Packing arborescences with matroid constraints via matroid intersection

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Combinatorial Geometries: Matroids, Oriented Matroids and Applications
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Outline

1. Summary of classical tree and arborescence packing results and the matroid structure behind them

packing

= edge-disjoint subgraphs

2. Some basics of rigidity theory
3. Summary of recent tree and arborescence packing result
4. New result: matroid structure behind the above packing results

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Theorem (Tutte, Nash-Williams)

In a graph $G = (V, E)$, there exists a packing of k spanning trees iff

$$e_G(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$$

holds for every partition $\mathcal{P}$ of V, where $e_G(\mathcal{P})$ denotes the number of edges that are not induced by any set of the partition.

Matroid structure behind

Union of k spanning trees = bases of the k-sum of the graphic matroid.

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Union of k spanning trees = bases of the k-sum of the graphic matroid.

s-arborescence = directed tree s.t. each node is reachable from its root on a one-way path

ϱ = the in-degree

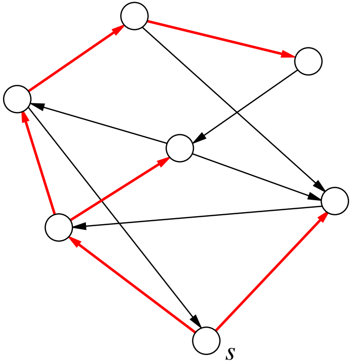

Figure: An *s*-arborescence

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In a rooted digraph $D = (V + s, A)$, there exists a packing of k spanning s-arborescences iff

$$\varrho(X) \geq k$$

holds for every $\emptyset \neq X \subseteq V$.

Remark

The Tutte–Nash-Williams-theorem follows from Edmonds' theorem by using Frank's orientation theorems.

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Matroid structure behind

Let the independent sets of $\mathcal{M}_0$ be the arc sets of D with maximum in-degree k on each $v \in V$ = direct sum of the uniform matroids of rank k on the incoming arc sets of all vertices in V.
Union of k spanning arborescences = common bases of the k-sum of the graphic matroid and $\mathcal{M}_0$. $\Rightarrow$ Efficient algorithm for the weighted problem through the weighted matroid intersection algorithm.

Kamiyama, Katoh, Takizawa: If there are no spanning arborescences... When is it possible to pack edge-disjoint “maximal” arborescences?

Reachability s-arborescence in D: an s-arborescence that spans each vertex which is reachable from s on a one-way path of D.

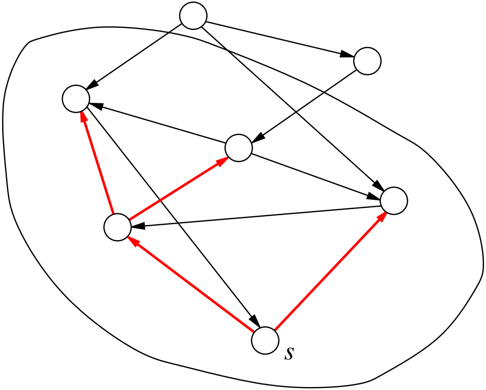

Figure: A reachability s-arborescence

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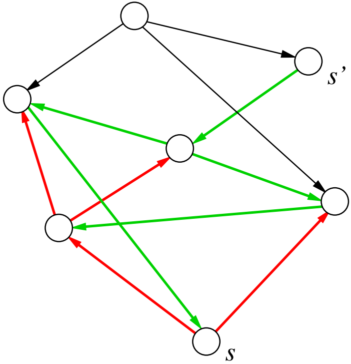

Figure: A packing of reachability s and s'-arborescences

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Theorem (Kamiyama, Katoh and Takizawa)

In a digraph $D = (V, A)$, let $R := \{s_1, ..., s_k\}$ be a multiset of vertices in V. There exists a packing of reachability s_i-arborescences in D $(i = 1, \ldots, k)$ iff

$$\varrho(X) \geq p'_R(X)$$

holds for every $X \subseteq V$ where $p'_R(X)$ denotes the number of s_i's for which X is reachable from s_i and $s_i \notin X$.

bar-joint framework: A graph $G = (V, E)$ and a map $p : V \to \mathbb{R}^d$ called a realization of G $\to$ vertices correspond to flexible joints that connect rigid bars.

infinitesimal motion of a bar-joint framework (G, p): $m : V \to \mathbb{R}^d$ s.t. $\langle m(u) - m(v), p(u) - p(v) \rangle = 0$ for every $uv \in E$.

rigidity matrix $R(G, p) \in \mathbb{R}^{d|V| \times |E|}$: each row corresponding to an edge uv is 0 except at the $d + d$ coordinates corresponding to u and v where its values are given by the $\mathbb{R}^d$ vectors $p(u) - p(v)$ and $p(v) - p(u)$, respectively $\to$ the space of infinitesimal motions is the kernel of $R(G, p)$

infinitesimally rigid bar-joint framework: the rank of $R(G, p)$ is $d|V| - \binom{d+1}{2}$, i.e. maximal as the isometries of $\mathbb{R}^d$ imply $\binom{d+1}{2}$ trivial infinitesimal motions

generic framework: the coordinates of p are algebraically independent over $\mathbb{Q}$ $\to$ when p is generic, rigidity does only depend on G

rigidity matroid: the linear matroid implied by the columns of $R(G, p)$ (if p is not given, then for generic p)

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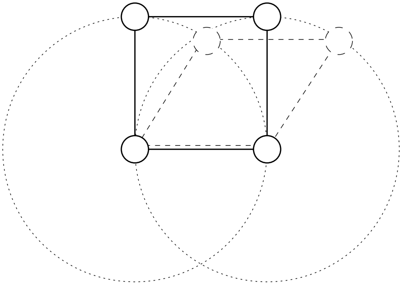

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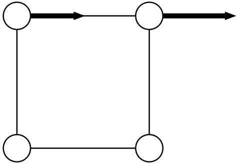

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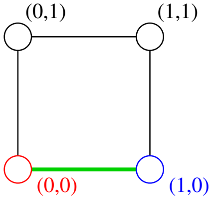

$$R(G,p) = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

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Theorem (Laman, Crapo)

Let $G = (V, E)$ be a loop-colored graph. Then the following are equivalent.

(i) For every/one generic realization p of G the corresponding bar-joint framework is minimally (infinitesimally) rigid.

(ii) (Laman) The following sparsity conditions hold:

(L1) $|E| = 2|V| - 3$,

(L2) $|F| \leq 2|V(F)| - 3$ *for every* $F \subseteq E$.

(iii) (Crapo) E has a partition $\{F_1, F_2, F_3\}$ where F_i is a forest such that each vertex is covered by exactly two of them and and no two subtrees span the same vertex set.

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Rank function for rigidity matroid

Theorem (Edmonds, Rota)

Let $h : 2^E \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ a monotone non-decreasing intersecting submodular set function. Then

$$\mathcal{I}_h = \{Y \subseteq S : |X| \leq h(X) \text{ for every } X \subseteq Y\}$$

forms the independent set family of a matroid $\mathcal{M}_h$ with rank function

$$r_h(Z) := \min\left\{\sum_{X \in \mathcal{P}} h(X) + |Z - \bigcup \mathcal{P}| : \mathcal{P} \text{ a subpart. of } Z\right\}.$$

Let $h(F) := 2|V(F)| - 3$ for $F \neq \emptyset$. As $h(e) = 1$ we can consider partitions instead of subpartitions.

Theorem (Lovász, Yemini)

The rank function of the rigidity matroid is the following:

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loop-colored graph: a multigraph $G = (V, E)$ without multiple edges with a coloring $c : L \to \{c_1, \ldots, c_k\}$ on its loop set $L \subseteq E$

bar-joint-slider framework on the plane: a bar-joint framework where a part of the joints are constrained by using line-sliders some of which may be parallel (i.e., the infinitesimal velocity on a joint constrained by a slider in direction x must by parallel to x)

loop=slider; color=direction

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Let $G = (V, E)$ be a loop-colored graph. Then the following are equivalent.

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(ii) The following sparsity conditions hold:

(KT1) $|E| = 2|V|$

(KT2) $|F| \leq 2|V(F)| - 3$ *for every* $F \subseteq E - L$,

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~~*(i) For every (injective) mapping of the colors $\{c_1 \dots, c_k\}$ to $\mathbb{S}^1$ and any generic realization p of G the corresponding bar-and-slider framework is minimally rigid.*~~

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Definition

1. matroid-rooted graph/digraph $(G = (V + s, E), \mathcal{M})$/ $(D = (V + s, A), \mathcal{M})$: a matroid $\mathcal{M}$ is given on the set of root edges/arcs (leaving s).

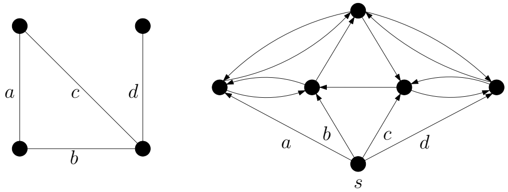

Figure: a matroid-rooted digraph

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2. $\mathcal{M}$-based packing of (s, t)-paths: if the root edges/arcs form a base of $\mathcal{M}$.

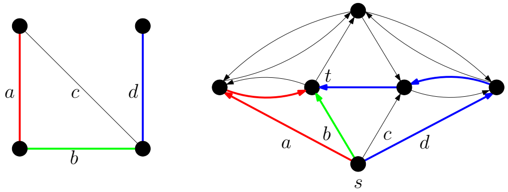

Figure: an $\mathcal{M}$-based packing of (s, t)-paths

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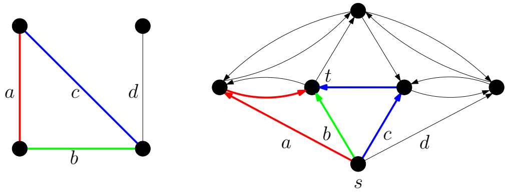

Figure: Not an $\mathcal{M}$-based packing of (s, t)-paths

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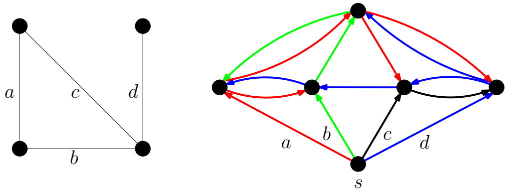

Figure: an $\mathcal{M}$-based packing of s-arborescences

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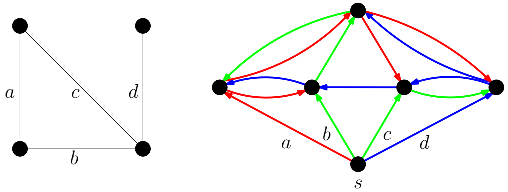

Figure: an $\mathcal{M}$-based packing of spanning s-arborescences

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1. matroid-rooted graph/digraph $(G = (V + s, E), \mathcal{M})/$ $(D = (V + s, A), \mathcal{M})$: a matroid $\mathcal{M}$ is given on the set of root edges/arcs (leaving s).
2. $\mathcal{M}$-based packing of (s, t)-paths: if the root edges/arcs form a base of $\mathcal{M}$.
3. $\mathcal{M}$-based packing of s-trees/arborescences: if the packing of (s, t)-paths provided by the trees/arborescences is $\mathcal{M}$-based $\forall t \in V$.

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Menger type characterization: $\exists$ an $\mathcal{M}$-based packing of (s, t)-paths iff $\varrho(X) \geq r(\partial_s(V)) - r(\partial_s(X))$ $(\forall t \in X \subseteq V)$.

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Question

Can the above theorems be extended for $\mathcal{M}$-based packings?

Theorem (Katoh, Tanigawa)

In a matroid-rooted graph $(G = (V + s, E), \mathcal{M})$ *there exists an* $\mathcal{M}$*-based packing of* ~~spanning~~ *s-trees iff*

$$e_G(\mathcal{P}) \geq r(\mathcal{M})|\mathcal{P}| - \sum_{X \in \mathcal{P}} r(\partial_s(X)) \text{ for every partition } \mathcal{P} \text{ of } V.$$

Matroid structure behind

Katoh and Tanigawa also proved that the $\mathcal{M}$-based packings of s-trees form the bases of the matroid induced by the following non-negative integer valued, monotone and intersecting submodular function:

$$b'(H) := k|V(H) - s| - k + r(H \cap \partial_s(V)) \;\; \forall \emptyset \neq H \subseteq A,$$

i.e. the matroid $\mathcal{M}_{b'}$ with independent sets

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Theorem (Durand de Gevigney, Nguyen and Szigeti)

In a matroid-rooted digraph $(D = (V + s, A), \mathcal{M})$ there exists an $\mathcal{M}$-based packing of ~~spanning~~ s-arborescences iff there exists an $\mathcal{M}$-based packing of (s, v)-paths for every $v \in V$, i.e.

$$\varrho(X) \geq r(\mathcal{M}) - r(\partial_s(X)) \ (\forall \emptyset \neq X \subseteq V).$$

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The Katoh–Tanigawa-theorem follows from this theorem by using Frank's orientation theorems.

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Edge sets of $\mathcal{M}$-based packing of s-arborescences = common bases of $\mathcal{M}_{b'}$ and $\mathcal{M}_0$.

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In a matroid-rooted digraph $(D = (V + s, A), \mathcal{M})$ there exists an $\mathcal{M}$-based packing of spanning s-arborescences iff

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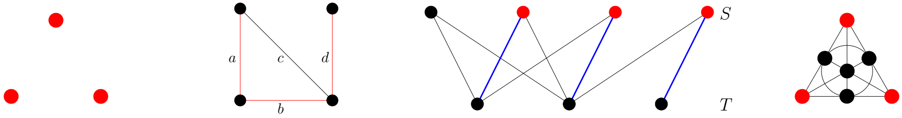
Examples

1. Free : all subsets of a set,
2. Graphic : edge-sets of forests of a graph,
3. Transversal: end-nodes in S of matchings of bipartite graph $(S, T; E)$
4. Fano: subsets of sets of size 3 not being a line in the Fano plane.



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Theorem (Fortier, K., Szigeti, Tanigawa)

The conjecture is true when the matroid $\mathcal{M}$

- *has rank at most 2 or*
- *is graphic or*
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The conjecture is false!
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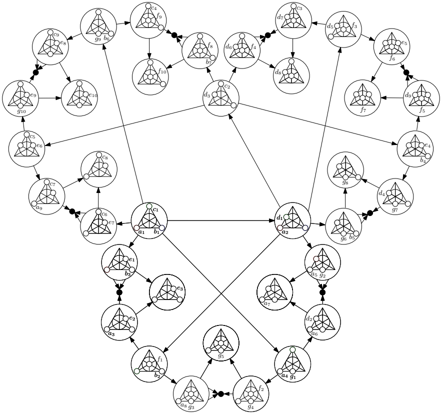
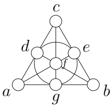
Counterexample

Digraph : acyclic, in-degree 3 for all $v \in V$, 46 nodes and 135 arcs,
Matroid : parallel extension of Fano with 64 elements,
Remark : matroid-based packing of (s, t)-paths exists for all t.





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Definitions

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Matroids from submodular bi-set functions

Theorem (Edmonds, Rota)

Let $h : 2^E \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ a monotone non-decreasing intersecting submodular set function. Then

$$\mathcal{I}_h = \{Y \subseteq S : |X| \leq h(X) \text{ for every } X \subseteq Y\}$$

forms the independent set family of a matroid $\mathcal{M}_h$.

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Matroid structure behind reachability-based packings

For simplicity we assume the following:

Assumption

(A1) $\partial_s(v)$ is independent in $\mathcal{M}$ for every $v \in V$.

This can be assumed by adding an extra vertex in the middle of each.

We define matroid $\mathcal{M}'_0$ (similarly to $\mathcal{M}_0$): for each $v \in V$ we take a matroid $\mathcal{M}'_v$ on the arcs entering v to be the direct sum of the free matroid on $\partial_s(v)$ and the uniform matroid of rank $m(v) - |\partial_s(v)|$ on $\partial_V(v)$ where $m(v) := r(\partial_s(P(v)))$. $\mathcal{M}'_0 := \bigoplus_{v \in V} \mathcal{M}'_v$.

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$$:= +\infty \quad (\forall X \notin \mathcal{F}).$$

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Let $B \subseteq A$ for a given $D = (V + s, A)$. The following two conditions are equivalent:

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$$|\partial_V^B(X)| \geq r(\partial_s^A(P_D(X))) - r(\partial_s^A(X)) \qquad (\forall X \subseteq V)$$
$$|\partial_V^B(X)| \geq r(I_X \cup J_X) - r(J_X) \qquad (\forall X \in \mathcal{F})$$

Theorem (K.)

In a matroid-rooted digraph $(D = (V + s, A), \mathcal{M})$ there exists an $\mathcal{M}$-reachability-based packing of s-arborescences iff

$$\varrho(X) = |\partial^A(X)| \geq r(\partial_s^A(P(X))) - r(\partial_s^A(X)) \ (\forall X \subseteq V).$$

Lemma

b is an intersecting submodular bi-set function.

$$\mathcal{I}^* := \{B \subseteq A : i_B(\mathrm{X}) \leq \mathrm{b}(\mathrm{X})\ \forall \mathrm{X} \in \mathcal{P}_2\}$$

forms the family of independent sets of a matroid $\mathcal{M}^*$ on A.

Theorem

Let $(D = (V + s, A), \mathcal{M})$ be a matroid-rooted digraph. Suppose that (A1) is satisfied. Then $B \subseteq A$ is the arc set of an $\mathcal{M}$-reachability-based packing of s-arborescences if and only if B is a common independent set of $\mathcal{M}_0'$ and $\mathcal{M}^*$ of size $\widetilde{m}(V)$.

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Theorem

Let $(D = (V + s, A), \mathcal{M})$ be a matroid-rooted digraph. There exists a polynomial algorithm to find an $\mathcal{M}$-reachability-based packing of s-arborescences in D of minimum weight if D has at least one such packing.

Theorem

Let $(D = (V + s, A), \mathcal{M}_1)$ be a matroid-rooted digraph with another matroid $\mathcal{M}_2 = \oplus_{v \in V} \mathcal{M}_v$ on A. There exists an $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescences in D if and only if

$$r(F) + r_2(\partial(X) - F) \geq r_1(\partial_s(P(X))) \text{ for all } X \subseteq V \text{ and } F \subseteq \partial_s(X).$$

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Thank you for your attention!