Highly connected rigidity matroids have unique underlying graphs

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Basic definitions

Let $G = (V, E)$ be an undirected graph and $p : V \to \mathbb{R}^n$ a function. We say (G, p) is a *bar and joint framework*.

(G, p) is *flexible*, if there exists a continuous deformation of (G, p).

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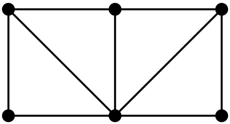

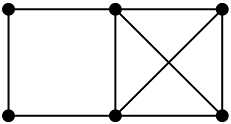

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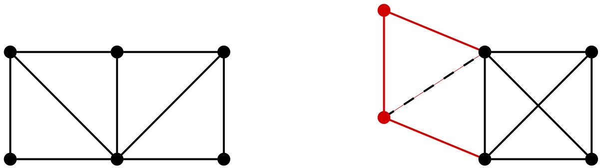

Infinitesimal rigidity

$m : V \to \mathbb{R}^d$ is an *infinitesimal motion* of (G, p) if for every edge uv

$$(m(u) - m(v))(p(u) - p(v)) = 0$$

holds.

An infinitesimal motion is trivial if it extends to an isometry of $\mathbb{R}^d$.

(G, p) is said to be *infinitesimally rigid* if it has no non-trivial infinitesimal motion.

(G, p) is *generic* if the coordinates of $p(v)$, $v \in V$ are algebraically independent over $\mathbb{Q}$.

Theorem (Asimov, Roth, 1978)

If p is generic then (G, p) is rigid $\iff$ it is infinitesimally rigid.

If p is generic then he rigidity of (G, p) depends only on G. Graph G is said to be *(generically) rigid* if (G, p) is rigid every generic p.

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Rigidity matroids

The d-dimensional rigidity matrix $R_d(G,p)$ of the framework (G,p) is the following $|E| \times d|V|$ matrix:
every edge has a row and every vertex has d columns. The row of edge uv is:

$$\left(0 \ldots 0 \quad \overbrace{p(u)-p(v)}^{u} \quad 0 \ldots 0 \quad \overbrace{p(v)-p(u)}^{v} \quad 0 \ldots 0\right)$$

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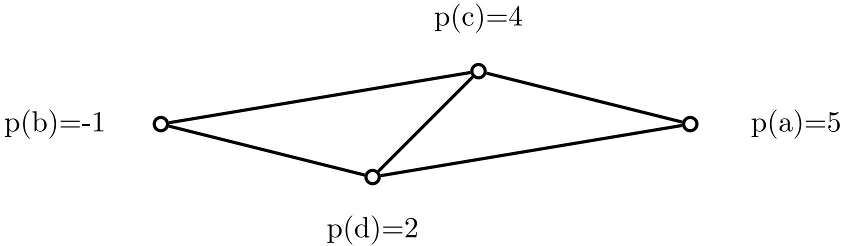
An example for $d = 1$



$$R(G,p) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 3 & 0 & 0 & -3 \\ 0 & -5 & 5 & 0 \\ 0 & -3 & 0 & 3 \\ 0 & 0 & 2 & -2 \end{pmatrix}$$

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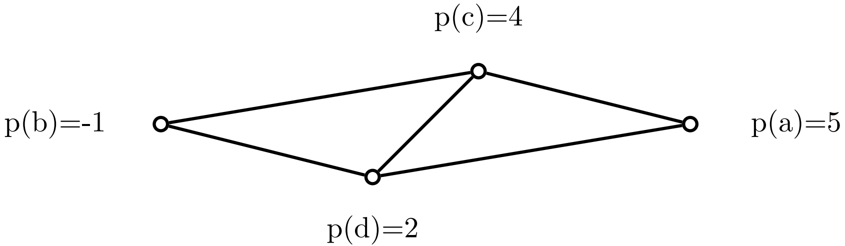

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Necessary conditions for independence in $\mathcal{R}_d(G)$

There are $\binom{n+1}{2}$ independent isometries in $\mathbb{R}^d$. They all generate a trivial infinitesimal motion in the kernel of $R_d(G,p)$.

Thus if G is independent in $\mathcal{R}_d(G)$ then for any subgraph $G' = (V', E')$ of G with at least $d+2$ vertices

$$|E'| \leq d|V'| - \binom{n+1}{2}$$

must hold.

For $d = 1, 2$ this characterises independence in $\mathcal{R}_d(G)$. For larger values of d this edge count does *not* define a matroid.

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Matroid connectivity

Let $\mathcal{M}$ be a matroid on ground set E with rank function r and let k be a positive integer.
We say that a partition (X, Y) of E is a *vertical k-separation* if

$$\min\{r(X), r(Y)\} \geq k, \quad \text{and}$$
$$r(X) + r(Y) \leq r(E) + k - 1.$$

The *vertical connectivity* of $\mathcal{M}$ is defined to be the smallest integer j for which $\mathcal{M}$ has a j-separation. If $\mathcal{M}$ has no vertical separations at all, we let $\kappa(\mathcal{M}) = r(E)$.

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It is known that graph G without an isolated vertex is k-vertex-connected if and only if its cycle matroid $\mathcal{M}(G)$ is k-connected.

It follows from Whitney's 2-isomorphism Theorem that if G and H are two graphs without isolated vertices, $\mathcal{M}(G)$ is 3-connected and $\mathcal{M}(G)$ and $\mathcal{M}(H)$ are isomorphic then G and H are isomorphic.

Open problem (B. and H. Servatius)

Is there a (smallest) constant k_d such that G is uniquely determined by $\mathcal{R}_d(G)$ provided that $\mathcal{R}_d(G)$ is k_d-connected?

$k_1 = 3$ follows from Whitney's Theorem.

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Highly connected graphs

Theorem (Jordán, K.)

Let G and H be two graphs without isolated vertices and suppose that $\mathcal{R}(G)$ is isomorphic to $\mathcal{R}(H)$. If G is 7-connected then G is isomorphic to H.

Sketch of the proof: One needs to extract information on the vertices. They correspond to subgraphs with co-rank 2 whose rigidity matroid is 2-connected.

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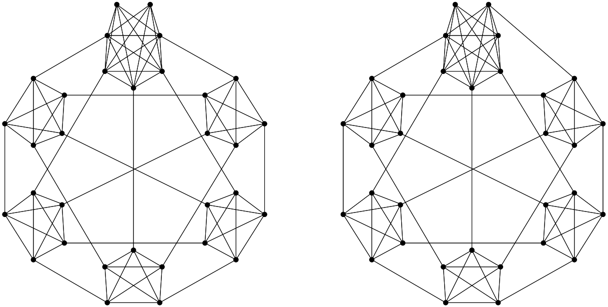
Spot the difference



Two non-isomorphic rigid 5-connected graphs with isomorphic rigidity matroids.

Highly connected matroids

Lemma (Jordán, K.)

Let G be a graph without isolated vertices and suppose that $\mathcal{R}_2(G)$ is $(2k-3)$-connected for some $k \geq 3$. Then G is k-connected.

Theorem (Jordán, K.)

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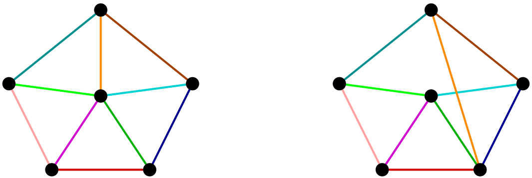

Then $3 \leq k_2 \leq 11$.

Higher dimensions?

There is no characterisation known for independence in $\mathcal{R}_d(G)$ for $d \geq 3$.
Some results used in the proofs have a higher dimensional conjecture version.

Theorem (Lovász, Yemini)

Every 6-vertex-connected graph is rigid in the plane.

They conjecture that there is a constant for every d. In $\mathbb{R}^3$ the smallest possible such constant could be 12.

Theorem (Jackson, Jordán)

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