Concatenation

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What is it?

It is a binary operation on collections of subsets of a linearly ordered set.

$(X, \mathcal{C})$, where $X = \{x_1, \ldots, x_n\}$, $\mathcal{C}$ is a collection of subsets of X

Given $W_1 = (X, \mathcal{C}_1)$, $W_2 = (X, \mathcal{C}_2)$, their *concatenation* is $W_1 \curlyvee W_2 = (X, \mathcal{D})$, where

$$\mathcal{D} = \{C_1 \cup C_2 : C_1 \in \mathcal{C}_1, C_2 \in \mathcal{C}_2, \text{ and the last element of } C_1$$

$$\text{is the first element of } C_2\}.$$

It is a construction method for various combinatorial objects.

an example

A	B	C	D	E	F	
1	-1	1	-1	$*$	$*$	$\{A, B\}$
$*$	1	1	-1	-1	$*$	$\{B, D\}$
$*$	$*$	-1	1	-1	1	$\{D, E\}$

two circuits: $\{A, B, D, E\}$, $\{A, B, C^*, D^*\}$

Where did it come from?

Las Vergnas extensions of oriented matroids

a Theorem of Las Vergnas: Unions of orientable matroids are orientable.

Lawrence and Weinberg: a union operation on oriented matroids that reduces to concatenation in the case of uniform oriented matroids

some of its relatives

ladder path matroids (Bonin and coauthors)

Richter-Gebert's *connected sum* operation

results relating to it

contractibility of realization space

(Mnev spectacularly showed this is not the case in general.)

few mutations: Richter-Gebert's construction

the Klee-Walkup counterexample to the Hirsch conjecture for unbounded polyhedra

(Santos has dispensed with the harder problem, the bounded case.)

Montejano and Ramirez Alfonsin and a conjecture of Roudneff

an example

A	B	C	D	E	F	
1	−1	1	−1	*	*	$\{A, B\}$
*	1	1	−1	−1	*	$\{B, D\}$
*	*	−1	1	−1	−1	$\{D, E\}$

two circuits: $\{A, B, D, E\}$, $\{A, B, C^*, D^*\}$

more examples (yielding well-known polytopes and oriented matroids)

$W = (X, \mathcal{C})$, where: $X = \{x_1, \ldots, x_n\}$; $\mathcal{C} =$ the set of 2-element subsets $\{x_i, x_j\}$ of X, for which exactly one of i, j is odd.

Then, combinatorially, $W \curlyvee W \curlyvee \ldots \curlyvee W$ ($n - r$ terms) is the set of facet complements of the cyclic d-polytope with n vertices.

Similarly: alternating oriented matroids

combinatorial pseudomanifolds, $(X, \mathcal{C})$

(where $\mathcal{C}$ is the set of complements of facets)

If $C \in \mathcal{C}$ and $p \in X \setminus C$ then there is a unique $D \in \mathcal{C}$ such that $p \in D \subseteq C \cup \{p\}$.

Easy: If W_1 and W_2 are pseudomanifolds then so is $W_1 \curlyvee W_2$.

uniform oriented matroids of rank r

$(E, \mathcal{C}, ^*)$, where

$E = \{x_1, x_1^*, \ldots, x_n, x_n^*\}$ and for $S \subseteq E$, $S^* = \{x^* : x \in S\}$.

$\mathcal{C}$ is a collection of subsets C of E, each having $r + 1$ elements, such that for each set $\bar{C} \subseteq E$ with $\bar{C}^* = C$ and $|\bar{C}| = 2(r + 1)$, there is a unique pair of circuits C and C^* contained in $\bar{C}$.

For each set $E_0 \subseteq E$ such that $E_0 \cap E_0^* = \emptyset$ and $E_0 \cup E_0^* = E$, with $\mathcal{C}_0 = \{C \in \mathcal{C} : C \subseteq E_0\}$, $(E_0, \mathcal{C}_0)$ is a pseudomanifold.

preservation

If W_1 and W_2 are combinatorial pseudomanifolds, so is $W_1 \curlyvee W_2$.

If W_1 and W_2 are combinatorial types of simplicial polytopes, so is $W_1 \curlyvee W_2$.

If W_1 and W_2 are uniform oriented matroids (as above), so is $W_1 \curlyvee W_2$.

If W_1 and W_2 are realizable uniform oriented matroids, so is $W_1 \curlyvee W_2$.

concatenation of rank 1 uniform oriented matroids

Γ and Φ

u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
1	-1	1	-1	1	-1	1	-1	1
1	-1	1	-1	1	-1	1	-1	1

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
u_6	u_2	u_5	u_3	u_4	u_7	u_1	u_8	u_9
u_9	u_8	u_7	u_1	u_4	u_3	u_5	u_2	u_6

some questions

Does the Hirsch conjecture hold for oriented matroid polytopes from Φ?

Characterize the classes Γ and Φ by excluded minors.

What are the mutation count matrices (or f-vectors) for elements of Γ and Φ?

What about Roudneff's conjecture, for Φ?