How many cubes are orientable?

Ilda da Silva

FCUL - Univ. of Lisbon

CIRM-Marseille, September 2018

What is a cube:

A **Cube** *over* $C^n = \{\pm 1\}^n$ *is a matroid,* $M = M(C^n)$*, satisfying two conditions:*

(i) Every (euclidean) **rectangle** *of* C^n *is a* **circuit** *of* M*.*

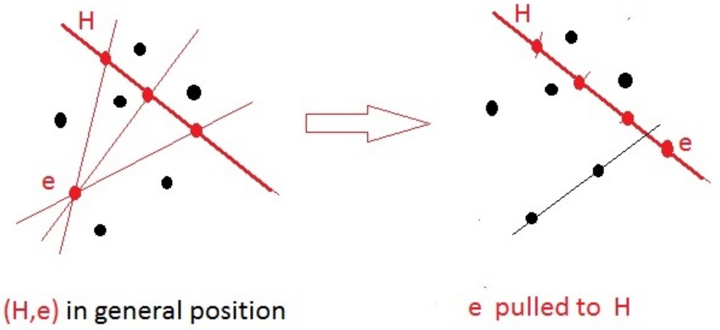
(ii) Every (euclidean) **facet** *and* **skew-facet** *of* C^n *is a* **hyperplane and a cocircuit** *of* M*.*

A **cube** *is a matroid whose underlying matroid is isomorphic to a cube over* C^n*.*

Examples. For every field F, $Aff_F(\{0,1\}^n)$ is a cube.

Generating new cubes from old ones

Theorem *The class of cubes is invariant under the operation of pulling points to hyperplanes.*



This operation does not, in general, preserve orientability or representabilty. **There are non-representable cubes.**
So far, **only ONE orientable cube is known:** $\mathcal{A}ff(C^n)$**, the real affine cube.**

How many cubes are orientable?

1) General results about orientable cubes.

2) Studying obstructions to orientability - new minor-minimal oriented matroids.

3) Recursivity and relation between rectangles and hyperplanes work in progress with **E. Gioan** - Counting orientable cubes and solving Las Vergnas cube conjecture in small dimensions.

(*) explicit descriptions of $\mathcal{A}ff(C^n)$ computed only for small dimensions. For $n \leq 8$ the total number of hyperplanes being aprox. 86×10^9 (Aicholzer+Aurenammer 96)

1. General results on orientable cubes

Theorem

Every orientable cube $M = M(C^n)$ has the following properties:

(i) M is a matroid of rank $n+1$.

(ii) **(Canonical orientation)** *Every class of orientations of M contains a unique orientation whose rectangles and cocircuits of the facets and skew-facets are signed as in, $\mathcal{A}ff(C^n)$.*

(iii) If $\mathcal{M}$ is a canonically oriented cube the **reorientations of $\mathcal{M}$ whose lattice of faces is isomorphic to the face lattice of the real cube** *are those obtained from $\mathcal{M}$ reversing signs on a facet of $\mathcal{M}$*.

Consequences:

1) (ii) + (iii) : the representation of OM's as highly symmetric euclidean objects inside the cube (topes/crinkled zonotope/Bergman complex...) is **independent of the orientable cube we may consider**.

1. General results on orientable cubes (cont.)

2) (ii) $\mathcal{M}(C^n)$ cube **canonically oriented**
$\mathcal{R} :=$ signed rectangles of $\mathcal{A}ff(\mathcal{C}^n)$
$\mathcal{F} :=$ signed cocircuits complementary of the facets and skew-facets

implies that:

(a) how many cubes are orientable?

$\Updownarrow$

how much of the real cube $\mathcal{A}ff(C^n)$ (hyperplanes + signed cocircuits) can be reconstructed from: $\mathcal{R} + \mathcal{F} +$ orientability?

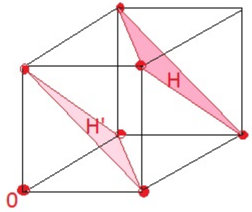
$+$

(b) whatever part of $\mathcal{A}ff(C^n)$ we might end up with it must have as symmetries the (affine and projective) symmetries of the real cube.

$+$

(c) affine cubes over finite fields, $Aff_F(C^n)$ are not orientable ($n \geq ch(F) + 2$).

2. Construction of obstructions to orientability inside a cube $C_0^n = \{0,1\}^n$



General idea :

Take $H := \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$
n points spanning a hyperplane,
not containing the vertices $\mathbf{0}, \mathbf{1} \in C_0^n$.
$H' := \{\mathbf{v'_1}, \ldots, \mathbf{v'_n}\} = \{\mathbf{1} - \mathbf{v_1}, \ldots, \mathbf{1} - \mathbf{v_n}\}$,
separates $\mathbf{0}$ from H.
$\mathbf{S_n} = Aff_{\mathbb{R}}(H \cup H')$ is a cross-polytope.
$\mathbf{N} := Aff_{\mathbb{R}}(H \cup H' \cup \mathbf{0})$

If $(H, \mathbf{0})$ is in general position in N, **pull 0 to H in N** obtain:

$$\mathbf{N}' = \mathbf{N}'(H \cup H' \cup \mathbf{0})$$

$\mathbf{N}'$ is a good candidate to non-orientable and $\mathbf{N}'/\mathbf{0}$ to minor-minimal non-orient.

New families of minor minimal non-orientable matroids $\mathcal{M}_{2n,n}$ generalizing Bland -Las Vergnas original family.

We can not argue in general.

3. Recursivity and relations between rectangles and hyperplanes in the real cube.

How much of $\mathcal{A}\mathit{ff}(C^n)$ can be reconstructed from:
$\mathcal{R} + \mathcal{F} +$ orientability ?

i.e. close the gap where to search for oriented cubes:

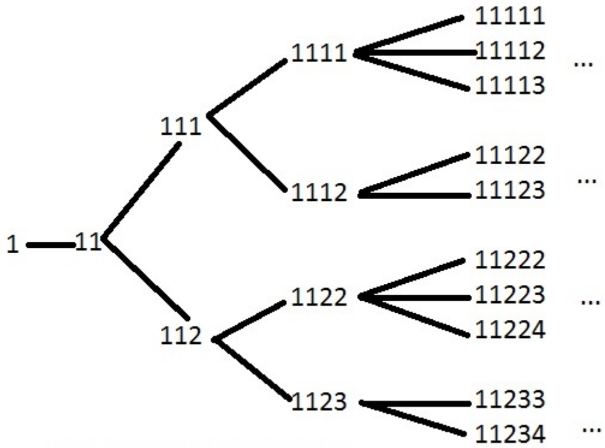
idea.
Find recursive families of hyperplanes of $\mathcal{A}\mathit{ff}(R^n)$:
1. reconstruct themhyperplanes + signed cocircuits.

For the variant

2. reconstruct only the signed cocircuits.
Las Vergnas cube conjecture *There is a unique way of signing all the cocircuits of $\mathcal{A}\mathit{ff}(C^n)$ from $\mathcal{R} + \mathcal{F} +$ orientability.*

Primitive vectors

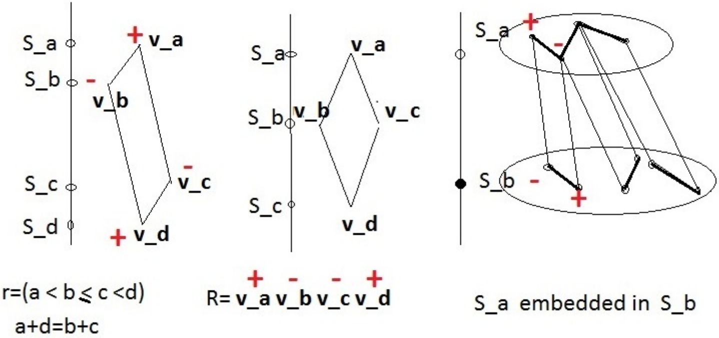
A recursive family of nonnegative integer vectors
$\mathbf{h} = (h_1 \leq h_2 \leq \ldots \leq h_n),\ 0 \leq h_1 < h_n = |\mathbf{h}| = \sum_i h_i$
such that $\forall i \quad h_{i-1} \leq h_i \leq \frac{\sum_{j<i} h_j}{2} + 1$



Positive primitive vectors

Primitive vectors, rectangles and hyperplanes

"parallel" strata $S_a(\mathbf{h}) := \{\mathbf{v} \in C^n : \mathbf{v}.\mathbf{h} = |\mathbf{h}| - 2a\}$.



If $\mathbf{h}$ is a primitive vector and S_b a hyperplane, such that $\forall a$ S_a *embedds in* S_b, then a recursive family of hyperplanes of every oriented cube starts there. By orthogonality with the signed rectangles of $\mathcal{R}$, must be signed as in $\mathcal{A}ff(C^n)$.

$\mathcal{A}ff(C^n)$ as a family of nonnegative integer vectors IS05

$\mathcal{H}_1 = \{$ (1,1) $\}$;
$\mathcal{H}_2 = \{$ (0,1,1)$\}$;
$\mathcal{H}_3 = \{$ (0,0,1,1), (1,1,1,1) $\}$;
$\mathcal{H}_4 = \{$ (0,0,0,1,1), (0,1,1,1,1),(1,1,1,1,2) $\}$;

$\mathcal{H}_5 = \{$(0,0,0,0,1,1), (0,0,1,1,1,1),(0,1,1,1,1,2), (1,1,1,1,1,1),
(1,1,1,1,1,3),(1,1,1,1,2,2), (1,1,1,2,2,3)$\}$;

$\mathcal{H}_6 = \{$(0,0,0,0,0,1,1), (0,0,0,1,1,1,1), (0,1,1,1,1,2), (0,1,1,1,1,1,1),
(0,1,1,1,1,1,3), (0,1,1,1,1,2,2), (0,1,1,1,2,2,3), (1,1,1,1,1,1,2),
(1,1,1,1,1,1,4), (1,1,1,1,1,2,3), (1,1,1,1,2,2,2), (1,1,1,1,2,2,4),
(1,1,1,1,2,3,3), (1,1,1,1,3,3,4), (1,1,1,2,2,2,3), (1,1,1,2,2,2,5),
(1,1,1,2,2,3,4), (1,1,1,2,3,3,5), (1,1,2,2,2,3,3), (1,1,2,2,3,3,4)
(1,1,2,2,3,4,5)$\}$; **all primitive!**

$|\mathcal{H}_7| = \mathbf{143} \longrightarrow |\tilde{\mathcal{H}}_7| = \mathbf{71.343.208}$ **very few non-primitive**

For small dimensions a unique canonically oriented cube

with a short proof:

Theorem (Bokovski etal 1996, Gioan-S 2017)
1) For $n \leq 7$, $\mathcal{A}ff(C^n)$ is the unique orientable cube.
2) **(Las Vergnas cube conjecture)** For $n \leq 7$, $\mathcal{A}ff(C^n)$, the oriented matroid $\mathcal{A}ff(C^n)$ has a unique class of orientations.

work in progress: Understanding how vaster classes of hyperplanes of the real cube behave with respect to net of rectangles.

How many cubes are orientable?

Many!

Find a pair (H, e) of a point in general position "near each other" in $\mathcal{A}\mathit{ff}(C^n)$. Then K. Fukuda, A. Tamura , 88 guarantees that the cube obtained by pulling e to H is oriented and not isomorphic to $\mathcal{A}\mathit{ff}(C^n)$.

on the other hand,

May be not so manny...

Probabilistic conjecture: (Odlyzco 88, Khan etal 95, Tao-Van Vu 07) the asymptotic behavior of the probability of a random $n \times n$ Bernoulli matrix M_n being singular is dominated by the probability that 3 out of 4 vertices forms the vertices of a rectangle.

Main references:

I.P.F. da Silva, Recursivity and geometry of the hypercube, *Lin. Alg. and its Appl.*, **397**(2005), 223-233.

I.P.F. da Silva, Orientability of cubes,*Discrete Maths*,**308**(2008), 3574-3585.

I.P.F. da Silva, On minimal non-orientable matroids with $2n$ elements and rank n,*Europ. J. of Combin.*, **30**(2009), 1825-1832.

E.Gioan, I.P. da Silva, Rectangles and hyperplanes of the hypercube, working paper 2018.