Pairs of Topes

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(joint work with Emanuele Delucchi.)

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Remark: If one fixes a tope U, there is a unique acyclic orientation of the tope graph with U as a source, giving a partial order $\leq_U$ on $\mathcal{T}$.

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If $\mathcal{F}$ is an orientation of M coming from a real realization $\mathcal{A}$, then the nerve of the poset $(\mathcal{Q}, \leq)$ is homotopy-equivalent to the arrangement complement X.

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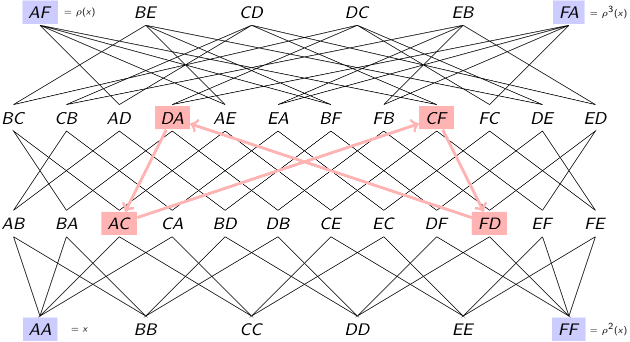
(nerve = order complex; simplices are chains in $\mathcal{Q}$.)

Example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[2] due to Priyavrat Deshpande

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- The poset $\mathcal{Q}$ is an alternative (with an easier description) to the *Salvetti poset* $\mathcal{S}$ associated with $\mathcal{F}$. There is an order-preserving map $\mathcal{Q} \longrightarrow \mathcal{S}$ to which the Quillen fiber lemma applies, yielding the theorem as a consequence of Salvetti's work.

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- $\mathcal{Q}$ has an additional feature, namely, there is a natural defined action of the circle S^1 (= the nerve of $\{\pm 1, \pm i\}$) that accurately models the diagonal action of $\mathbb{C}^\times$ on X. (No such action is apparent for $\mathcal{S}$.)

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- There is a notion of complexification of a pseudo-hyperplane arrangement[2]; the result holds in that generality.

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The space of orbits of the action of W on the nerve of $\mathcal{Q}$:

- *is the nerve of a category $\mathcal{W}$ with set of objects W, generated by the union of the left and right weak (Bruhat) orders, and*
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Morphisms $v \xrightarrow{g} w$ in $\mathcal{W}$ are labeled by group elements $g \in W$ that satisfy $gv \preceq_R w$.

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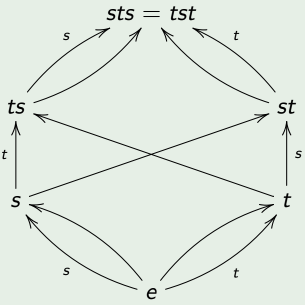

Figure: A model for the braid group on three strands.

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Theorem (Delucchi-F, *PAMS* 2017)

Let $\mathcal{F}$ be an orientation of P*, and $\mathcal{Q}$ the associated tope-pair poset. The nerve of $\mathcal{Q}$ is not homotopy equivalent to the complement of any complex hyperplane arrangement.*

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Problem: Reconstruct P from the polymatroid of $\mathcal{R}^1(\mathrm{P})$.

Elements of $\mathcal{R}^1$

[3] or, "tropically," by $\{0, 1\}$ matrices

Elements of $\mathcal{R}^1$

- elements of $\mathcal{R}^1$ are subspaces of $\Delta \cap \mathbb{C}^I$, where

$$\Delta = \{x \in \mathbb{C}^n \mid \sum_{i=1}^{n} x_i = 0\}$$

and $I \subseteq [n]$, and are defined by $\{0, \pm 1\}$ matrices[3];

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- small submatroids (e.g., $M(K(_4)$, rank-three whirl and one of its single element extensions) are determined by their $\mathcal{R}^1$'s.

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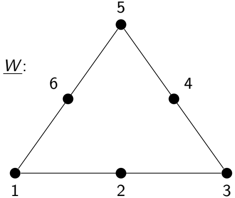
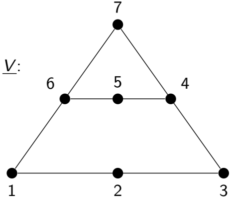
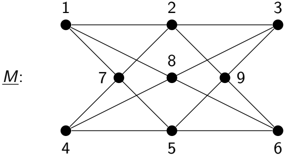
Pictures







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Problem

Give a description of $\mathcal{R}^1(\mathrm{M})$ *for graphic matroids.*