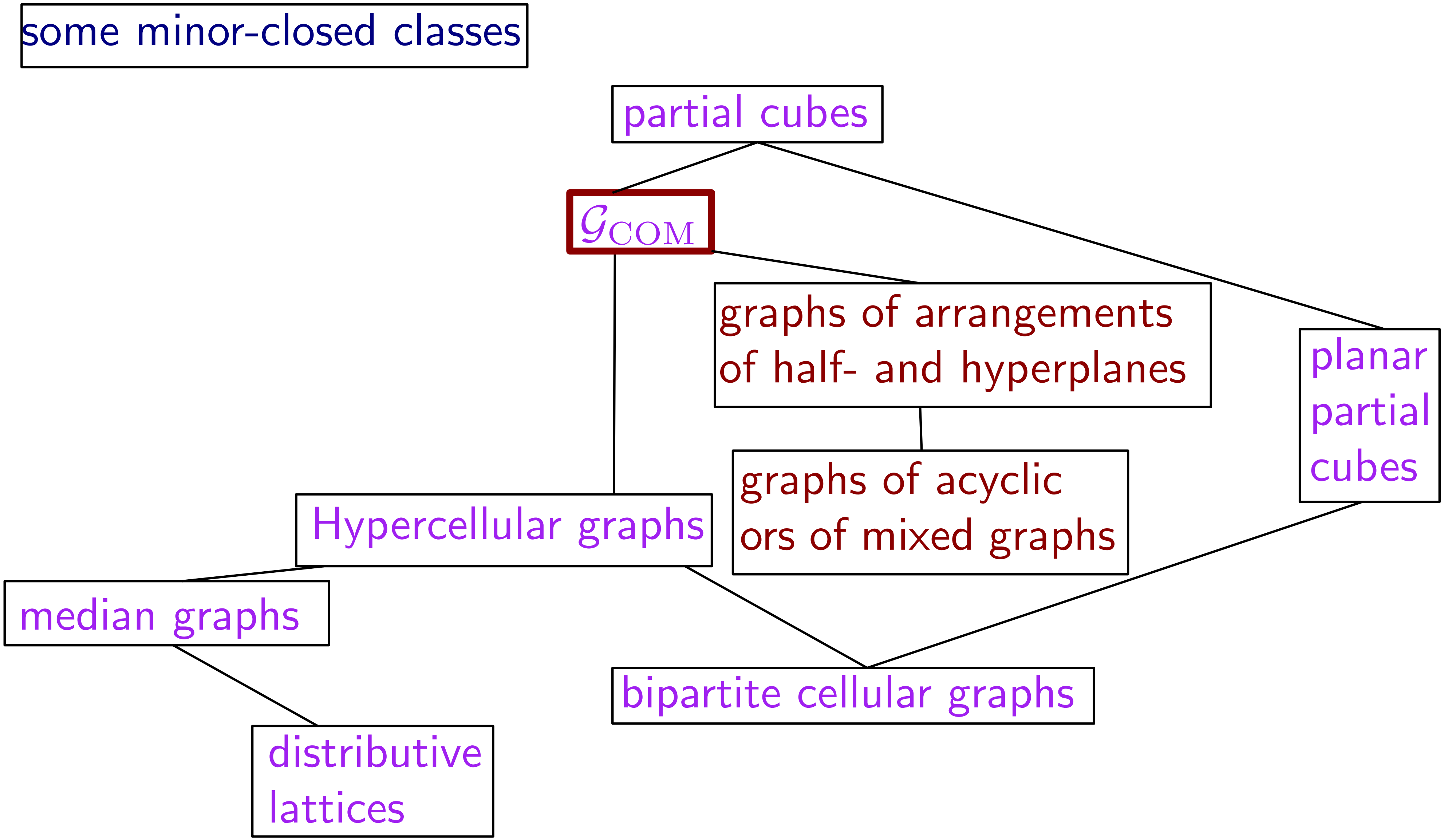
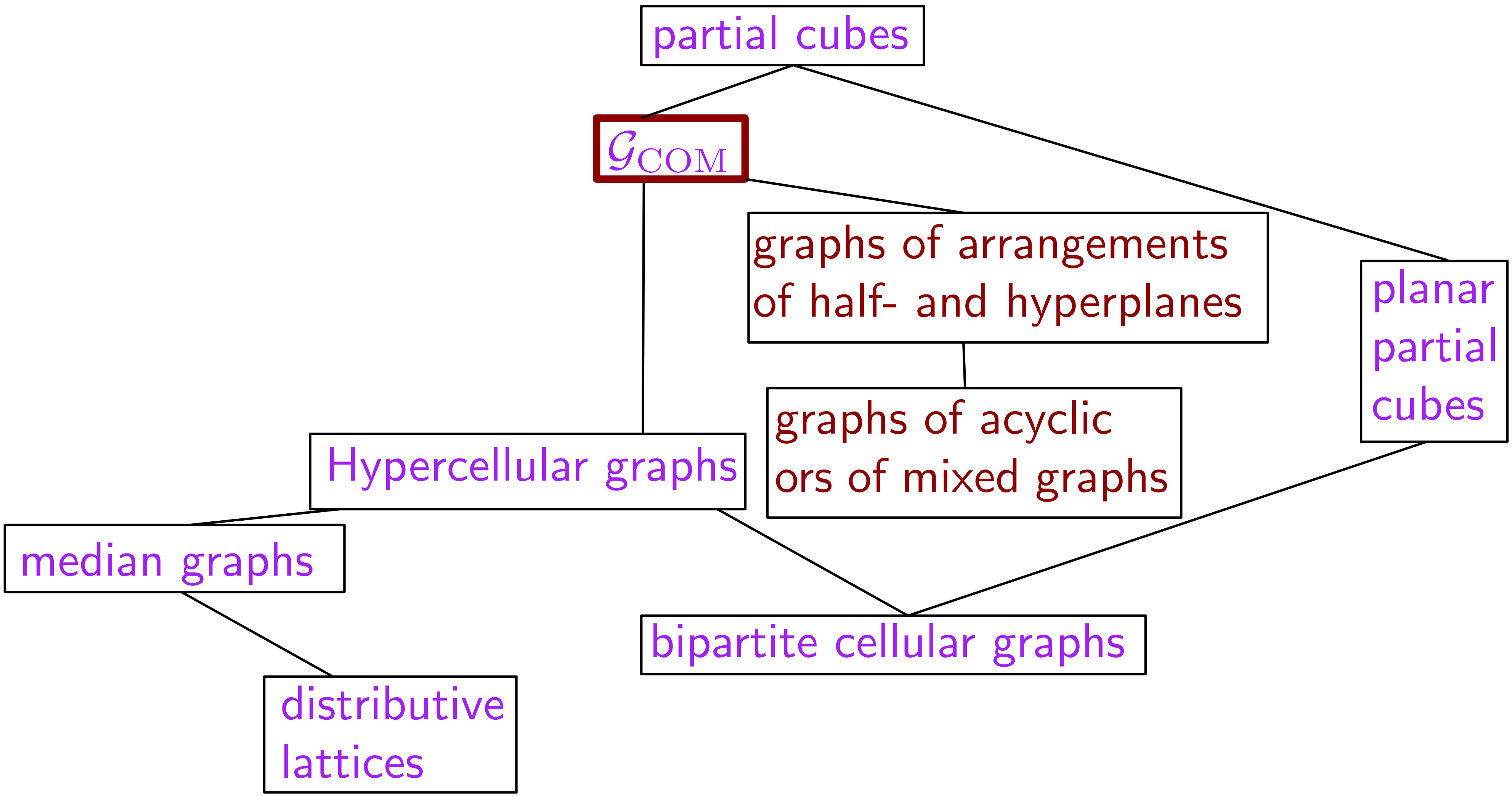


Partial cube minors

some minor-closed classes



each has a family of excluded minors

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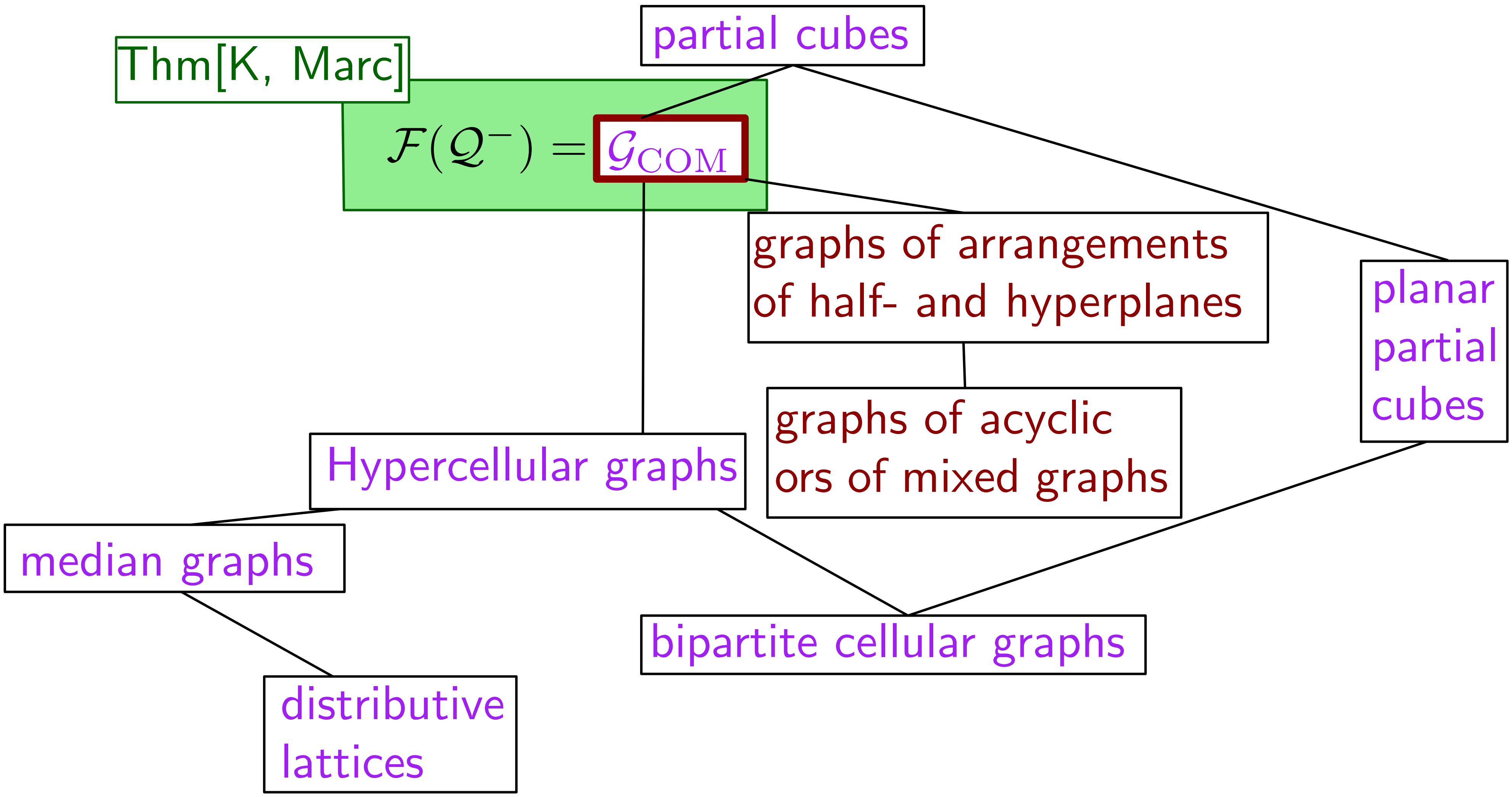

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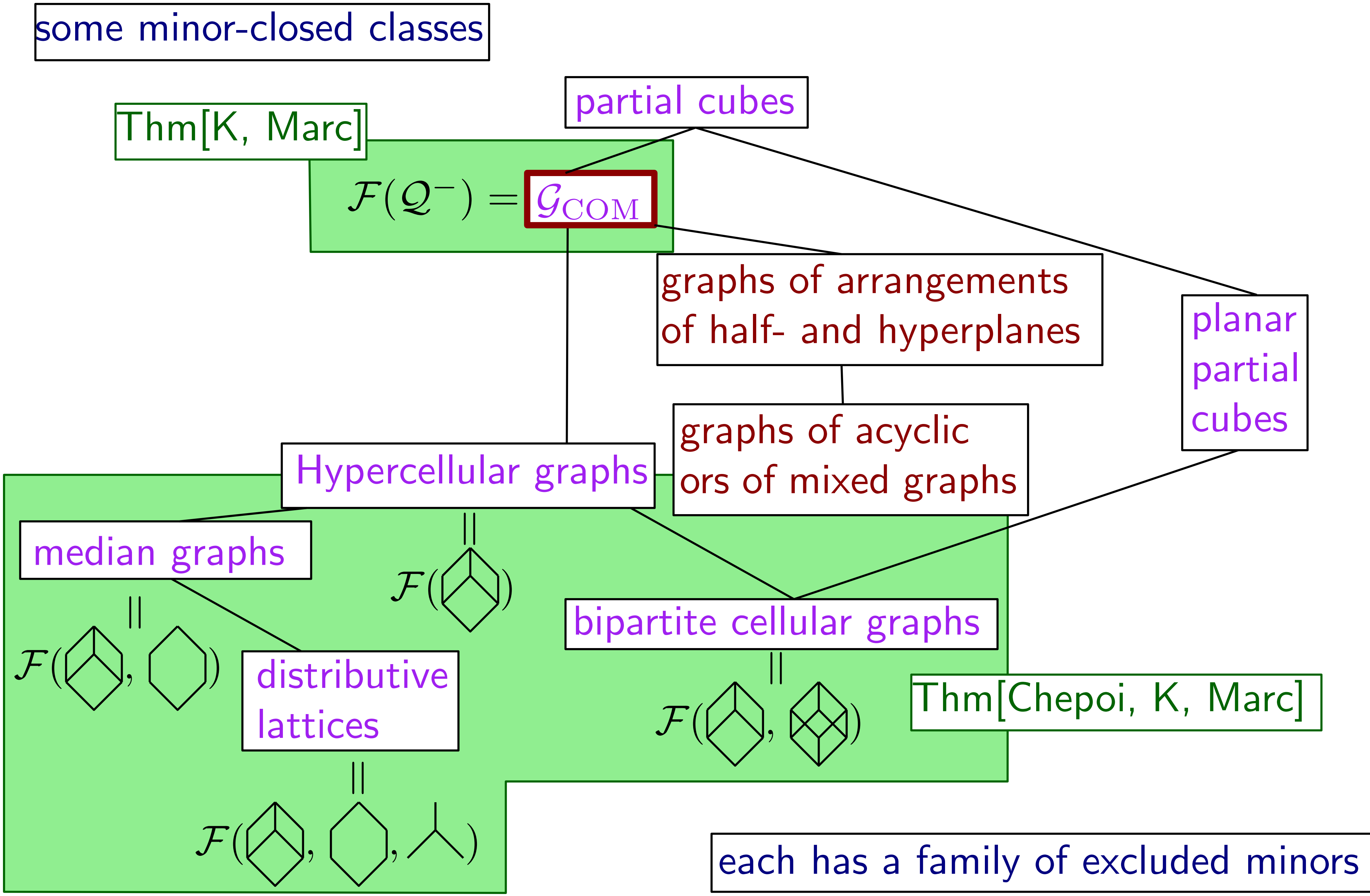

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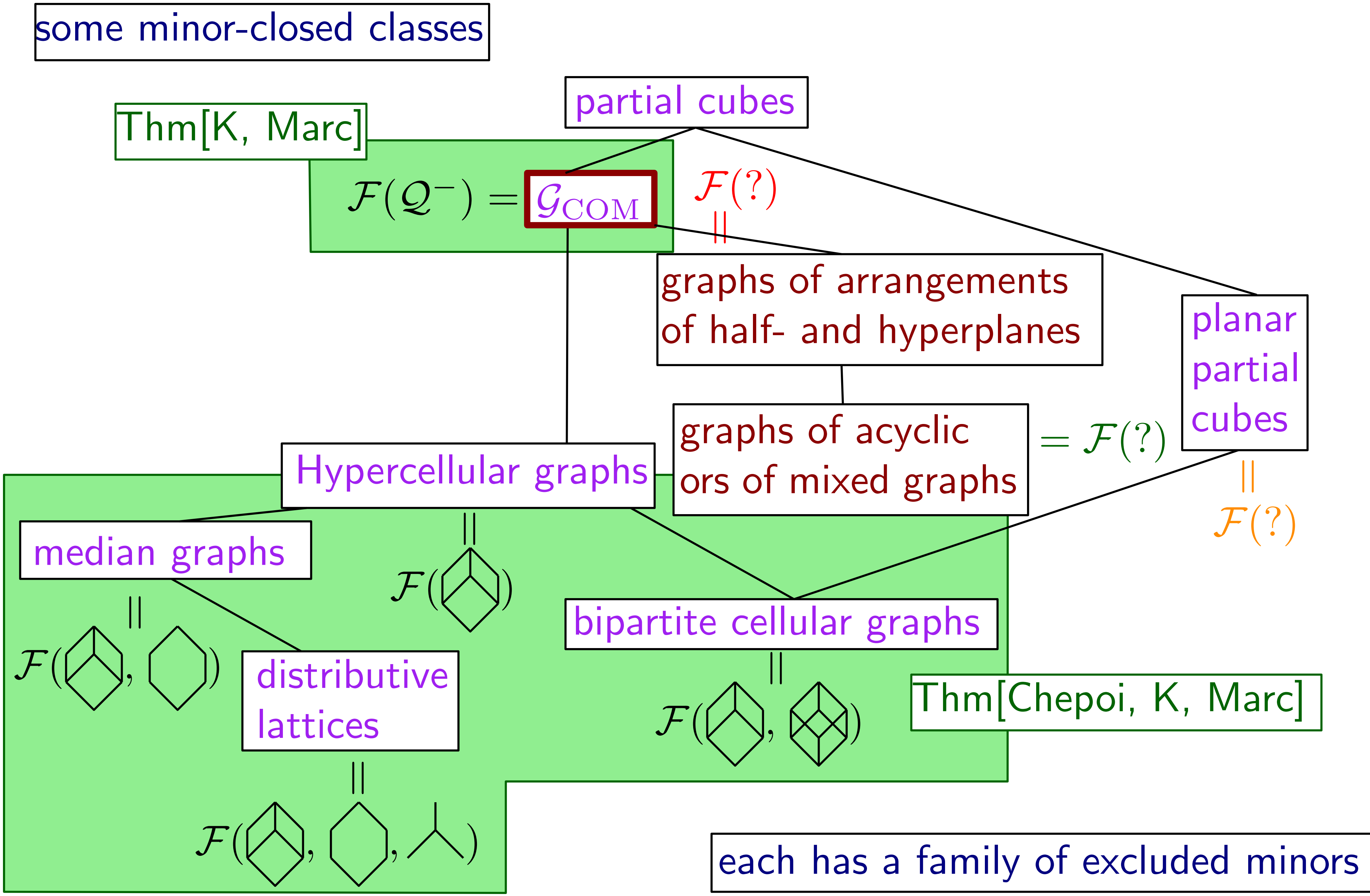

From partial cubes to sign vectors

Let G partial cube, then $G' \subset G$ convex $\iff$ G' restriction of G

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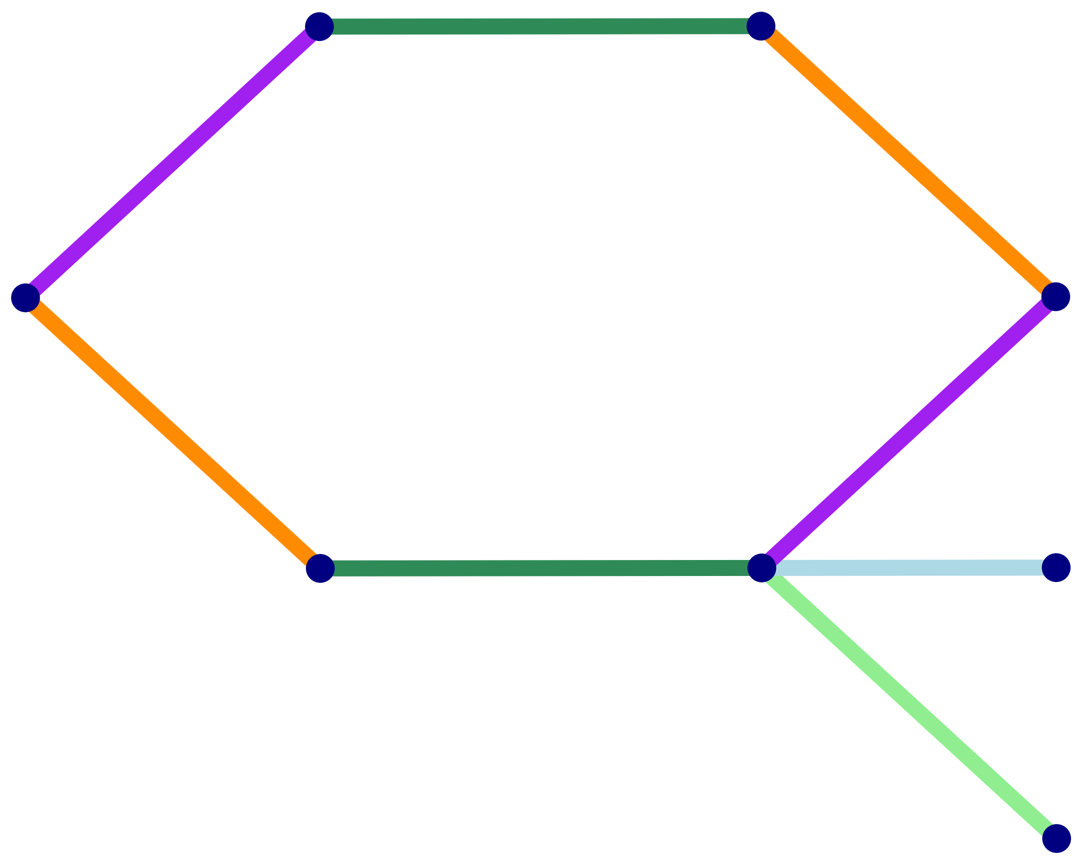

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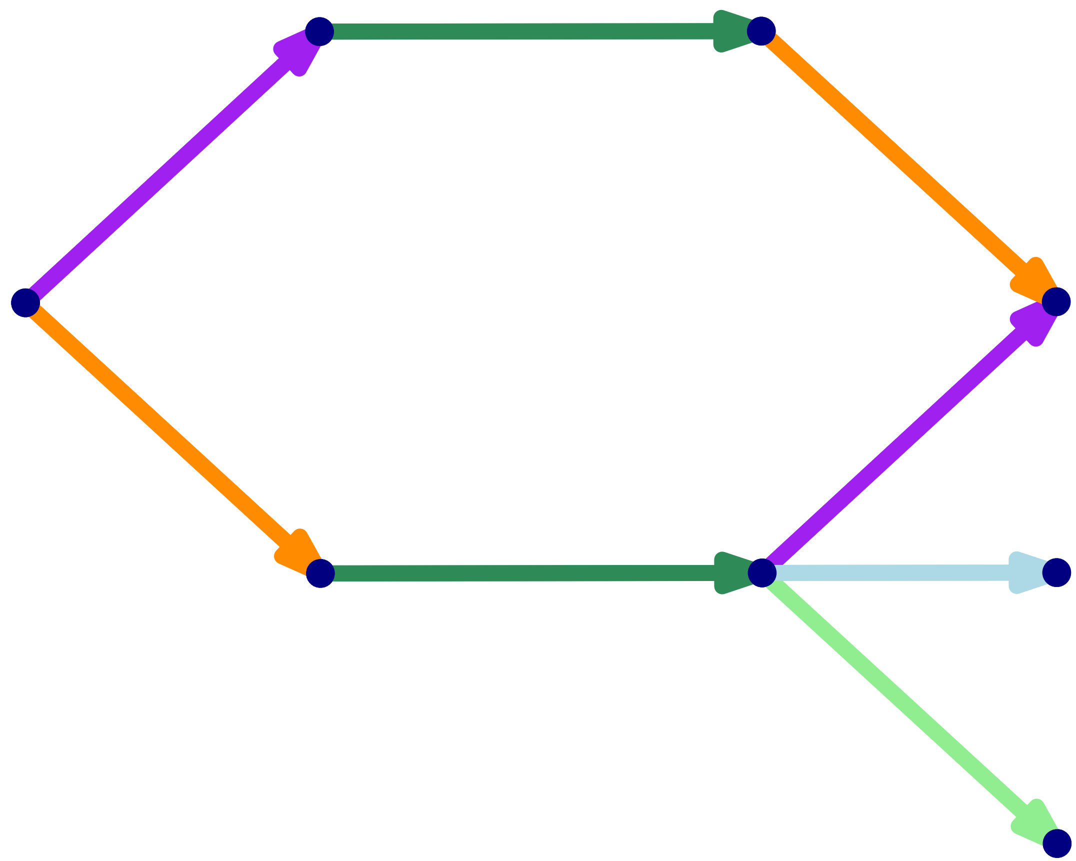

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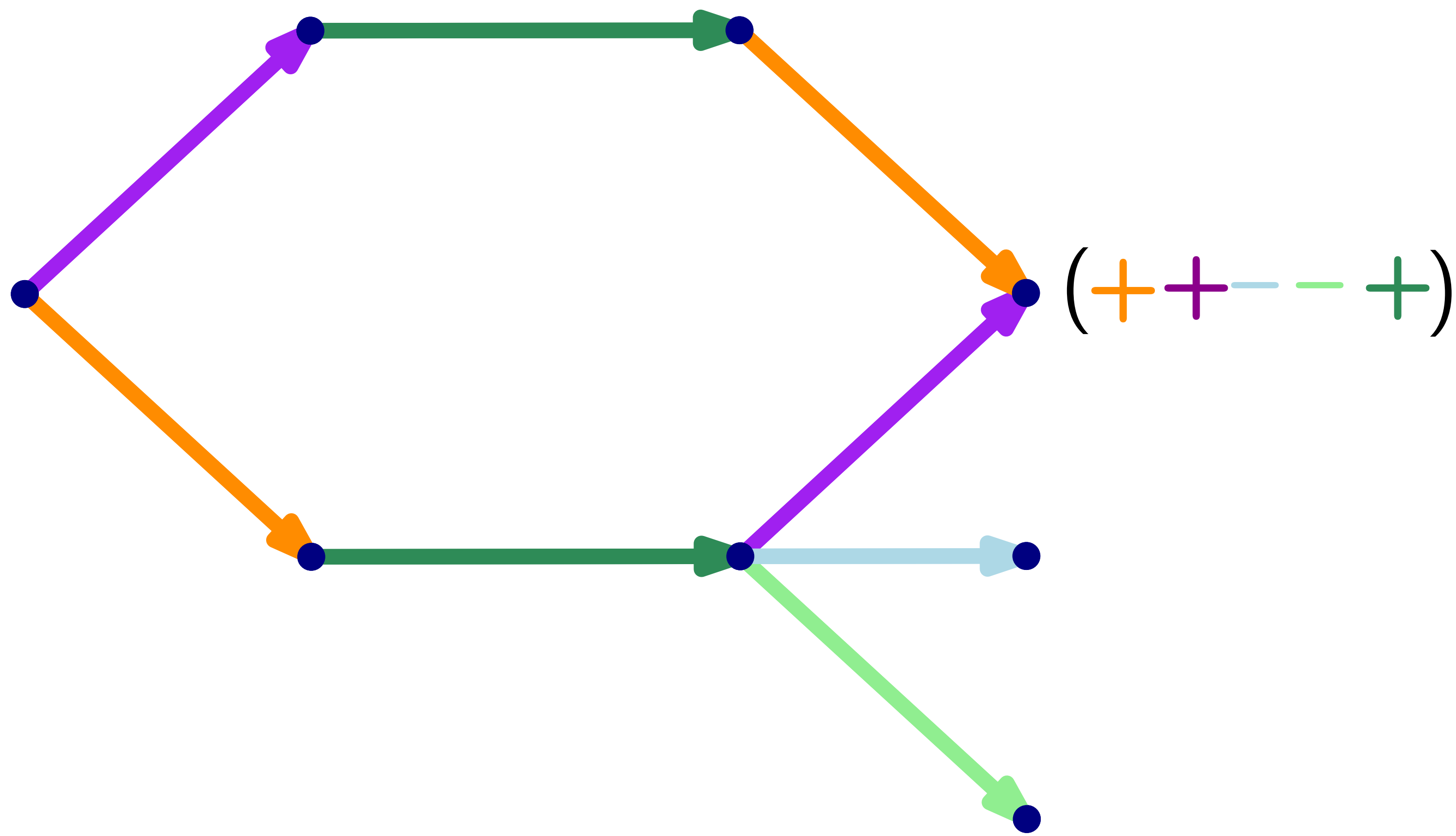

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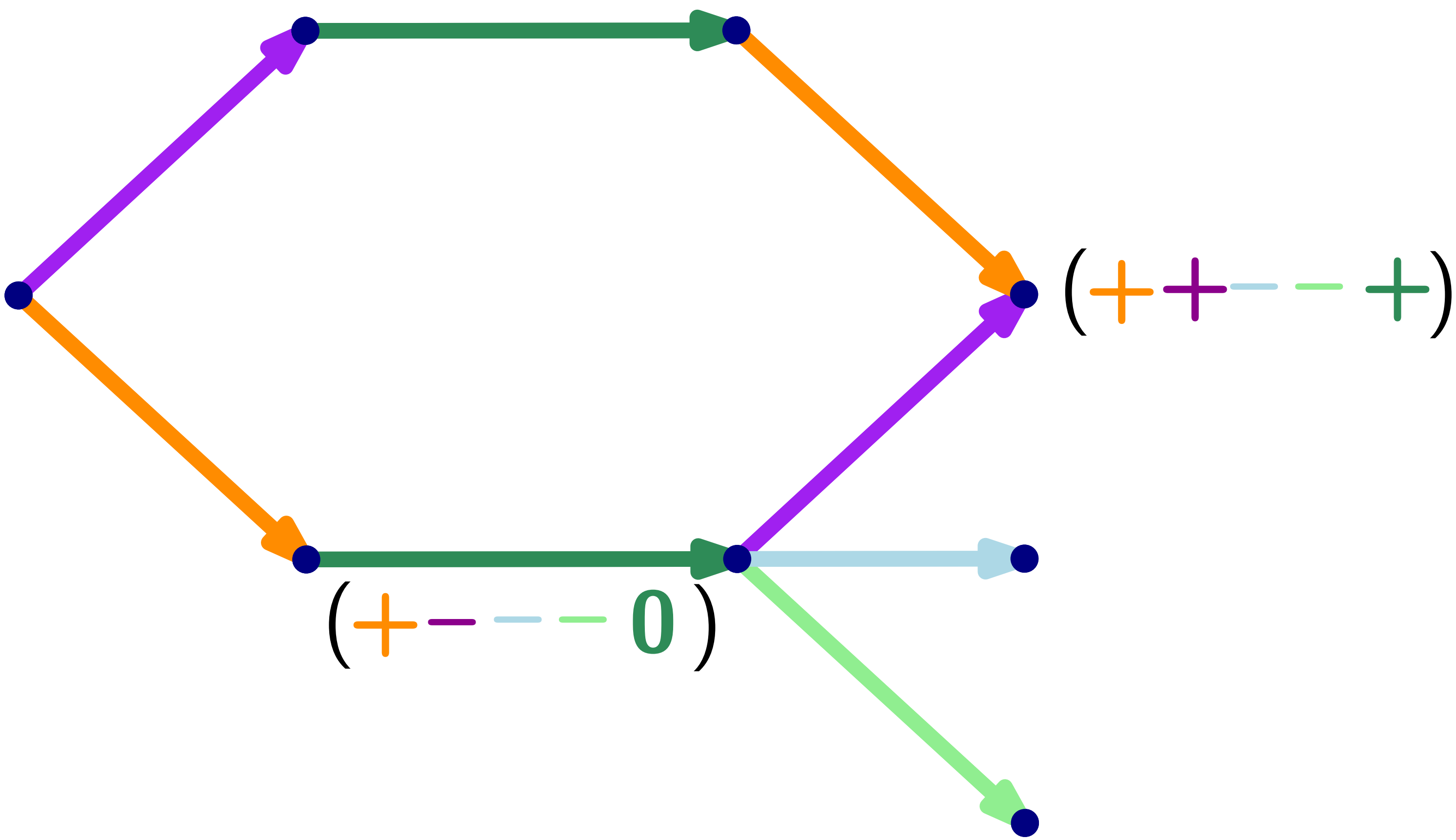

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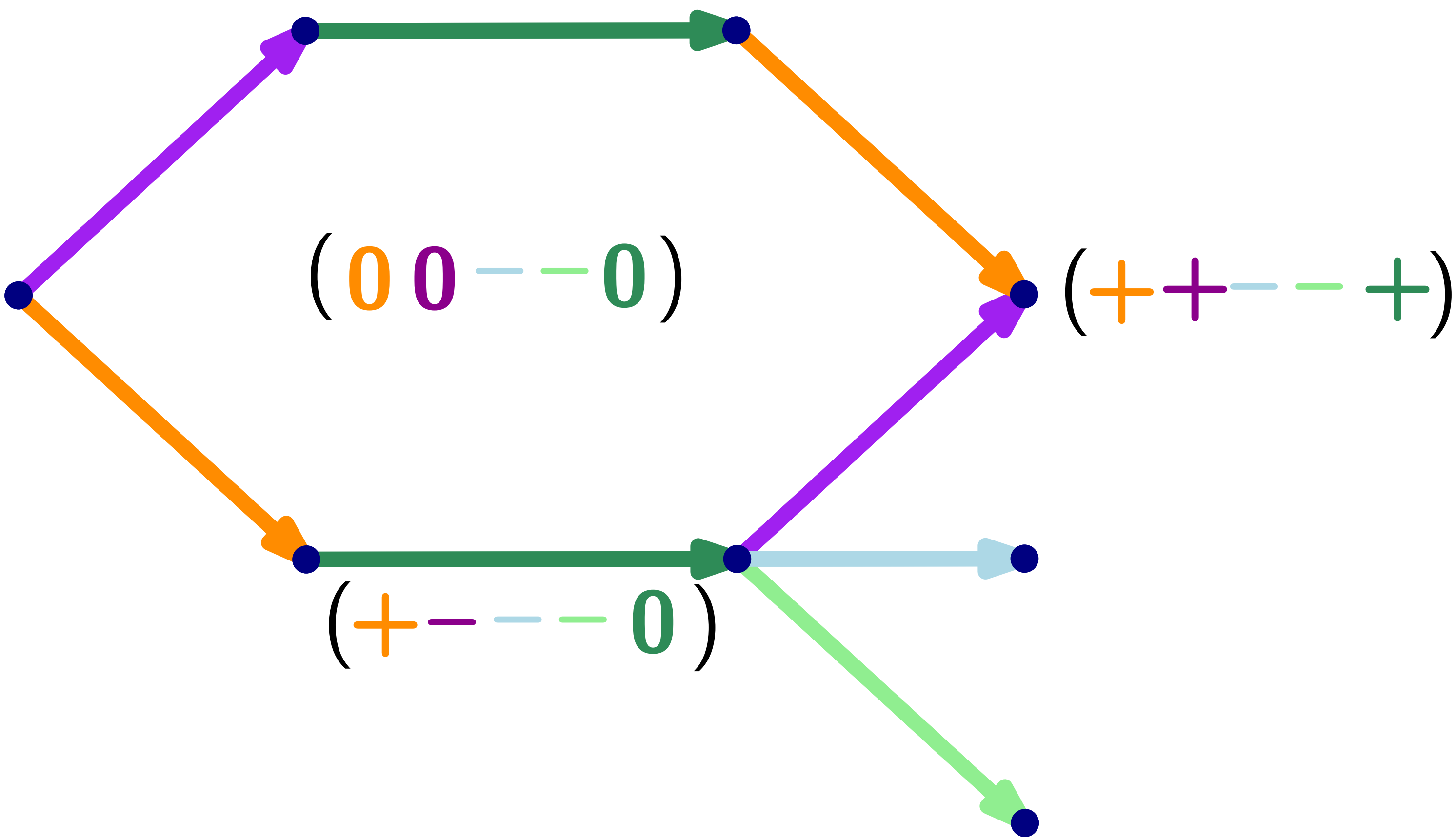

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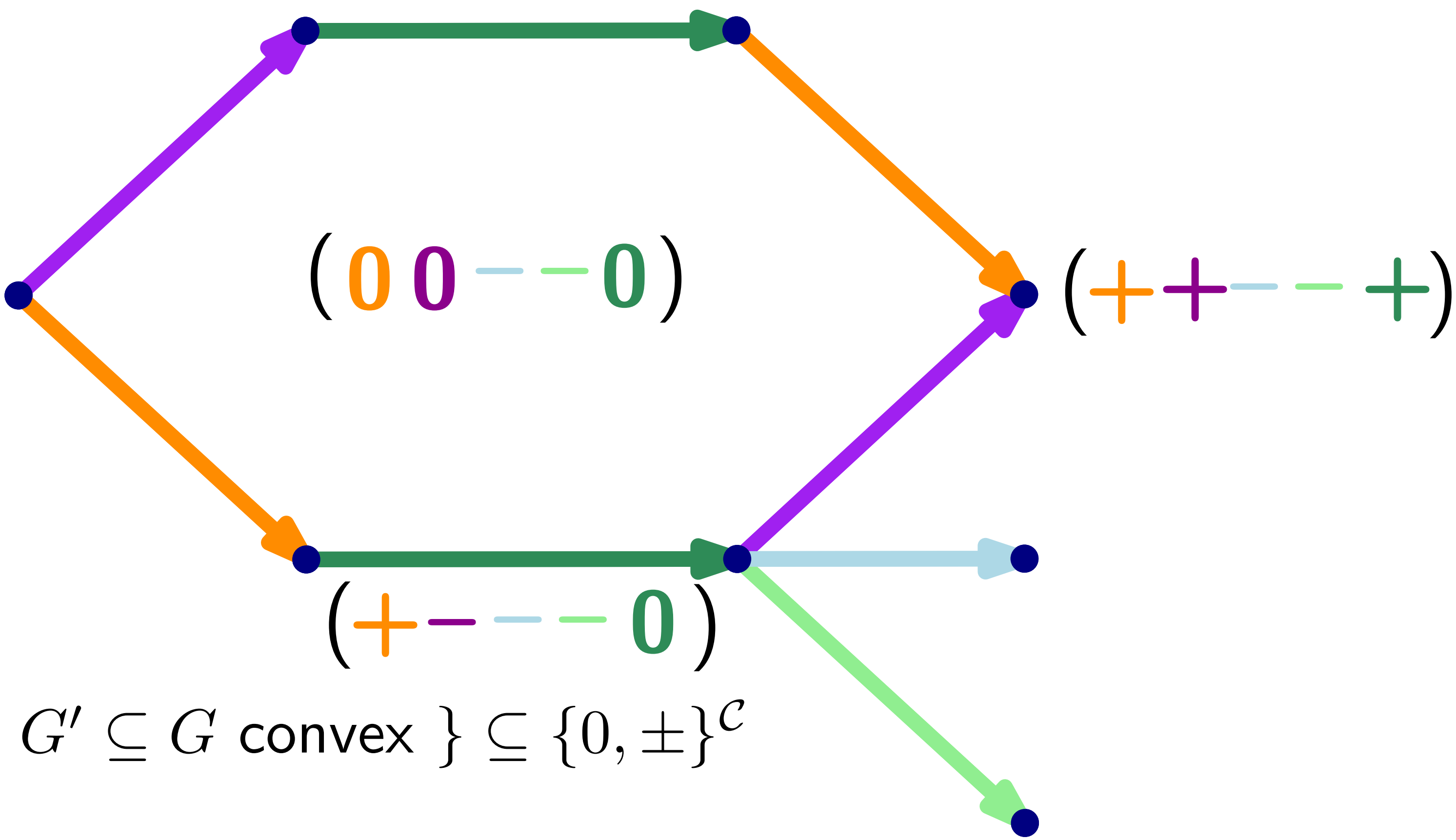

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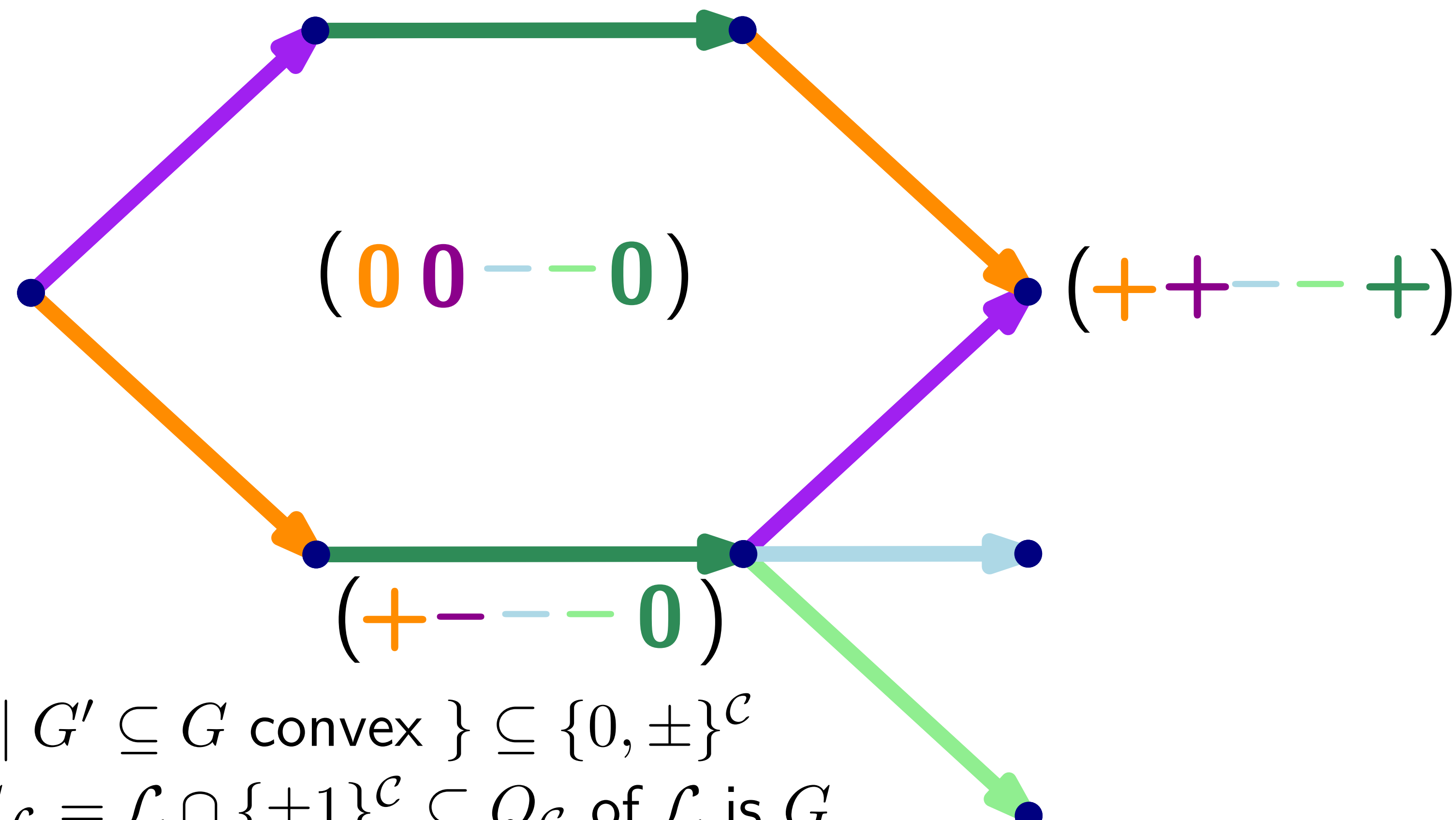

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Gated subgraphs

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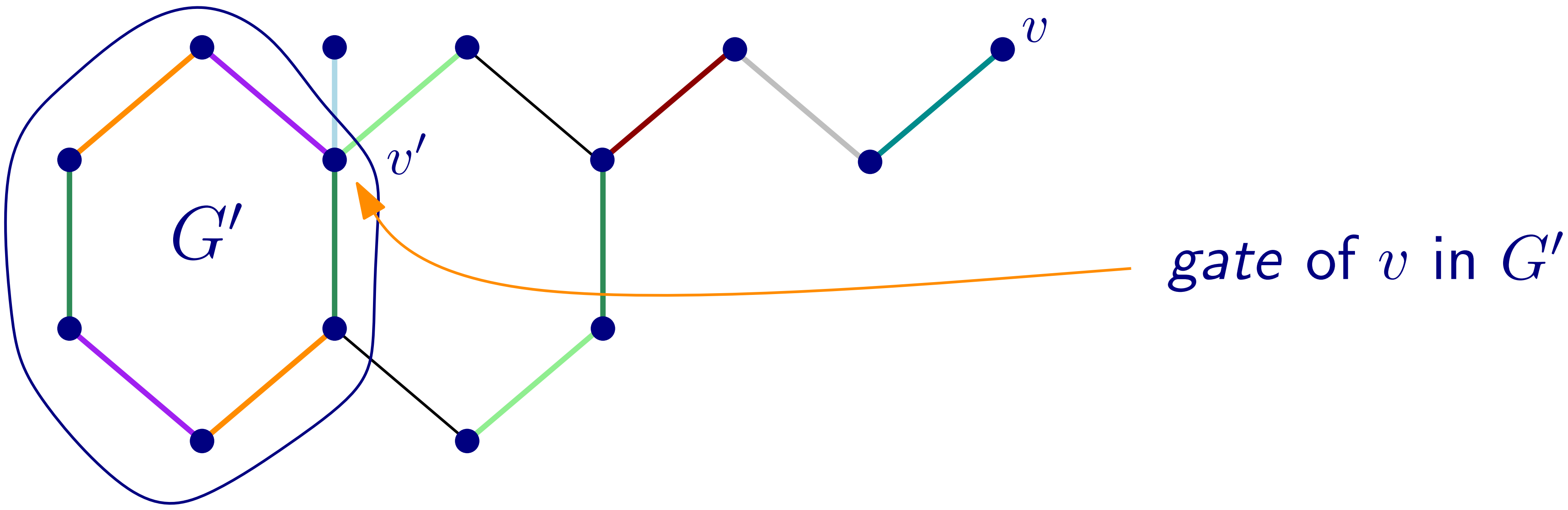

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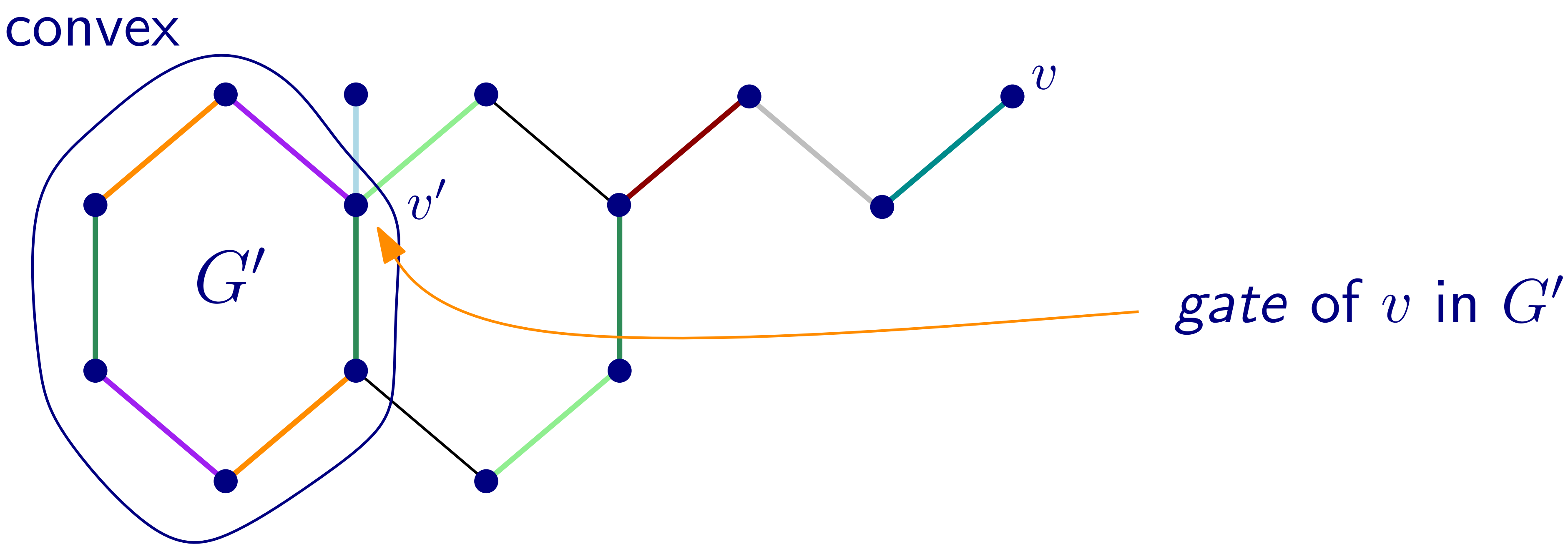

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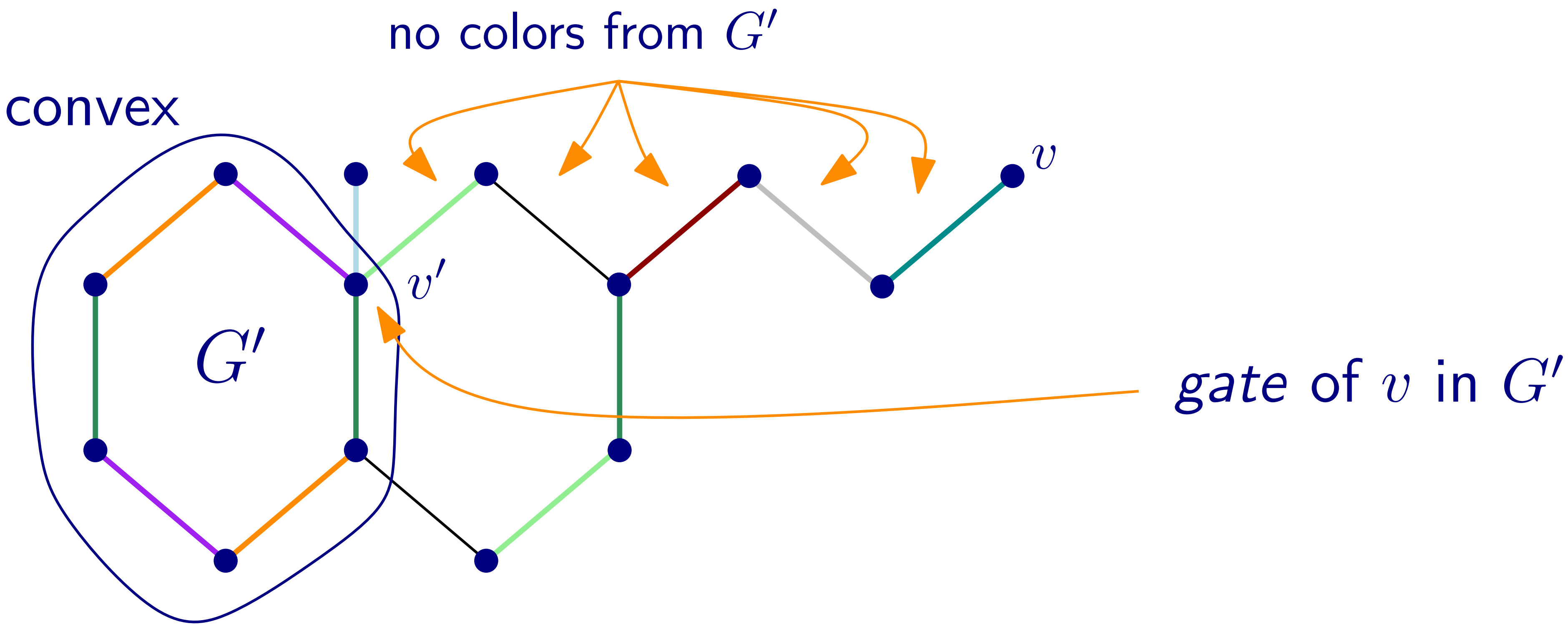

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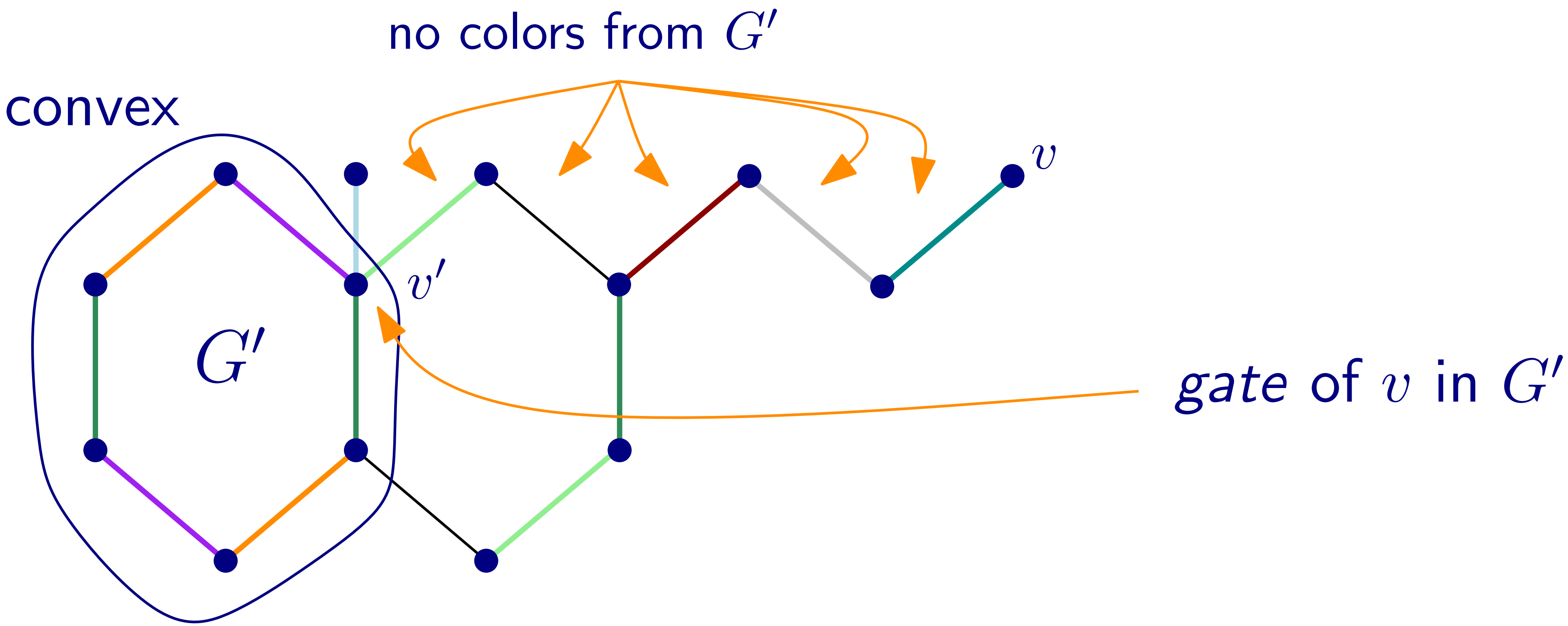

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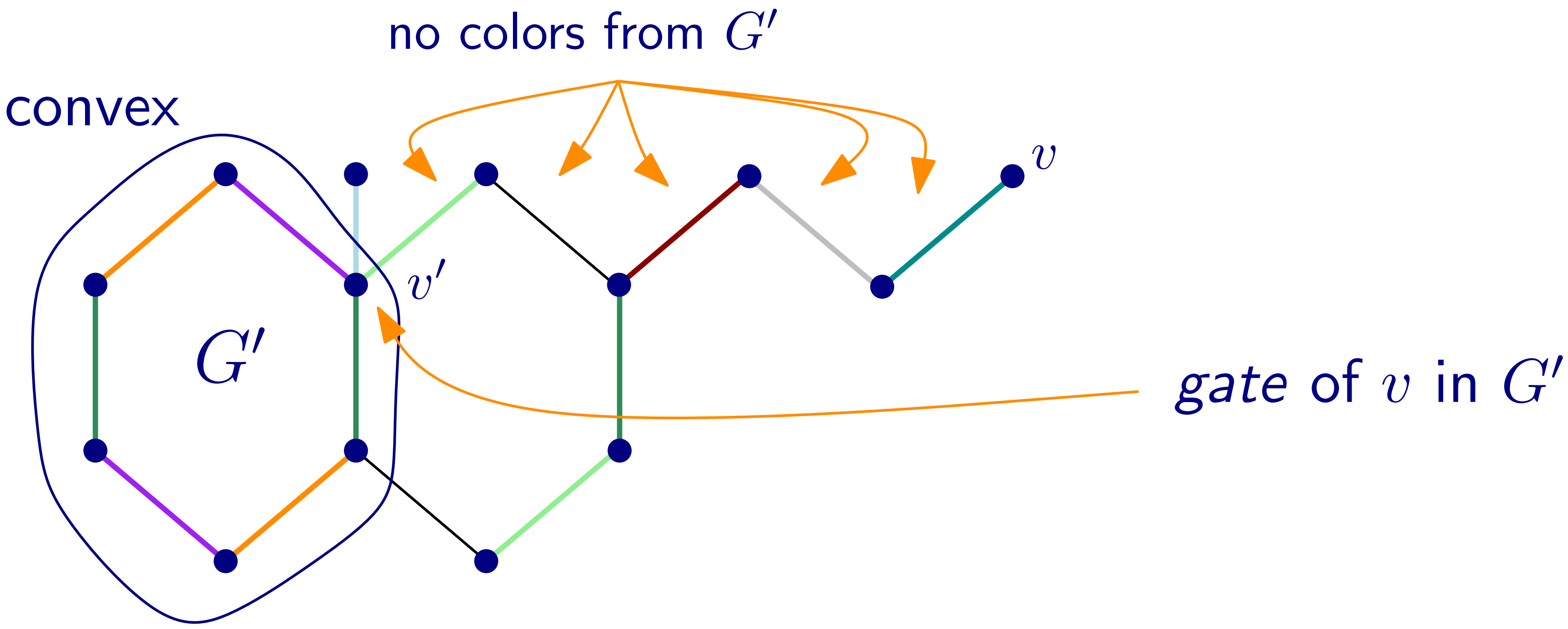

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Antipodal gated subgraphs

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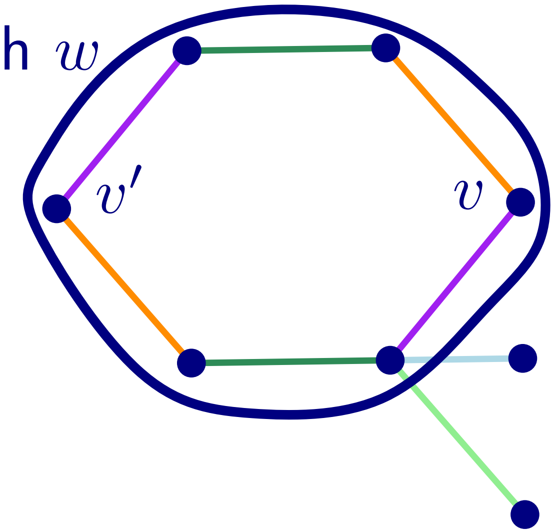

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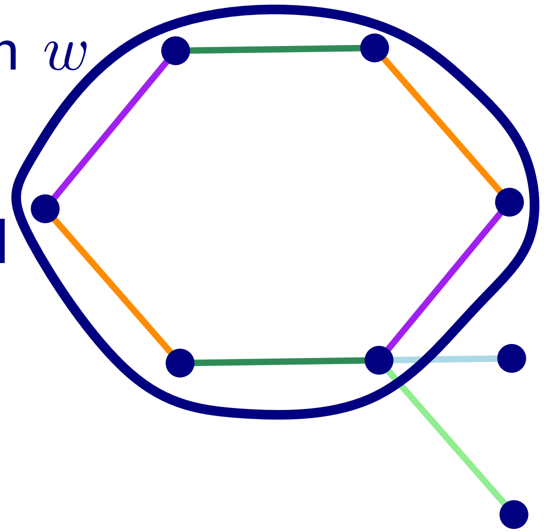

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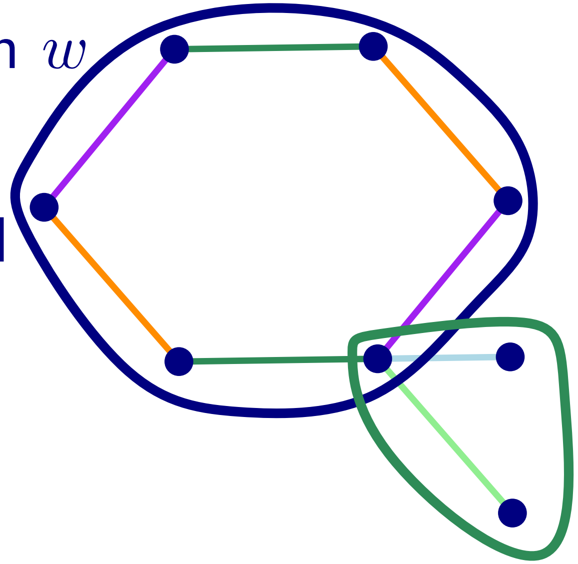

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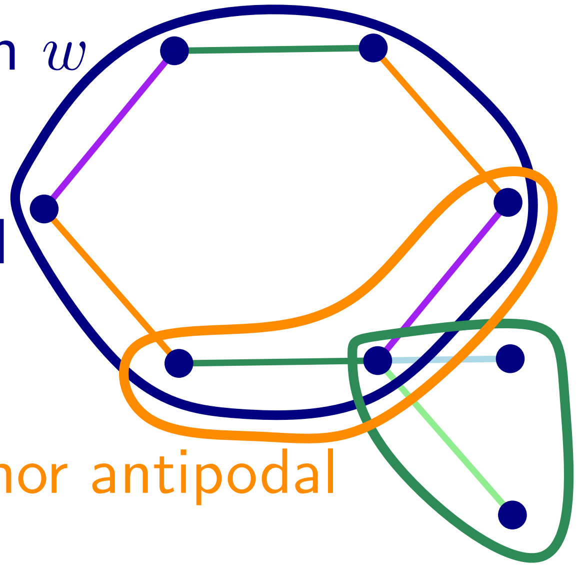

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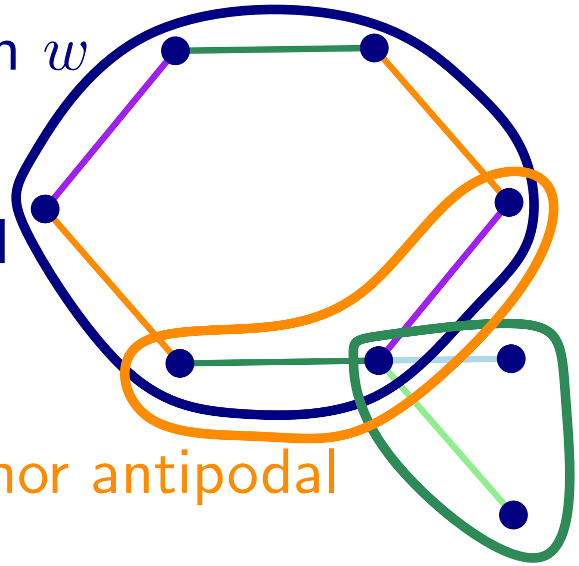

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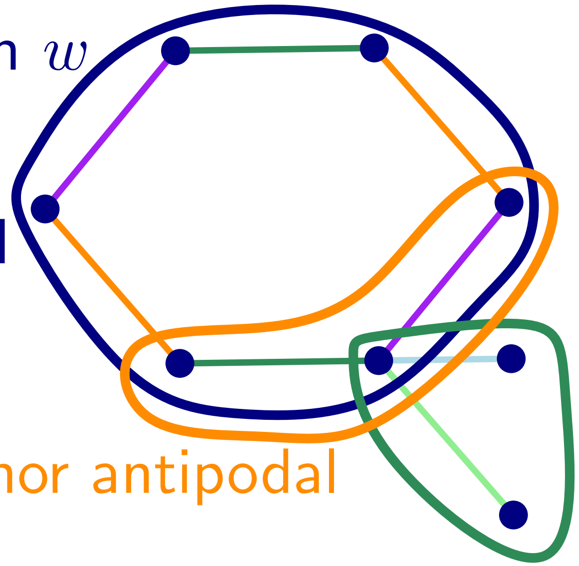

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$\rightsquigarrow$ G tope graph of COM
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AG$= \{G$ partial cube$|$ all antipodal subgraphs gated$\}$

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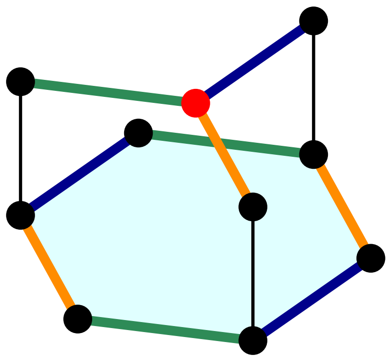

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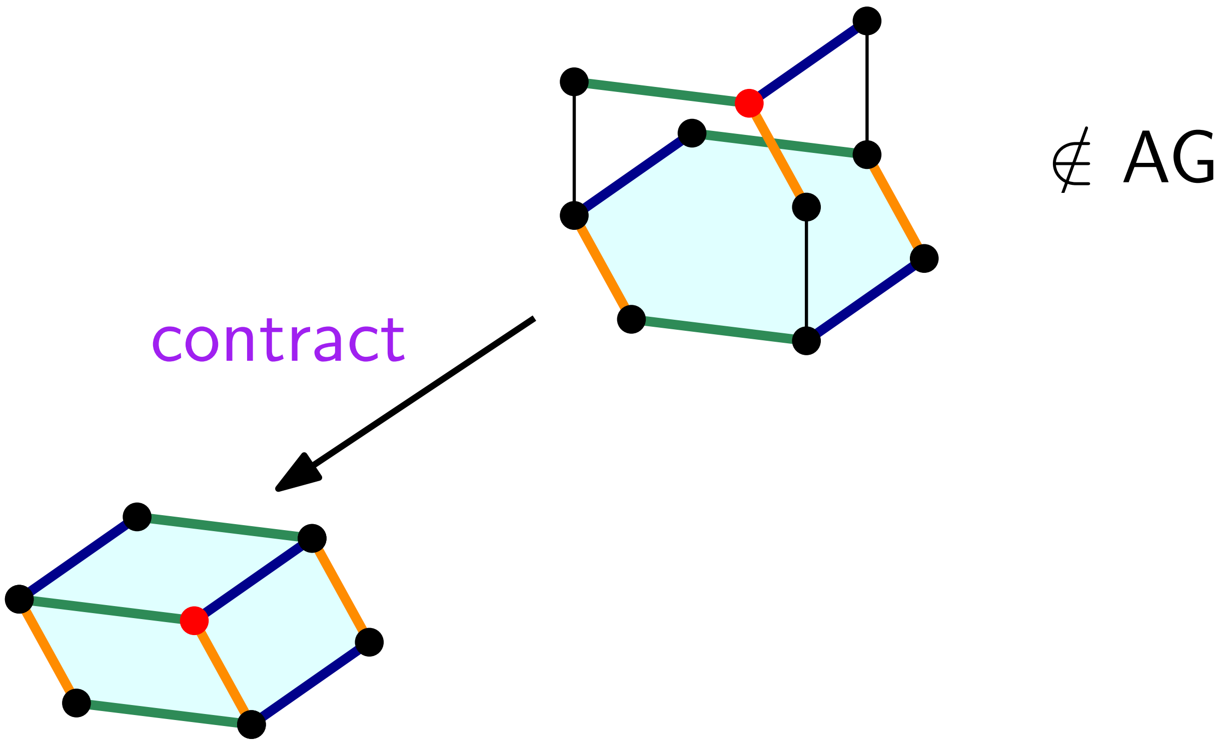

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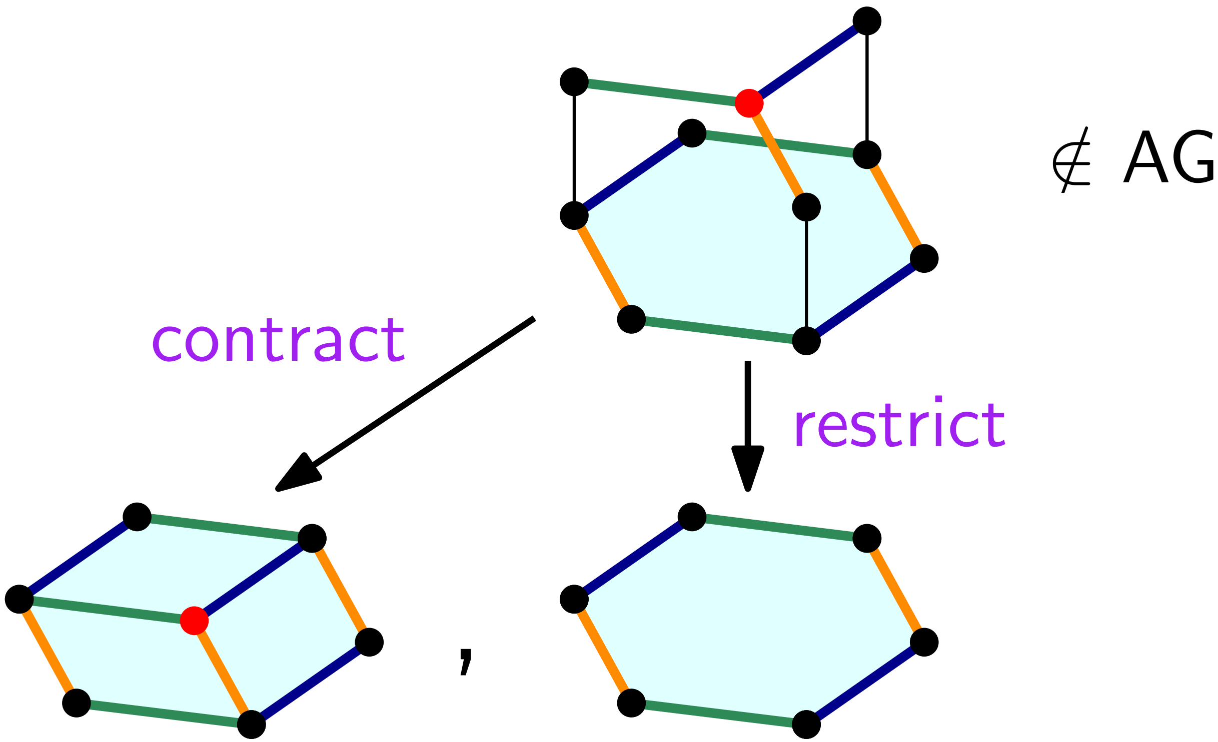

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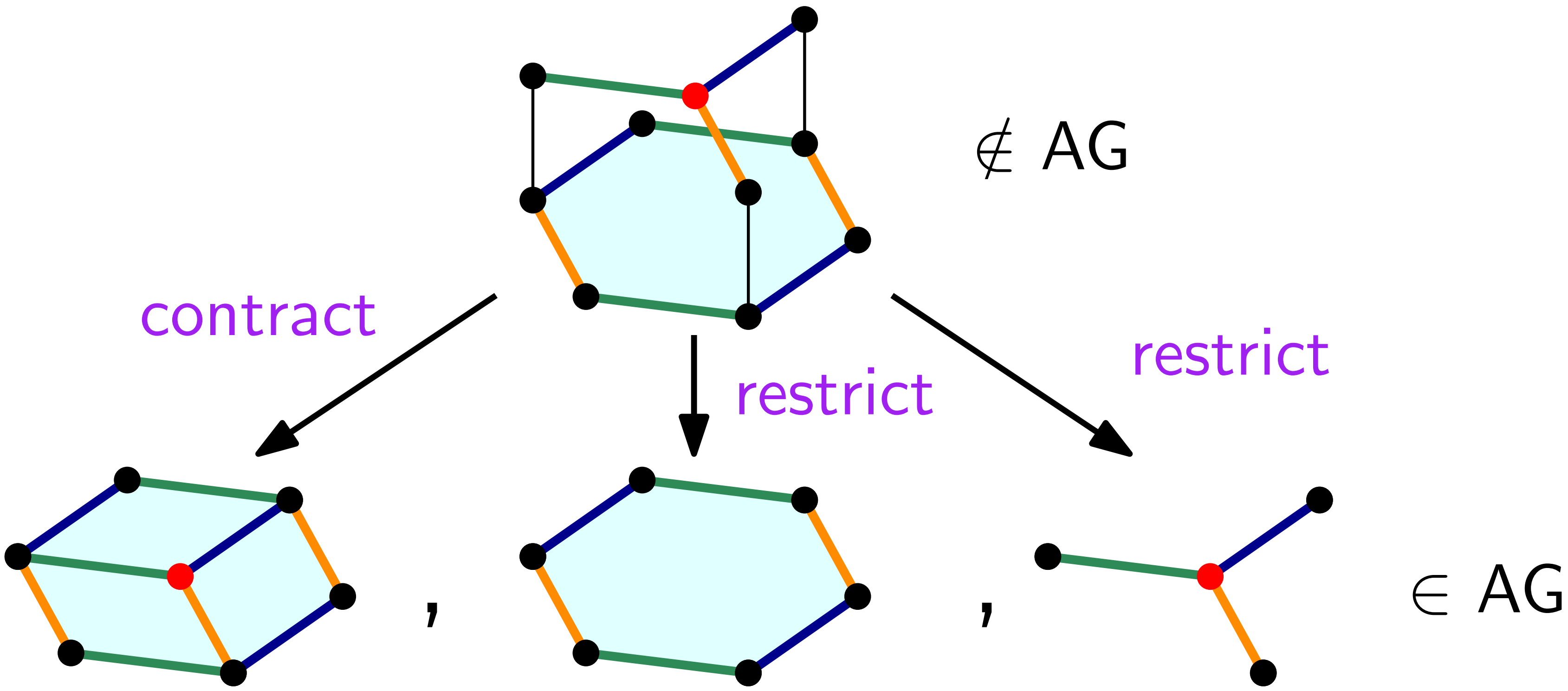

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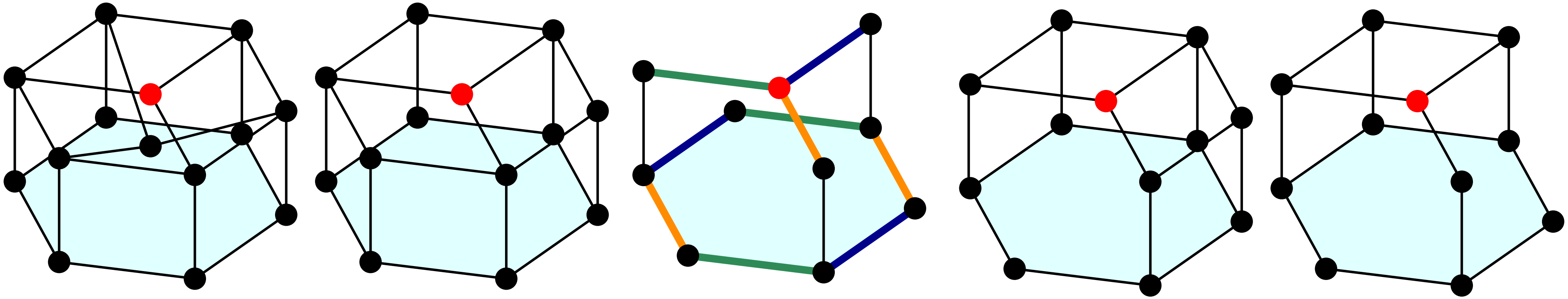

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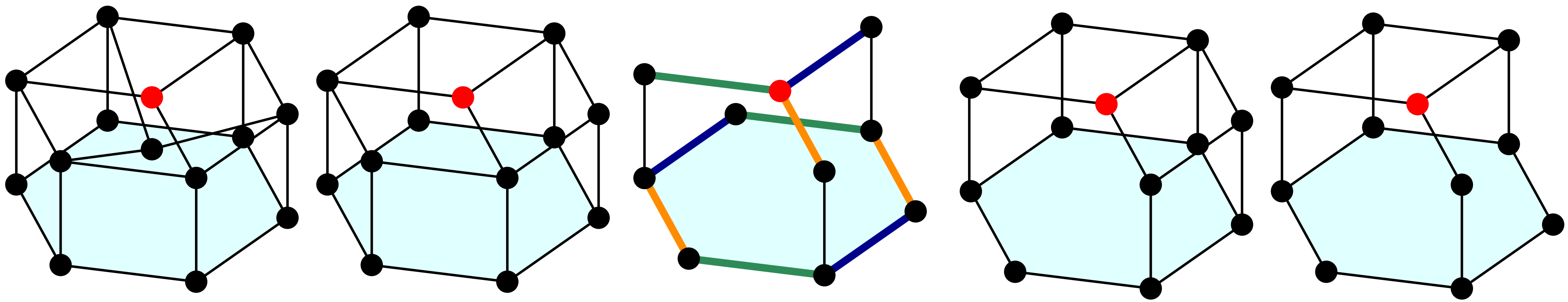

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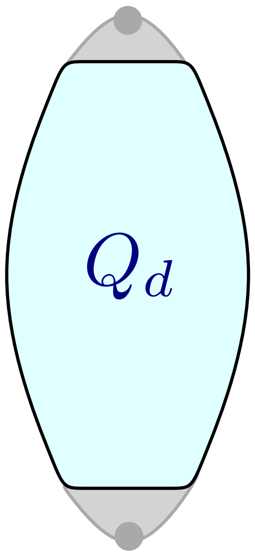

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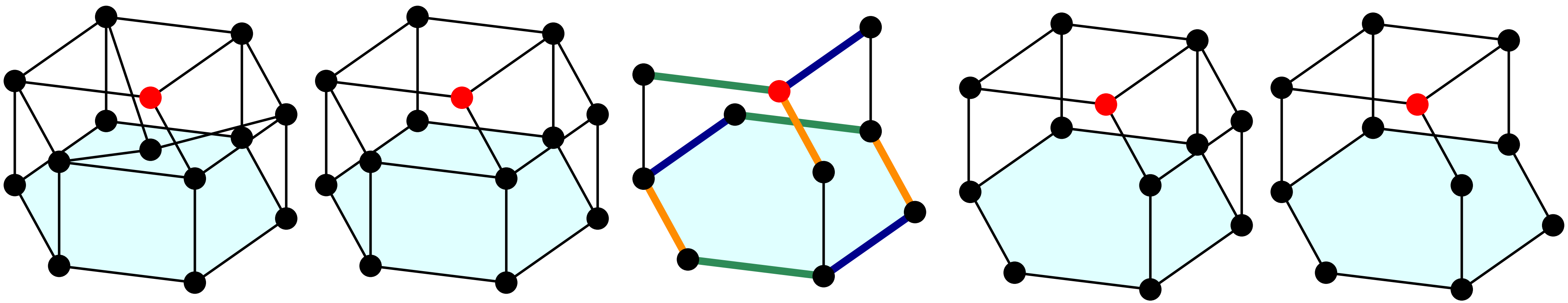

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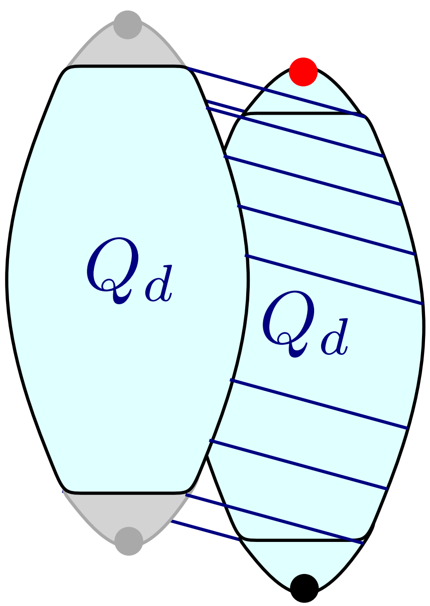

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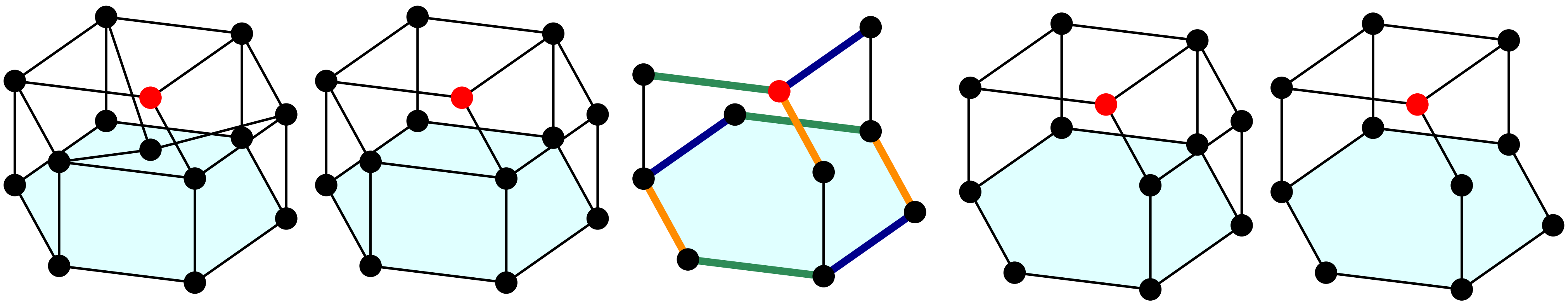

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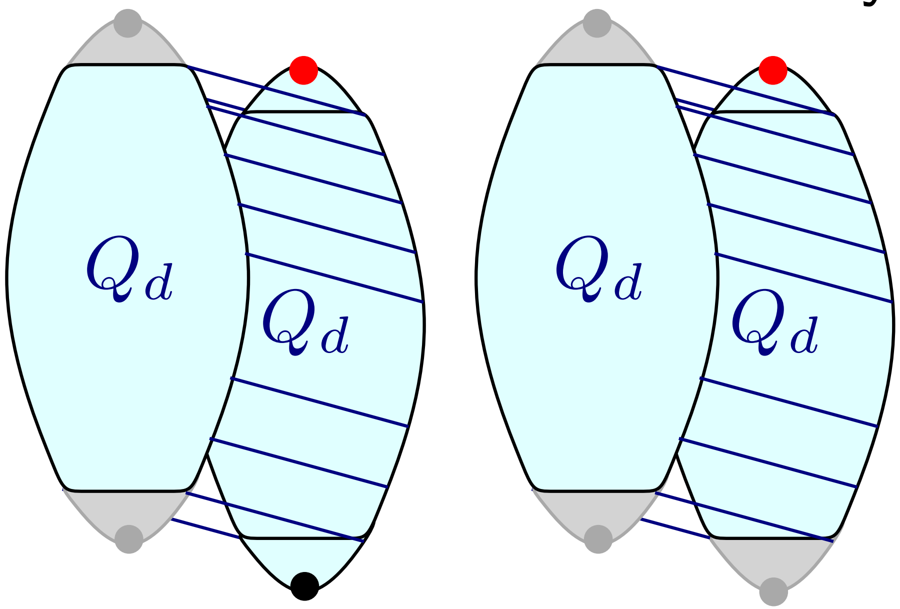

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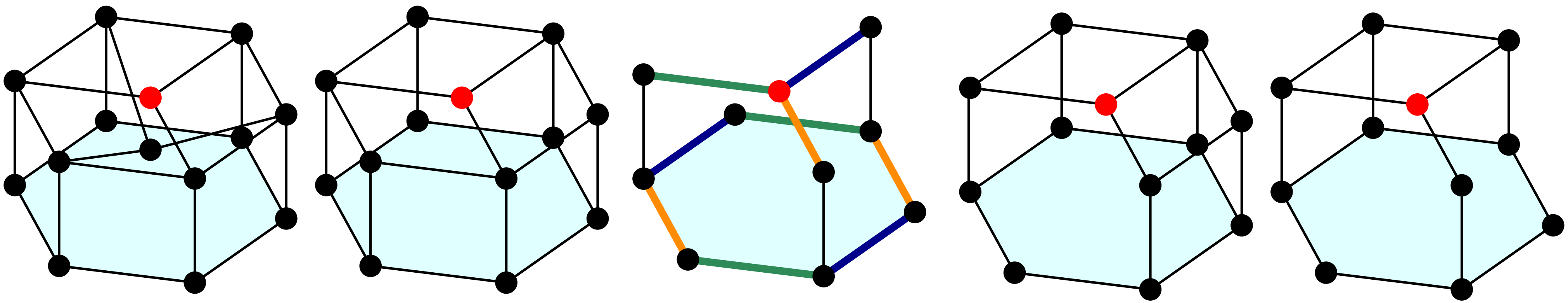

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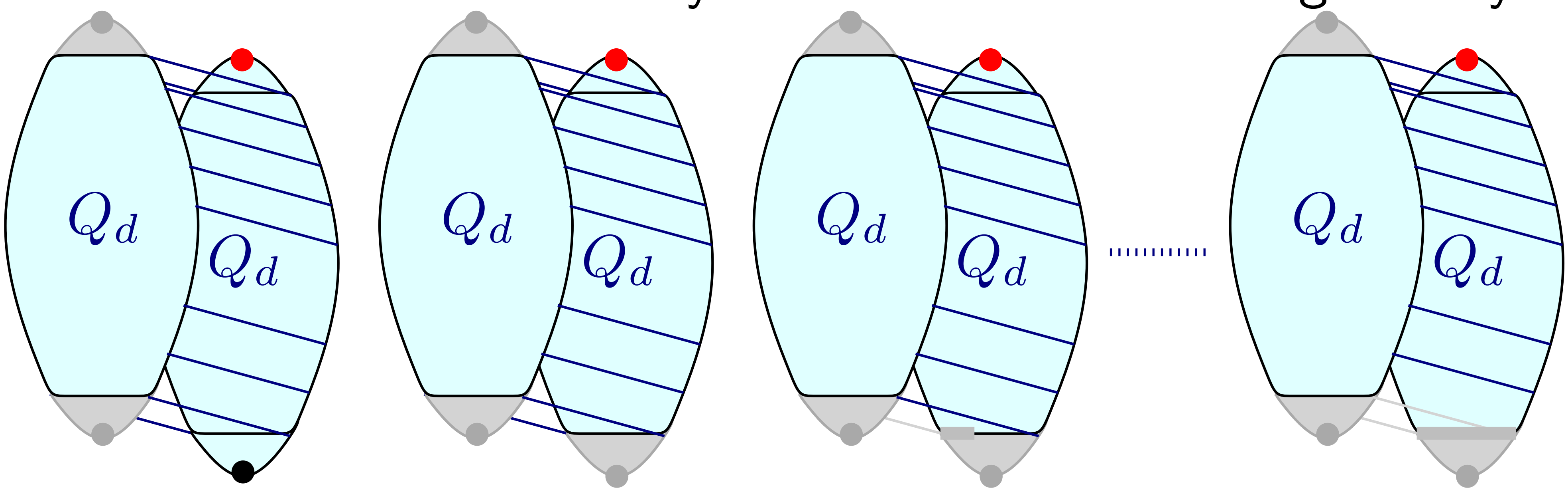

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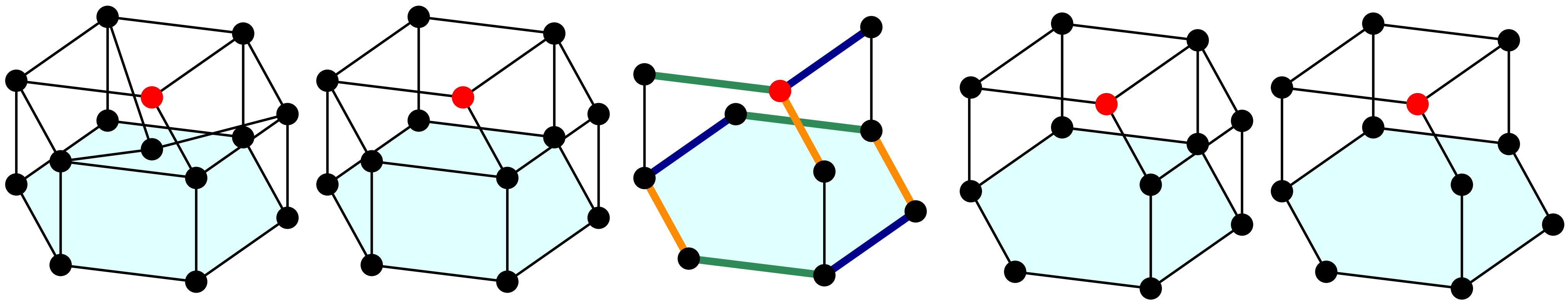

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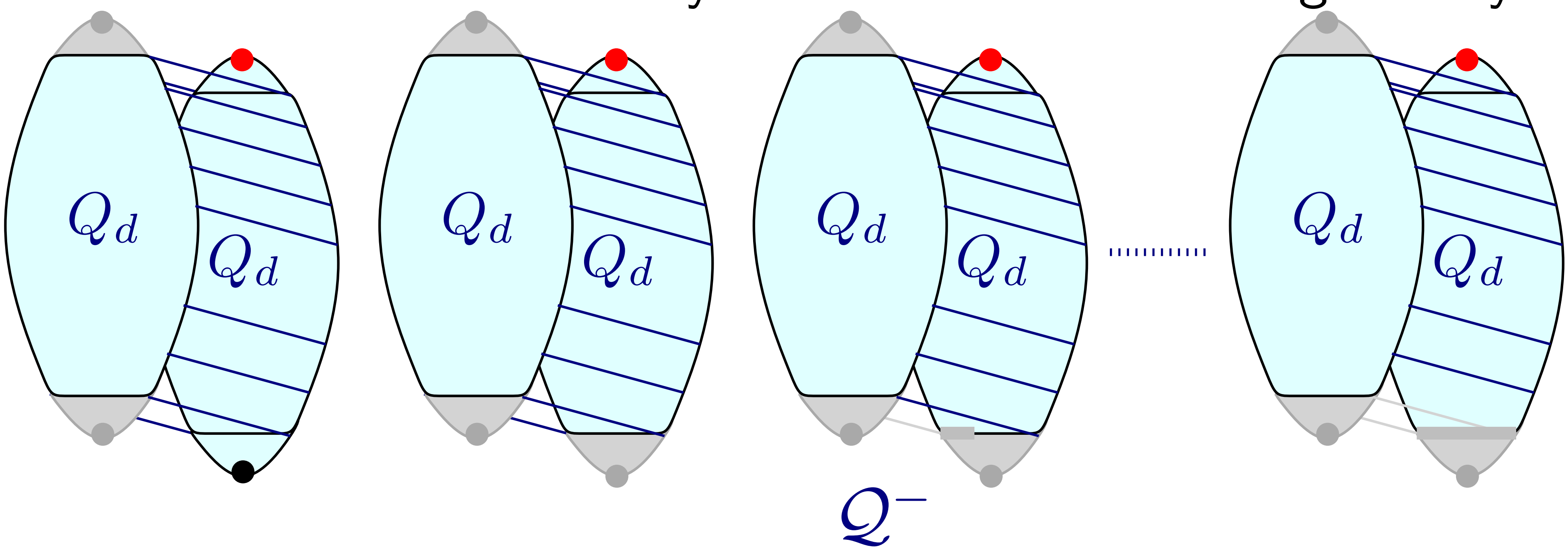

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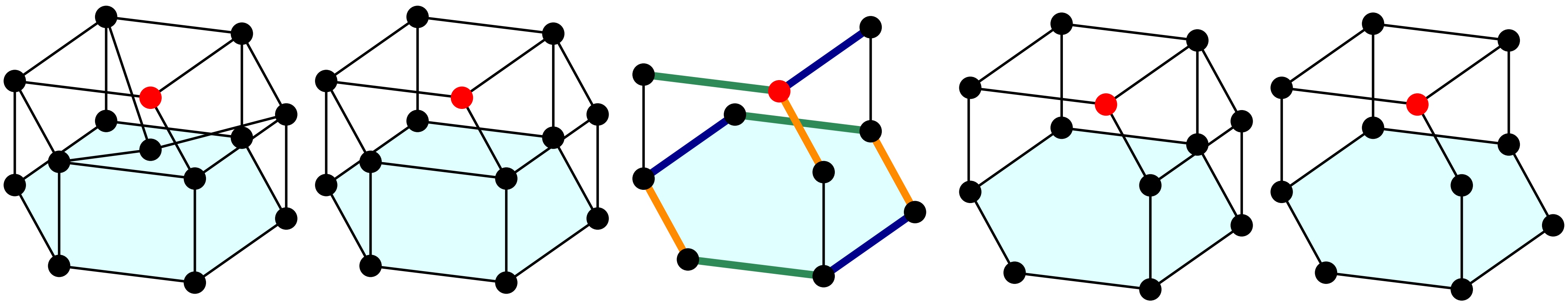

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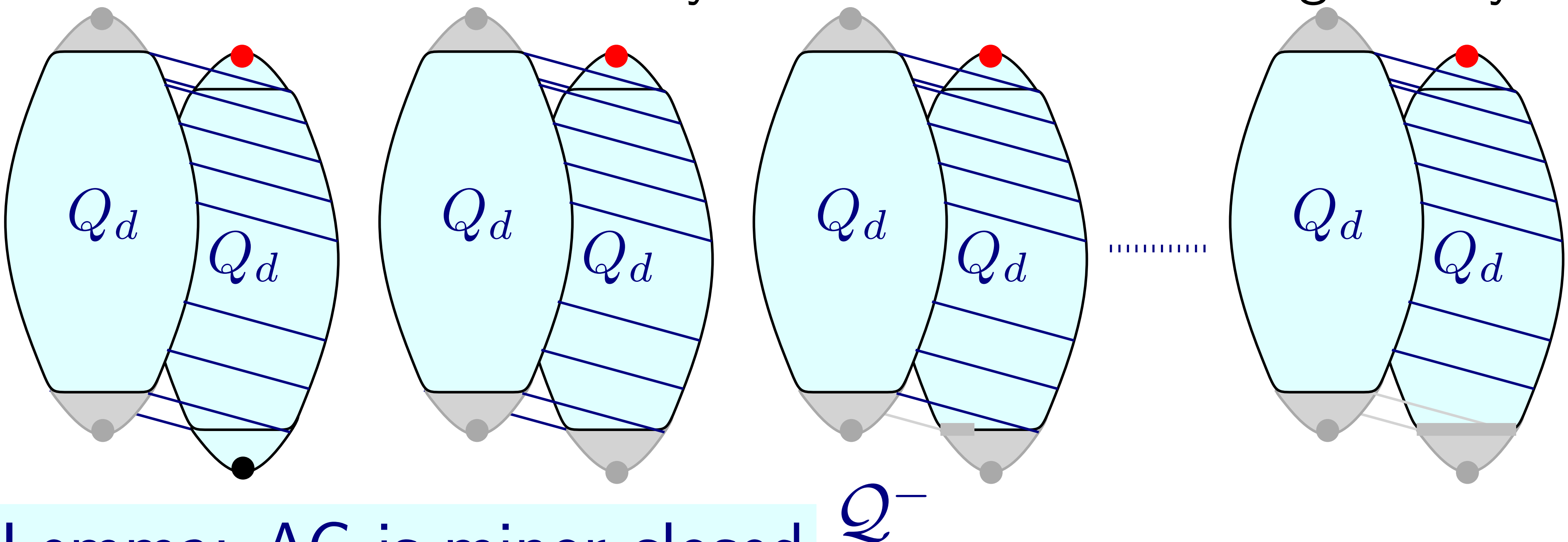

Lemma: AG is minor-closed

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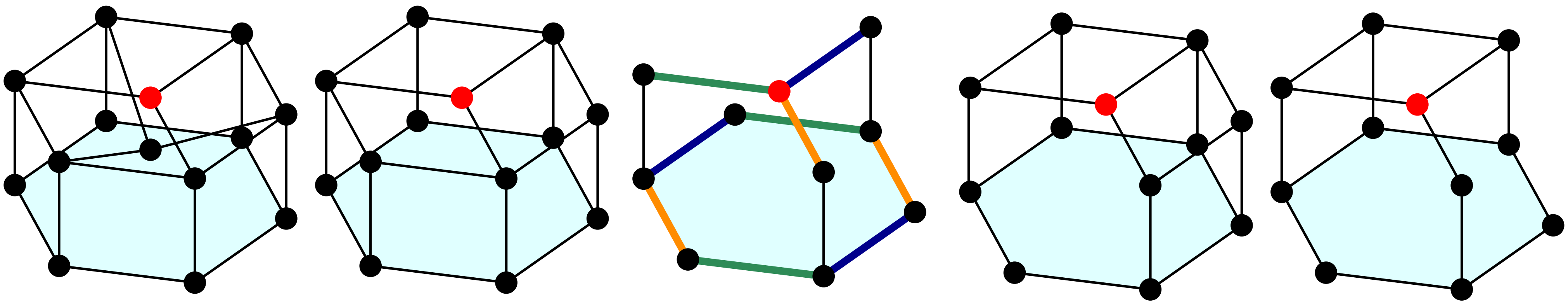

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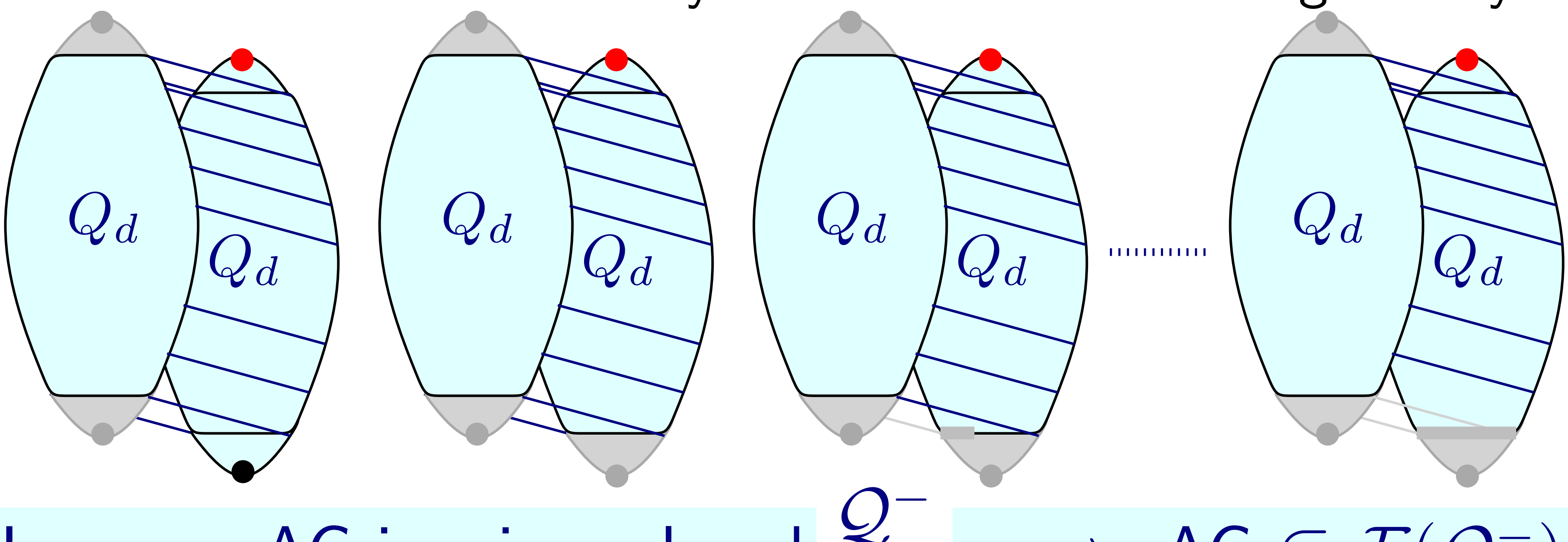

Lemma: AG is minor-closed $\implies$ AG $\subseteq \mathcal{F}(\mathcal{Q}^-)$

Characterization

THM[K, Marc '17]:
for a partial cube G the following are equivalent:

- G is tope graph of a COM
- all antipodal subgraphs of G are gated
- G has no partial cube minor from $\mathcal{Q}^-$
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A common generalization

THM[K, Marc 17]:
G tope graph of COM iff G partial cube such that all antipodal subgraphs gated.

COR:
G tope graph of OM iff G *antipodal* partial cube such that all antipodal subgraphs gated. *da Silva*

COR:
Lawrence
G tope graph of LOP iff G partial cube and all antipodal subgraphs hypercubes.

COR:
G tope graph of AOM iff G *affine* partial cube such that all antipodal and *conformal* subgraphs gated.

Recognition

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naive polytime alogrithm

- check if partial cube $O(n^2)$
- find antipodal subgraphs $O(n^2)$ shortest path intervals
  - check if antipodal
- for each check if gated do some distances

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Conjecture [Bandelt, Chepoi, K '15]: every G_{COM} is convex subgraph of G_{OM}.

would yield

- Topological Representation Theorem with pseudohyperplanes and pseudohalfspaces for COMs
- ideas for duality