Generalizations of Crapo's Beta Invariant

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Outline

Outline

Henry Crapo

Henry Crapo's House

Centre de Recherche du Larzac Méridional, or Les Moutons Matheux
La Vacquerie et Saint Martin de Castries, France

Matroid catalog

Matroid catalog

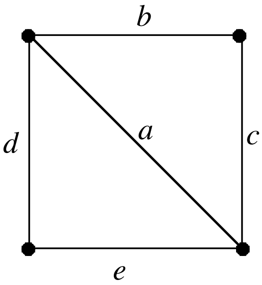
The Henry Crapo group presents the incredible catalog of 8 point geometries. See single element extensions grow before your eyes.

β for Graphs

Definition

$G = (V, E)$ a graph, $A \subseteq E$. Define $r(A)$, the rank of A, by

$$r(A) = \max_{F \subseteq A}\{|F| \mid F \text{ is acyclic}\}.$$



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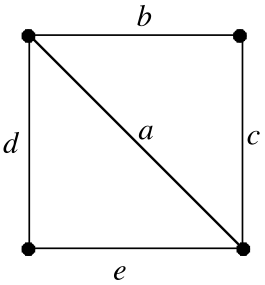


Subset	$\emptyset$	singleton	pair	triple	any 4	E
Rank	0	1	2	2 or 3	3	3

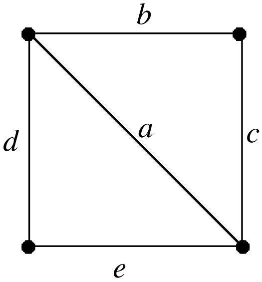
Graph example

$$r(A) = \max_{F \subseteq A}\{|F| \mid F \text{ is acyclic}\}$$

Definition

Let G be a graph. Then the **beta invariant** $\beta(G)$ is defined by

$$\beta(G) = (-1)^{r(E)} \sum_{A \subseteq E} (-1)^{|A|} r(A).$$



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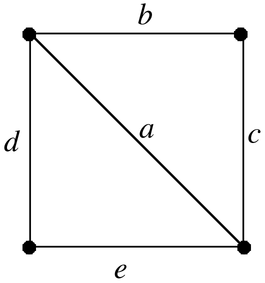

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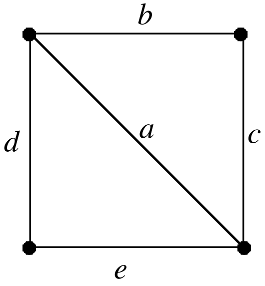

Subset	$\emptyset$	$\{x\}$	$\{x, y\}$	$\{a, b, c\}$ or $\{a, d, e\}$	Other triples	any 4	E
Rank	0	1	2	2	3	3	3
$(-1)^{\lvert A\rvert} r(A)$	0	-1	2	-2	-3	3	-3

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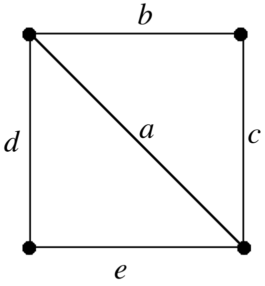

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$$\beta(G) = (-1)^3 (0 - 5 \cdot 1 + 10 \cdot 2 - 2 \cdot 2 - 8 \cdot 3 + 5 \cdot 3 - 1 \cdot 3) = 1.$$

Crapo's motivation

JOURNAL OF COMBINATORIAL THEORY 2, 406-417 (1967)

A Higher Invariant for Matroids

HENRY H. CRAPO

University of Waterloo,

Waterloo, Ontario, Canada

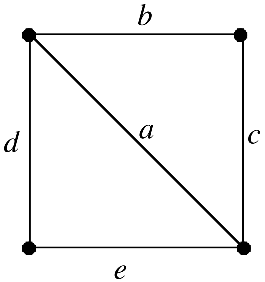
Communicated by Gian-Carlo Rota

ABSTRACT

The Möbius invariant μ, essential to the classification of surfaces, is less useful in the study of exchange geometries (matroids) because it undergoes sizeable fluctuations as a result of minor structural changes, such as the lengthening of an arc. The number β, investigated here, is not only a geometric invariant, like μ, but is also a duality invariant, and provides a complete determination of separability.

Tutte polynomial connection

$T(G; x, y) = x^3 + 2x^2 + 2xy + 1x + y^2 + 1y$



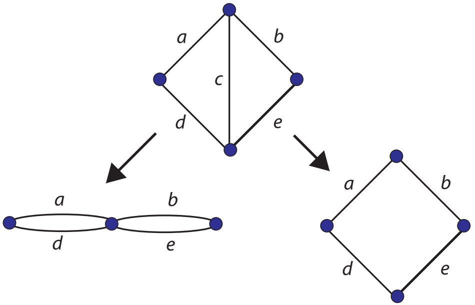
Theorem

$\beta(G)$ equals the coefficient of x in the Tutte polynomial $T(G; x, y)$.

- If G has more than one edge, then the coefficient of x equals the coefficient of y in the Tutte polynomial.

Deletion-contraction

- Deletion-contraction: $\beta(G) = \beta(G - e) + \beta(G/e)$.



Delete and contract edge c.

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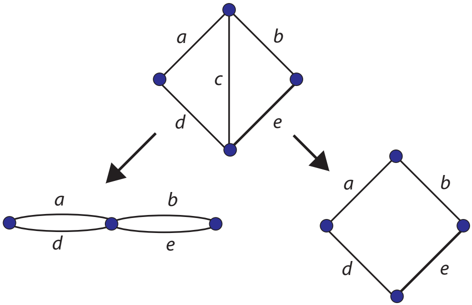

Delete and contract edge c.

- $\beta(G/c) = 0$ and $\beta(G - c) = 1$.

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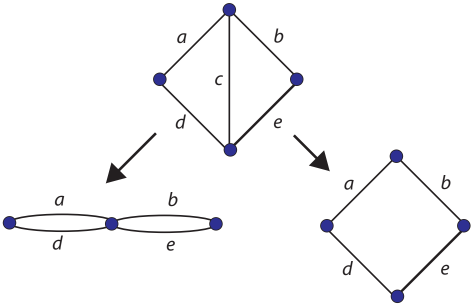


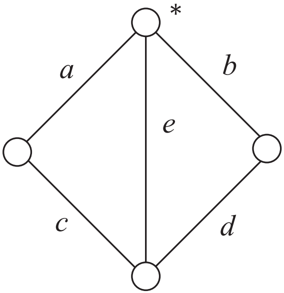
Delete and contract edge c.

- $\beta(G/c) = 0$ and $\beta(G - c) = 1$.
- Consequence: For all graphs G, $\beta(G) \geq 0$.

Rooted Graphs

G is a graph with a distinguished vertex.

- $r(A) = \max_{F \subseteq A} \{|F| \mid F \text{ is a rooted tree}\}$
- $\beta(G) = (-1)^{r(E)} \sum_{A \subseteq E} (-1)^{|A|} r(A)$.



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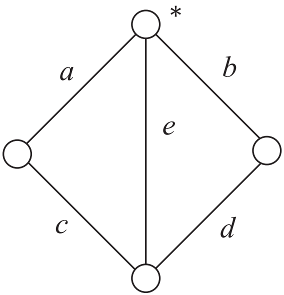

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Subset	$\emptyset$	$\{x\}$	$\{x, y\}$	$\{x, y, z\}$	any 4	E
Rank	0	0 or 1	0, 1 or 2	2 or 3	3	3
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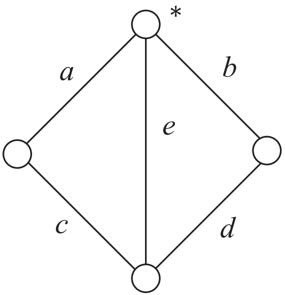

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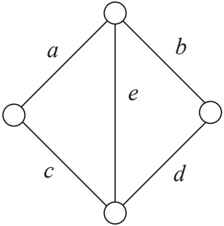
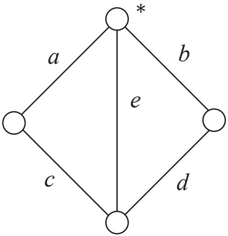
$$\beta(G) = 3.$$

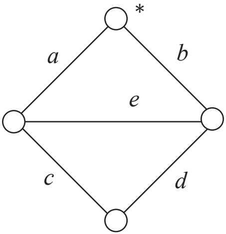
Rooted vs. unrooted



$\beta = 1$

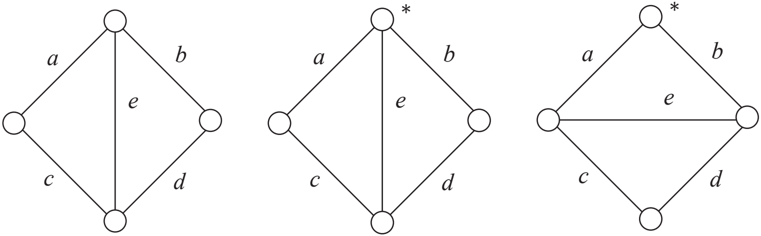


$\beta = 3$



$\beta = 4$

Rooted vs. unrooted



$\beta = 1$ $\beta = 3$ $\beta = 4$

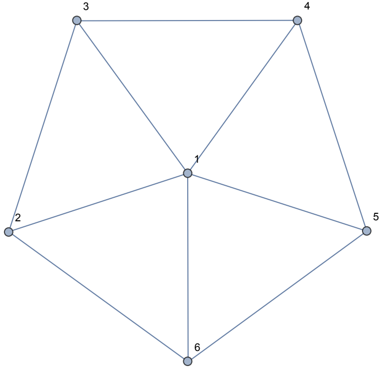
- Deletion-contraction: Assume e is incident to the root. Then

$$\beta(G) = \beta(G - e) + \beta(G/e).$$

- So $\beta(G) \geq 0$ for rooted graphs, too.

Rooted vs. unrooted

	Trees	C_n	Fans	W_n	K_n
Unrooted	0	1	1	$n-1$	$(n-2)!$
Rooted	0 or 1	$n-1$	$n-1$	$n(n-1)$	$(n-1)!$



Wheel W_5

Outline

Beta for matroids

Definition

Let M be a matroid. Then $\beta(M) = (-1)^{r(E)} \sum_{A \subseteq E} (-1)^{|A|} r(A)$.

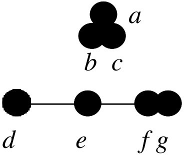
- Deletion-contraction: $\beta(M) = \beta(M - e) + \beta(M/e)$.
- Non-negativity: $\beta(M) \geq 0$.
- Direct sum: $\beta(M_1 \oplus M_2) = 0$.
- Dual: If $|E| > 1$, then $\beta(M) = \beta(M^*)$.

Beta for matroids

$$\beta(M) = (-1)^{r(E)} \sum_{A \subseteq E} (-1)^{|A|} r(A).$$

Theorem (Crapo '67)

A matroid M with more than 1 point is disconnected if and only if $\beta(M) = 0$.



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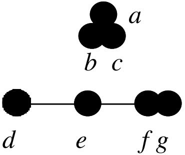
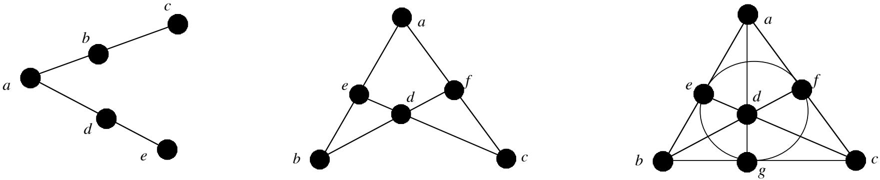


Theorem (Brylawski '71)

A matroid M with more than 1 point is a series-parallel network if and only if $\beta(M) = 1$.

Three matroids



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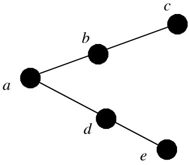
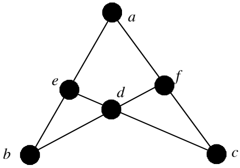
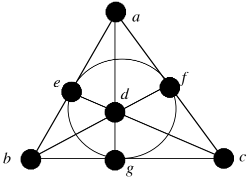


$\beta = 1$

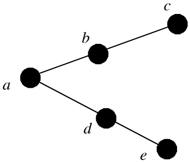


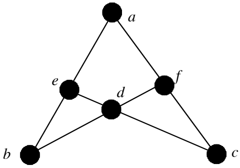
$\beta = 2$

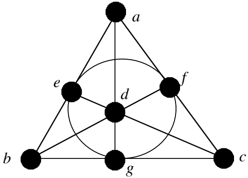


$\beta = 3$

Three matroids



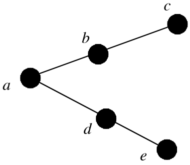


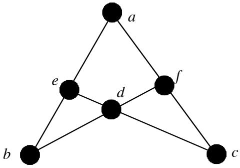


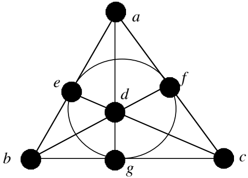
$\beta = 1$ $\beta = 2$ $\beta = 3$

Series-parallel $M(K_4)$: Not series-parallel Fano

Three matroids







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Series-parallel $M(K_4)$: Not series-parallel Fano

- Oxley (1982) characterized the matroids M with $\beta(M) \leq 4$.

Outline

Closure, matroids and antimatroids

Definition

A closure operator on a set E is a function $2^E \to 2^E$ satisfying, for all $A \subseteq E$,

- $A \subseteq \overline{A}$
- If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$,
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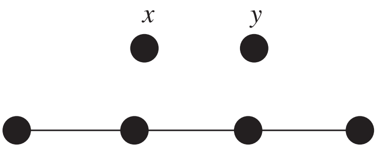
- Matroid: If $x, y \notin \overline{A}$ and $y \in \overline{A \cup x}$, then $x \in \overline{A \cup y}$. MacLane-Steinitz exchange



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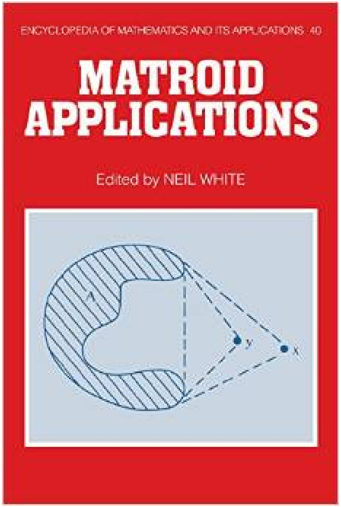
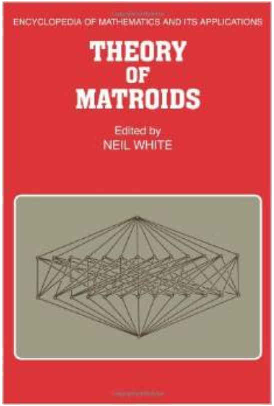
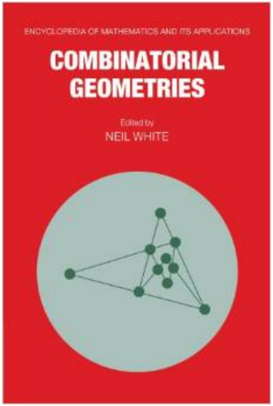
$\boxed{\text{Matroids : Affine closure :: Antimatroids : Convex closure}}$

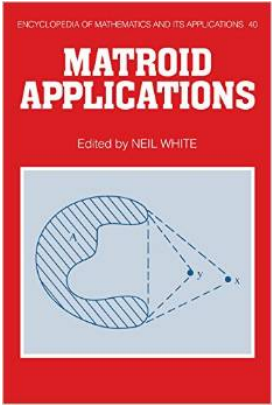


If $x, y \notin \overline{A}$ and $y \in \overline{A \cup x}$, then $x \notin \overline{A \cup y}$.

Matroid texts edited by Neil White







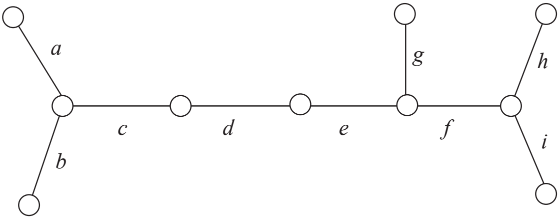
Neil White (1945 – 2014)

Robert MacPherson, Neil White, Richard Stanley, 2004.

Antimatroids – (Wave to Michael Falk)

“Antimatroid” coined by Robert Jamison (1980).

- Discovered [invented?] by Dilworth (1940)
- Avann (1960’s) Lower-semidistributive [LSD] lattices.
- More independent discoveries:
  - 1960’s: Boulaye, Bennett, Pfaltz
  - 1970’s: Greene & Markowsky, Jamison, Edelman



Trees are antimatroids

Convex sets, feasible sets and rank

Definition

Let G be an antimatroid with ground set E with a (convex) closure operator.

- Convex sets: C is convex if $\overline{C} = C$.
- Feasible sets: $F \subseteq E$ is feasible if $E - F$ is convex.
- Rank function: Let $A \subseteq E$. Then $r(A) = \max_{F \subseteq A}\{|F| \mid F \text{ feasible}\}$.

Beta for antimatroids

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$$\beta(G) = (-1)^{r(E)} \sum_{A \subseteq E} (-1)^{|A|} r(A).$$

Definition

- C is free convex (or simply free) if every subset of C is convex.

Theorem

For an antimatroid G,

$$\beta(G) = \sum_{C \text{ free}} (-1)^{|C|-1} |C|.$$

Various expansions

- Subset expansion by rank

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Various expansions

- Subset expansion by rank

$$\beta(G) = (-1)^{r(E)} \sum_{A \subseteq E} (-1)^{|A|} r(A).$$

- Free convex set expansion

$$\beta(G) = \sum_{C \text{ free}} (-1)^{|C|-1} |C|.$$

- Subset expansion by closure

$$\beta(G) = \sum_{S \subseteq E} (-1)^{|S|-1} |\overline{S}|$$

- Others exist (Möbius function, Boolean, and a few more).

Recursions

- Deletion-contraction

$$\beta(G) = \beta(G/x) - \beta(G - x).$$

$\beta(G)$ may be negative!

Recursions

- Deletion-contraction

$$\beta(G) = \beta(G/x) - \beta(G - x).$$

$\beta(G)$ may be negative!

- Direct sum

$$\beta(G_1 \oplus G_2) = 0.$$

Outline

Strategy

$$\beta(G) = \sum_{C \text{ free}} (-1)^{|C|-1}|C|.$$

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Strategy

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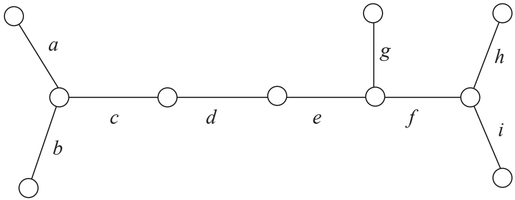
1. Given a class of antimatroids, interpret the free convex sets combinatorially.
2. Compute $\beta(G)$ somehow.
3. Interpret $\beta(G)$ combinatorially.
4. Remove $\beta(G)$.

Strategy

$$\beta(G) = \sum_{C \text{ free}} (-1)^{|C|-1}|C|.$$

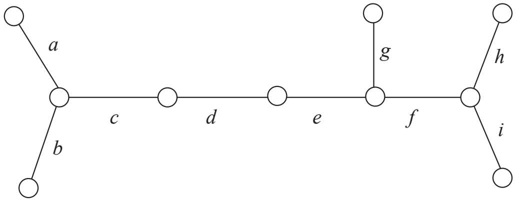
1. Given a class of antimatroids, interpret the free convex sets combinatorially.
2. Compute $\beta(G)$ somehow.
3. Interpret $\beta(G)$ combinatorially.
4. Remove $\beta(G)$.
5. Show everybody your new theorem.

Trees



Trees

- Antimatroid ground set $\leftrightarrow$ edges of T.
- C is convex if C forms a subtree.



Trees

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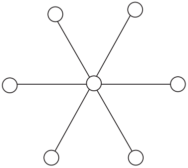
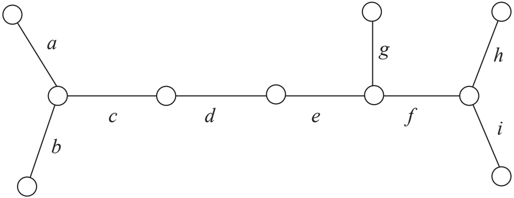


Typical free set



Trees

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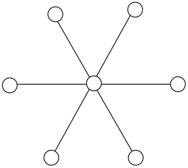

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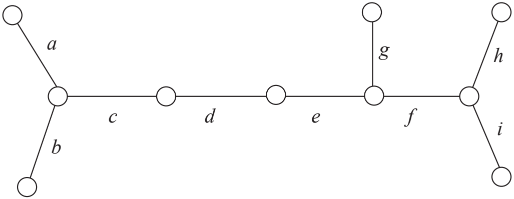

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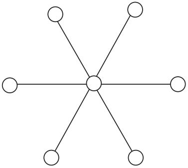

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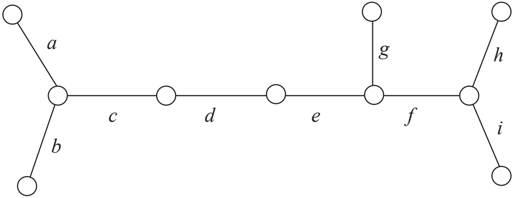

size of free set	1	2	3
# free sets			

Trees

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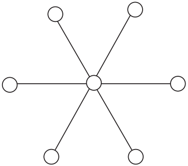

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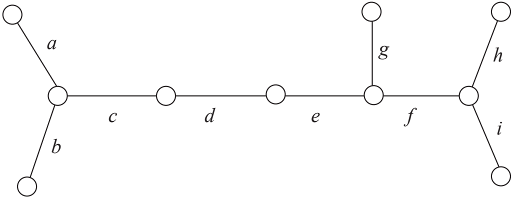

size of free set	1	2	3
# free sets	9		

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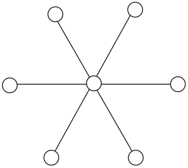
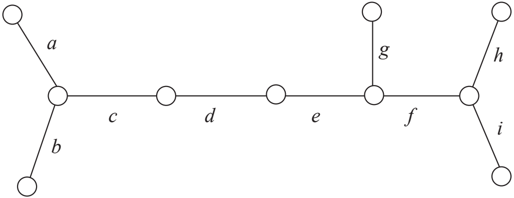


Typical free set



size of free set	1	2	3
# free sets	9	11	

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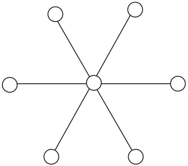

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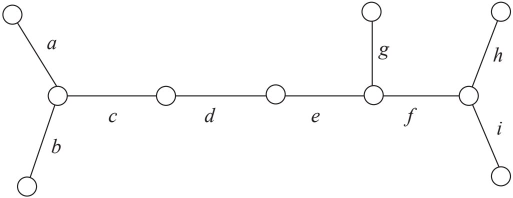

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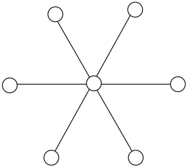

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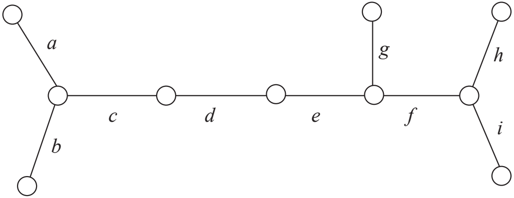
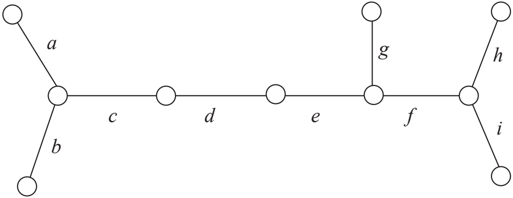


size of free set	1	2	3
# free sets	9	11	3

$$\beta(G) = 1 \cdot 9 - 2 \cdot 11 + 3 \cdot 3 = -4.$$

Trees

- $\beta(G) < 0$ for trees.



Trees

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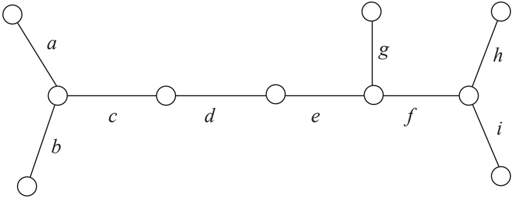
- Goal: Interpret 4 combinatorially.



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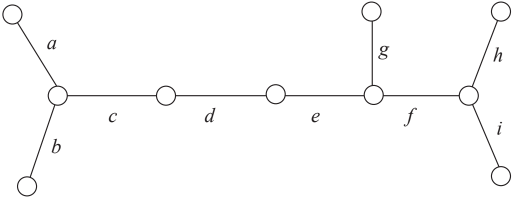

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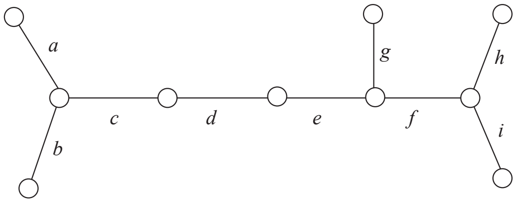

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Theorem

Let T be a tree.

$$\sum_{s \text{ a star}} (-1)^{|S|}|S| = \#\ \textit{interior edges.}$$

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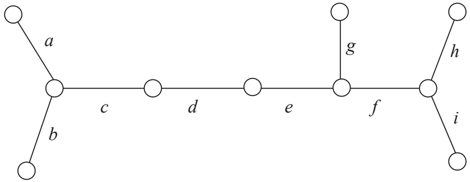

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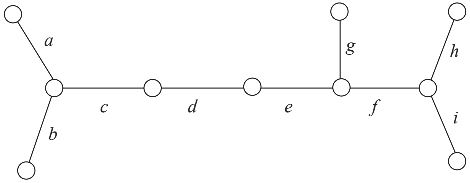

You can rewrite the count in terms of the degree sequence of the tree:
Let T be a tree with degree sequence $d_1, d_2, \ldots, d_n$. Then

$$\#\text{ leaves } = \sum_{i=1}^{n} \sum_{k=1}^{d_i} (-1)^{k-1} k \binom{d_i}{k}.$$

Trees

Theorem

Let T be a tree. Then $\beta(T) = -\#$ interior edges.

Compare this theorem to the original beta invariant.

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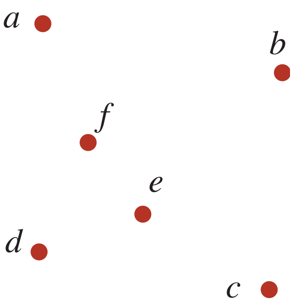
Theorem

Let T be a tree. Then $\beta_{Crapo}(T) = 0$.

Prototypical antimatroid – Finite subsets of $\mathbb{R}^n$

Let E be a finite subset of $\mathbb{R}^n$.

- $C \subseteq E$ is convex if $Hull(C) \cap E = C$.
- C is free if C is an empty polygon.



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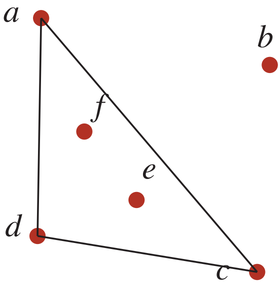

$A = \{a, c, d\}$

Convex? Free?

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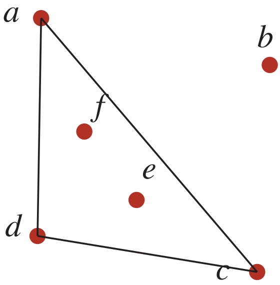

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Convex? No

Free? No

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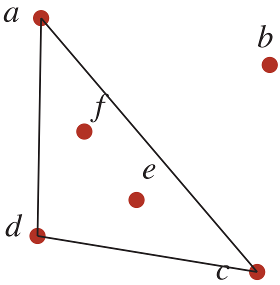

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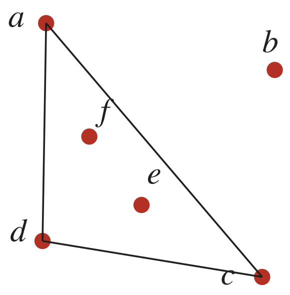
Convex? Free?

Yes No



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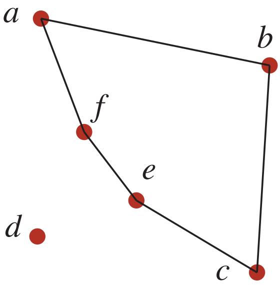

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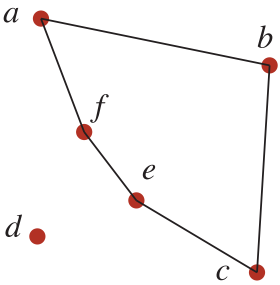

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Convex? Free?

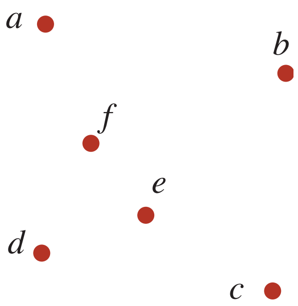
Yes Yes

$\beta(G)$ for finite point sets

$$\beta(G) = \sum_{K \text{ free}} (-1)^{|K|-1}|K|$$

Let f_i be the number of free sets of size i.

i	1	2	3	4	5
f_i	6	15	15	6	1



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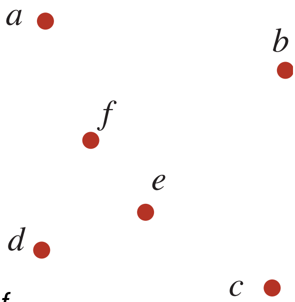

$$\beta(G) = f_1 - 2f_2 + 3f_3 - 4f_4 + 5f_5$$

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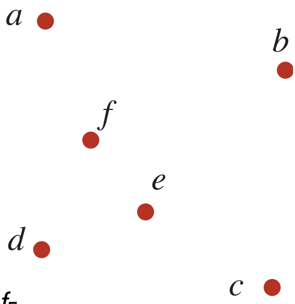

$$\begin{aligned} \beta(G) &= f_1 - 2f_2 + 3f_3 - 4f_4 + 5f_5 \\ &= 6 - 2\cdot 15 + 3\cdot 15 - 4\cdot 6 + 5\cdot 1 \end{aligned}$$

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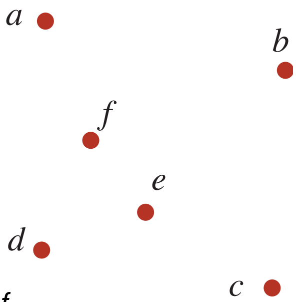
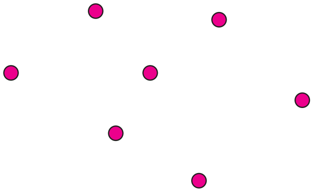


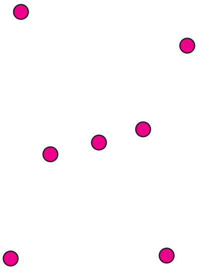
$$\begin{aligned}\beta(G) &= f_1 - 2f_2 + 3f_3 - 4f_4 + 5f_5 \\ &= 6 - 2 \cdot 15 + 3 \cdot 15 - 4 \cdot 6 + 5 \cdot 1 \\ &= 2.\end{aligned}$$

$\beta(G)$ for finite point sets

$$\beta(G) = \sum_{K \text{ free}} (-1)^{|K|-1}|K|$$



$\beta = 1$



$\beta = 3$

$\beta(G)$ for finite point sets

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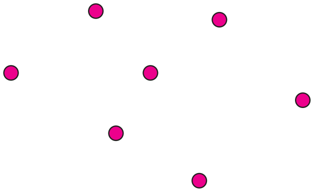
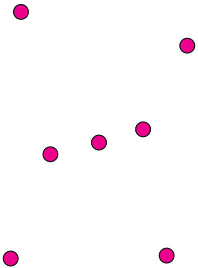


$\beta = 1$



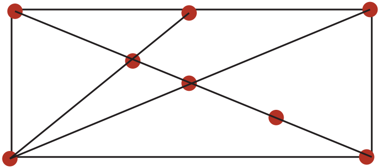
$\beta = 3$

$\boxed{\text{What is } \beta(G) \text{ counting?}}$

$$\beta(G) = \sum_{K \text{ free}} (-1)^{|K|-1}|K|$$

Theorem (Ahrens, G. and M. (1999))

Let E be a finite subset of $\mathbb{R}^2$. Then $\beta(G)$ is the number of interior points.



$\beta(G) = 3.$

The points do not need to be in general position.

Generalization to $\mathbb{R}^n$

Theorem (Lots of other people)

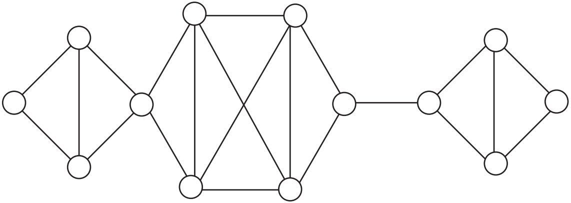
Let E be a finite subset of $\mathbb{R}^n$. Then

$$\sum_{K \text{ empty}} (-1)^{|K|-1}|K| = (-1)^n|int(E)|.$$

1. Edelman & Reiner (2000): Combinatorial topology.
2. Klain (2000): Valuations on lattices.
3. Bárány and Valtr (2004): Elementary geometric arguments.
4. Pinchasi, Radoičić, and Sharir (2006): Similar to above.

Motivation: Finding empty hexagons.

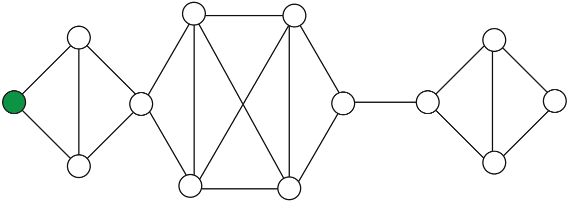
Chordal graphs



- G is chordal if it has no chord-free cycles.
- v is a simplicial vertex if its neighbors form a clique.

If G is chordal, we can shell the simplicial vertices, one-by-one, eventually removing them all.

Chordal graphs



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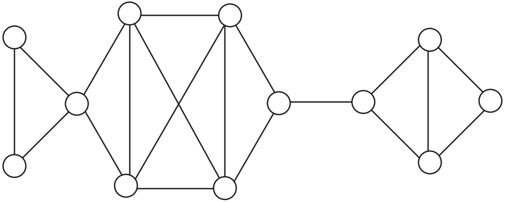

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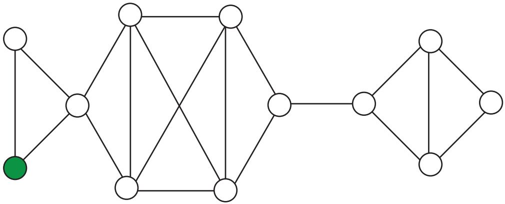

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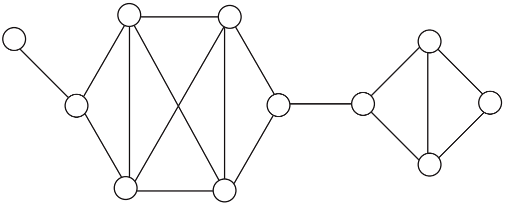

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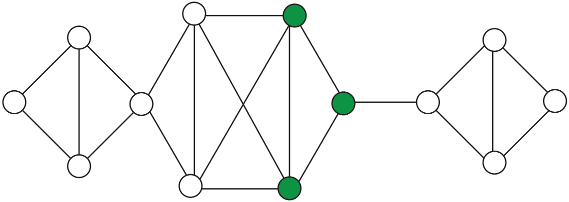
Chordal graphs

Antimatroid structure:

- E = vertices of G.
- C is convex if $E - C$ is a simplicial sequence.
- C is free if C forms a clique.



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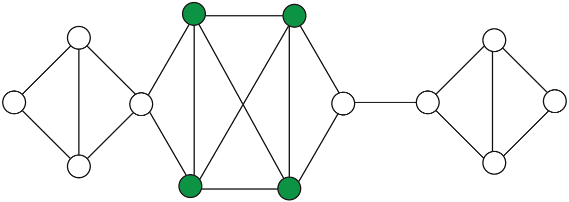
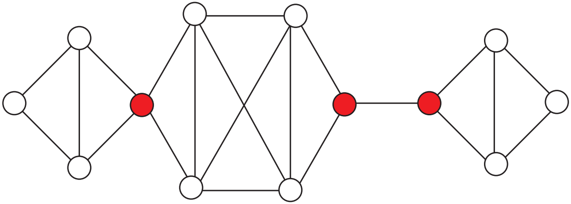


Chordal graphs

Theorem

Let G be a chordal graph. Then

$$\beta(G) = \sum_{c \text{ a clique}} (-1)^{|C|-1}|C| = -\#\ \textit{cut vertices}.$$

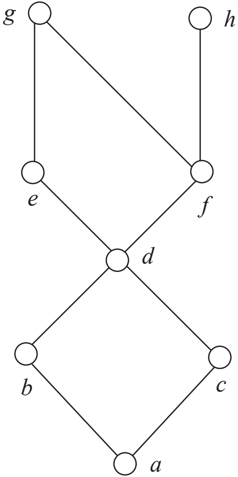


$$\beta(G) = -3.$$

Posets

Antimatroid structure:

- E = elements of P.
- F is feasible if F can be formed by repeatedly removing maximal and minimal elements [Double-shelling antimatroid].
- C is free if C contains no chains of length ≥ 3.



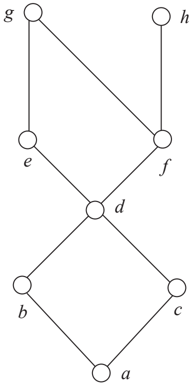
$$\beta(G) = -1.$$

Posets

Theorem (Edelman & Reiner)

Let P be a poset. Then

$$\beta(G) = \sum_{C \text{ free}} (-1)^{|C|-1}|C| = -\#\text{ bottlenecks.}$$



Other antimatroids, greedoids and things

Moral: If $G = (E, r)$ is any combinatorial object with a reasonable rank function, then we can define a beta invariant.

- Trees [vertex ground set]
- Rooted trees
- Gaussian elimination greedoids
- Rooted graphs [Vertex search]
- …