Characteristic polynomials and chambers for cones in hyperplane arrangements

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Combinatorial Geometries, 2018

Overview

Characteristic Polynomial of a Hyperplane Arrangement

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$ and let $\mathcal{L}(\mathcal{A})$ denote the set of intersections of $\mathcal{A}$, ordered by reverse inclusion.

Definition

Then the characteristic polynomial $\chi_{\mathcal{A}}(t)$ of $\mathcal{A}$ is

$$\chi_{\mathcal{A}}(t) = \sum_{x \in \mathcal{L}(\mathcal{A})} \mu(\hat{0}, x) t^{\dim x} = \sum_{k=0}^{n} (-1)^k w_k t^{n-k}$$

where w_k denotes the signless Whitney number of the 1st kind of $\mathcal{L}(\mathcal{A})$.

Zaslavsky's Theorem

Theorem (Zaslavsky)

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$. Let $\chi_{\mathcal{A}}(t)$ be the characteristic polynomial of $\mathcal{A}$. Then

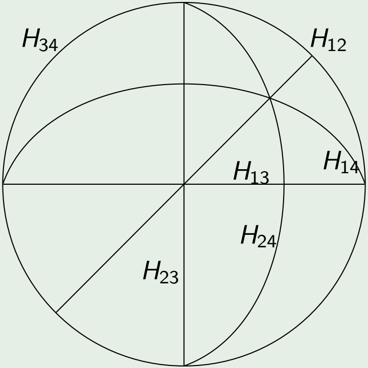
$$\chi_{\mathcal{A}}(-1) = (-1)^n \, \# \begin{pmatrix} \textit{chambers of the} \\ \textit{arrangement} \end{pmatrix}$$

In other words

$$\begin{aligned} \# \begin{pmatrix} \text{chambers of the} \\ \text{arrangement} \end{pmatrix} &= \pm \sum_{x \in \mathcal{L}(\mathcal{A})} \mu(\hat{0}, x)(-1)^{\dim x} \\ &= \sum_{x : \mathrm{rk} x = 1} |\mu(\hat{0}, x)| \\ &= w_0 + w_1 + \cdots + w_n \end{aligned}$$

Example

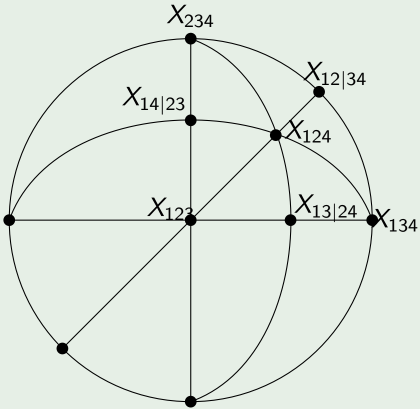
Consider the Type A reflection arrangement in $\mathbb{R}^4$. I've drawn a snapshot of a linearly-equivalent arrangement in $\mathbb{R}^3$ (see note).



Note: All the hyperplanes pass through span $\left(\vec{1}\right)$, so we project into the orthogonal complement of span $\left(\vec{1}\right)$.

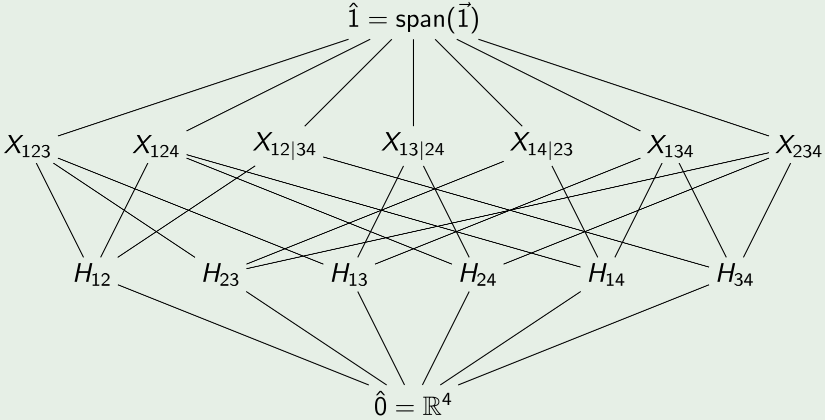
Example

Some of the intersections of this arrangement:



Example

The intersection poset is



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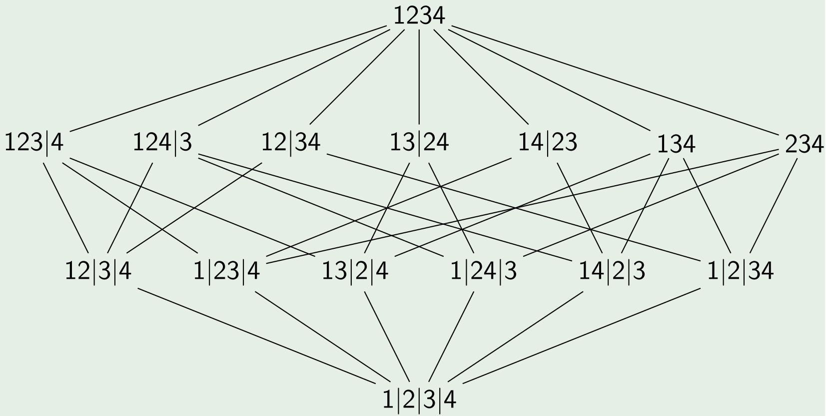

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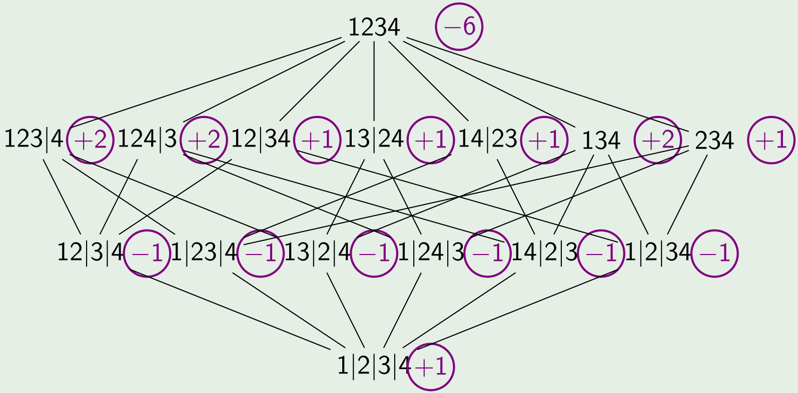
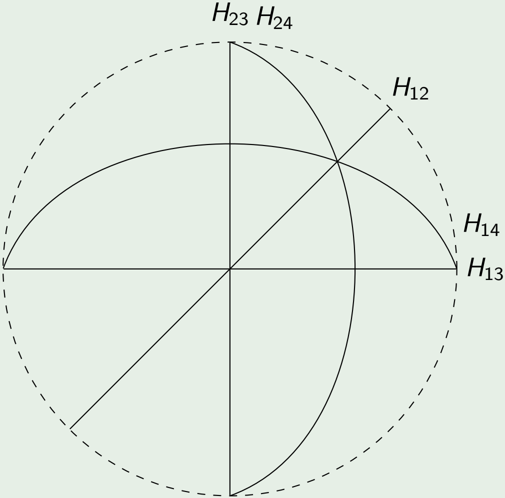


Characteristic Polynomial: $\chi_{\mathcal{A}}(t) = t^4 - 6t^3 + 11t^2 - 6t$
Evaluated: $\chi_{\mathcal{A}}(-1) = 1 + 6 + 11 + 6 = 24$

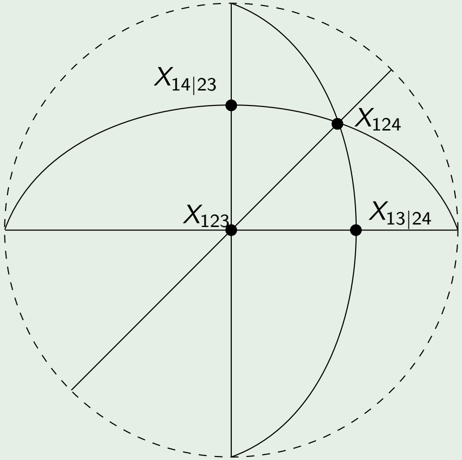
Example

Consider the affine arrangement where H_{34} is the line at infinity.



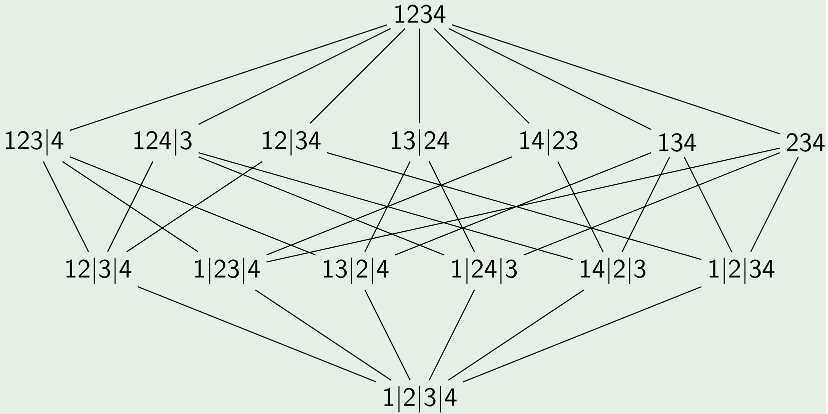
Example

Consider the affine arrangement where H_{34} is the line at infinity.



Example

We already computed the intersection poset of this affine arrangement when we did the previous example...



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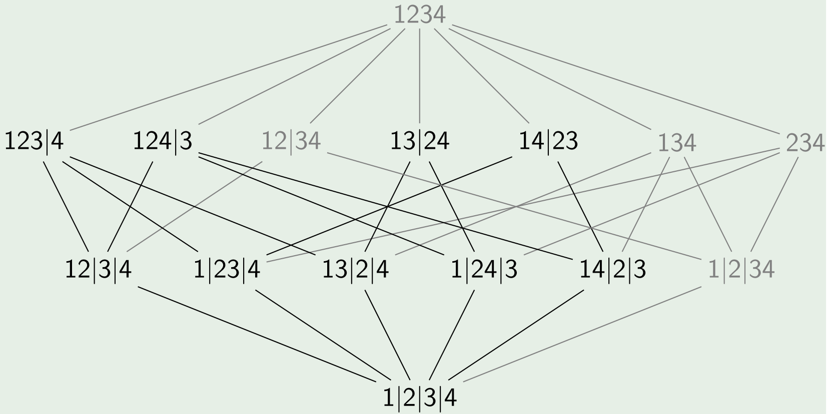

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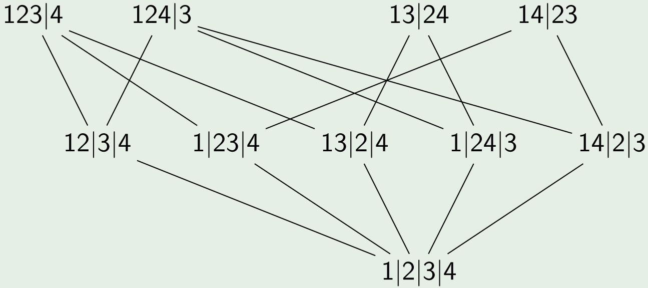
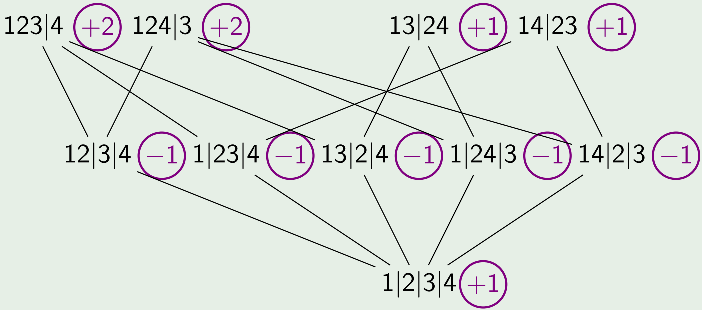


Example

We can obtain the characteristic polynomial from the Möbius function...



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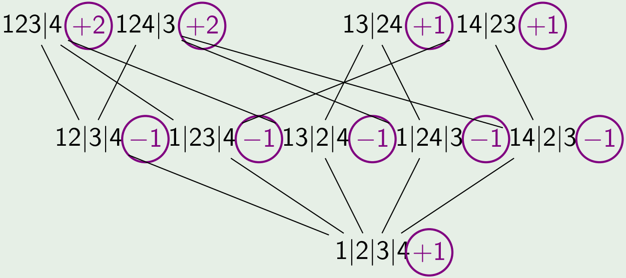


Characteristic Polynomial: $\chi_{\mathcal{A}}(t) = t^4 - 5t^3 + 6t^2$

Evaluated: $\chi_{\mathcal{A}}(-1) = 1 + 5 + 6 = 12$

Cone of an Arrangement

Let $\mathcal{A}$ be an arrangement with chambers $\mathcal{C}(\mathcal{A})$.

Definition

Let $\mathcal{A}$ be an arrangement of hyperplanes. Let $\mathcal{A}'$ be a central subarrangement of $\mathcal{A}$ and let $\mathcal{C}(A')$ the set of chambers of $\mathcal{A}'$. Then a *cone* $\mathcal{K}$ is an element of $\mathcal{C}(A')$.

Interior Intersections

Let $\mathcal{L}^{\text{int}}(\mathcal{A}, \mathcal{K})$ denote the set of insersections touching the interior of the cone and w_k denote the kth signless Whitney number of the first kind.

Definition

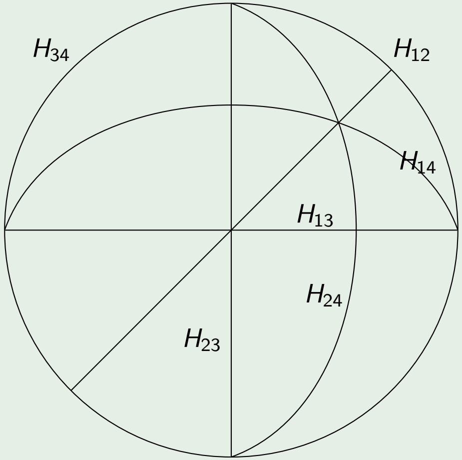
Let $\mathcal{A}$ be a hyperplane arrangement and $\mathcal{K}$ a cone of $\mathcal{A}$. Then the *characteristic polynomial* of $\mathcal{K}$ is

$$\chi_{\mathcal{A}}(\mathcal{K}, t) = \sum_{x \in \mathcal{L}^{\text{int}}(\mathcal{A}, \mathcal{K})} \mu(\hat{0}, x) t^{\dim x}$$

$$= \sum_{k=0}^{n} (-1)^{k+1} w_k t^{n-k}$$

Example

Let's consider a cone $\mathcal{K}$ defined by H_{12} and H_{34} in



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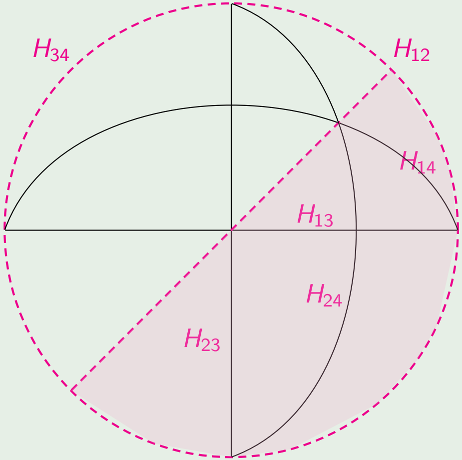

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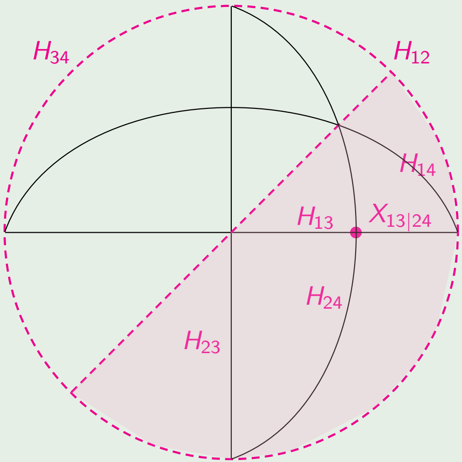
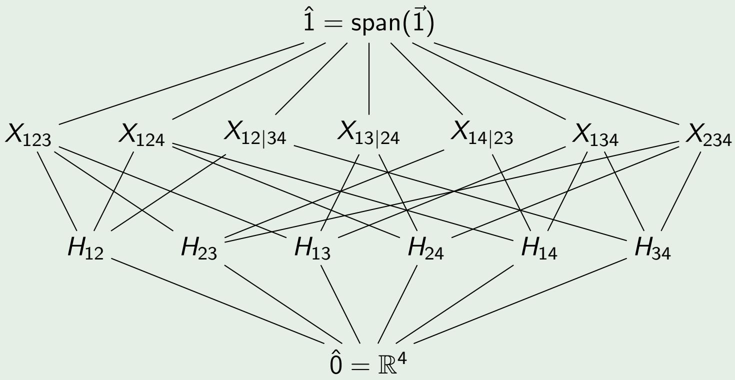


Example

The intersection lattice of this cone is



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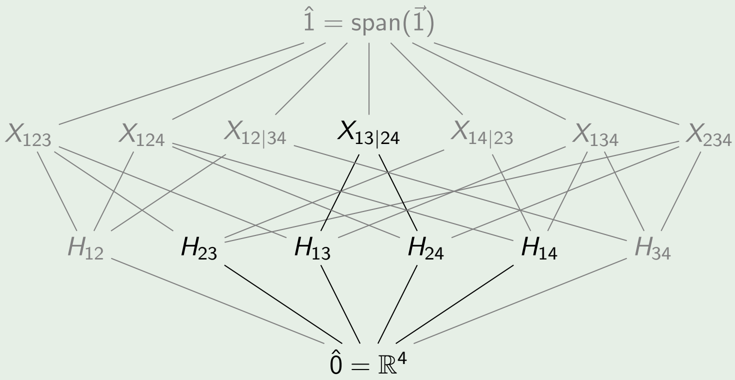

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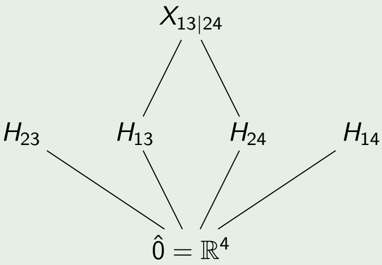

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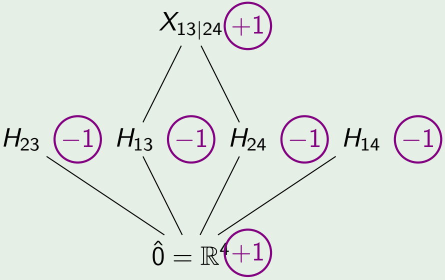

Characteristic Polynomial: $\chi_{\mathcal{A}}(\mathcal{K}, t) = t^4 - 4t^3 + t^2$

Zaslavsky's Theorem, revisited

Theorem (Zaslavsky, 1977)

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$ and $\mathcal{K}$ be a cone of $\mathcal{A}$. Let $\chi_{\mathcal{A}}(\mathcal{K}, t)$ be the characteristic polynomial of the cone. Then

$$\chi_{\mathcal{A}}^{int}(\mathcal{K}, -1) = (-1)^n \, \# \binom{\text{chambers interior}}{\text{to the cone } \mathcal{K}}.$$

In other words

$$\begin{aligned} \# \binom{\text{chambers of the}}{\text{arrangement}} &= \pm \sum_{x \in \mathcal{L}(\mathcal{A},\mathcal{K})} \mu(\hat{0}, x)(-1)^{\dim x} \\ &= \sum_{x : \mathrm{rk} x = 1} |\mu(\hat{0}, x)| \\ &= w_0 + w_1 + \cdots + w_n \end{aligned}$$

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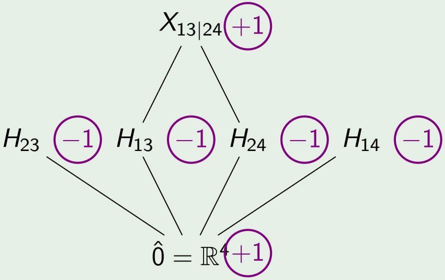
$$\chi_{\mathcal{A}}^{int}(\mathcal{K}, -1) = (-1)^n \, \# \begin{pmatrix} \text{chambers interior} \\ \text{to the cone } \mathcal{K} \end{pmatrix}.$$

Note:

- This is implicit in Brown's work on BHR random walks in hyperplane arrangements and cones (2000).
- $\mathcal{L}^{\text{int}}(\mathcal{A}, \mathcal{K})$ appears independently in work of Gente (2013) and Aguiar-Mahajan (2017) on Varchenko's determinant for cones.

Example

The intersection lattice of this cone is



Characteristic Polynomial: $\chi_{\mathcal{A}}(\mathcal{K}, t) = t^4 - 4t^3 + t^2$

Evaluated: $\chi_{\mathcal{A}}(\mathcal{K}, -1) = 1 + 4 + 1 = 6$

The Type A Reflection Arrangement

Definition

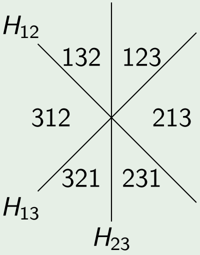
The *Type A Reflection Arrangement* A_{n-1} is the arrangement with hyperplanes $H_{ij} = \{\vec{x} \in \mathbb{R}^n \mid x_i - x_j = 0\}$ for all $1 \leq i < j \leq n$.

Linear Extensions and Chambers

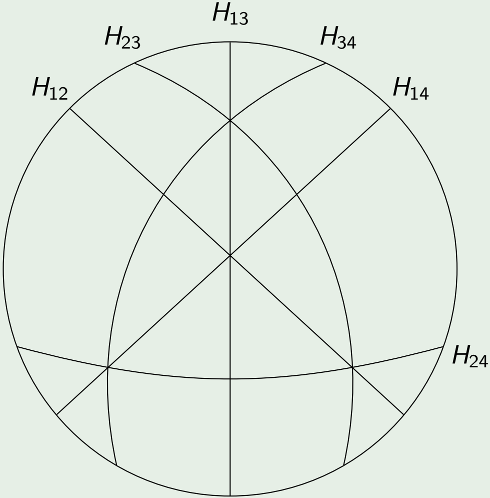
Chambers of A_{n-1}

The chambers A_{n-1} can be labelled by permutations on $[n]$ in which if i appears before j in the permutation, then $x_i < x_j$.

Example (A_{3-1}, projected into $\mathbb{R}^2$)



Example (A_{4-1},projected into $\mathbb{R}^4$)



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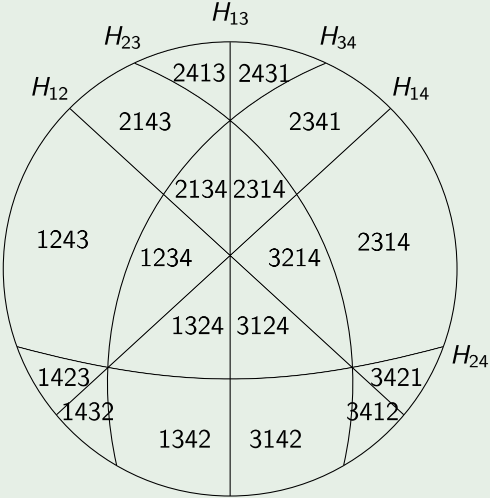


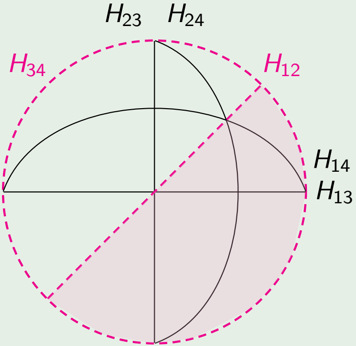
Cones of the Type A Reflection Arrangement

Cones of A_{n-1}

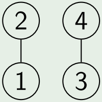
Cones of A_{n-1} can be encoded as posets P on $[n]$ by the rule if $x_i < x_j$ in the cone, then $i < j$ in P.

- If we have a cone of A_{n-1} defined by P, we'll call it $\mathcal{K}_P$.
- The chambers of $\mathcal{K}_P$ can be labelled by linear extensions of P.

Example (A_{4-1})



$\leftrightarrow \quad P =$

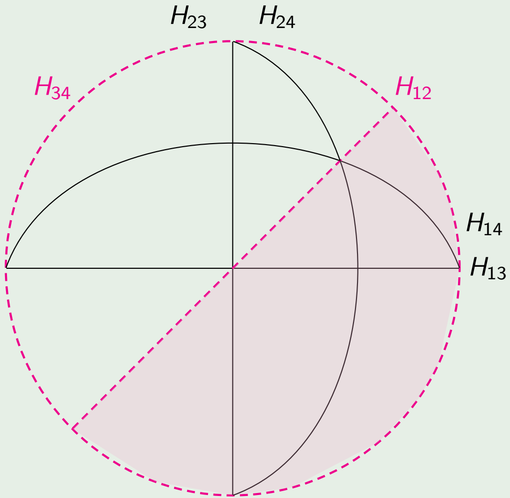


The chambers are labelled by linear extensions of P:

$$1234, \quad 1324, \quad 1342, \quad 3124, \quad 3142, \quad 3412$$

Example (A_{4-1})

We can label the chambers of $\mathcal{K}_P$ by linear extensions of P.



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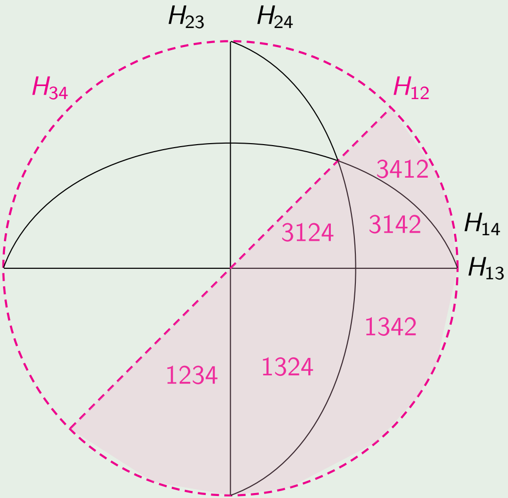


Posets & Characteristic Polynomials

Let P be any poset on $[n]$ and let $\mathrm{LinExt}(P)$ denote the set of linear extensions of P. Then

$$\begin{aligned} w_0 + w_1 + \cdots + w_n &= (-1)^n \; \chi_{\mathcal{A}}^{\mathrm{int}}(\mathcal{K}_P, t) \,|_{t=-1} \\ &= \# \; \mathrm{LinExt}(P). \end{aligned}$$

where w_k is the kth Whitney number of $\mathcal{L}(\mathcal{A}, \mathcal{K})$.

Question

What do the unsigned Whitney numbers $w_0, w_1, \ldots, \; w_n$ count?

Rephrasing: Fibres of Maps

Let P be any poset on $[n]$ and let $\mathrm{LinExt}(P)$ denote the set of linear extensions of P.

Question

Can we find a map $\varphi : LinExt(P) \to \mathcal{L}^{int}(\mathcal{A}, \mathcal{K}_P)$ such that for $x \in \mathcal{L}^{int}(\mathcal{A}, \mathcal{K}_P)$, the cardinality of the preimage is precisely $|\mu(\hat{0}, x)|$? That is

$$\#\varphi^{-1}(x) = |\mu(\hat{0}, x)|$$

Then the Whitney numbers are precisely

$$w_k = \#\{\sigma \in \mathrm{LinExt}(P) \mid \mathrm{rk}\varphi(\sigma) = k\}$$

Family 1: Width 2 Posets

Recall Dilworth's theorem (the width of a poset):

Theorem (Dilworth, 1950)

Let P be a poset and $A \subseteq P$ be an antichain of largest cadinality. Then A has the same number of elements as a minimum chain decomposition of P, called the **width** *of a poset.*

Family 1: Width 2 Posets

Proposition (GDB, 2018)

If P is a width 2 poset then $\#LinExt(P) = \#\mathcal{L}^{int}(\mathcal{A}, \mathcal{K}_P)$.

Proof.

Since no antichain has more than two elements, the Möbius function values of $X \in \mathcal{L}^{\text{int}}(\mathcal{A}, \mathcal{K}_P)$ are ± 1. □

Theorem (GDB, 2018)

For a choice of decomposition $P = P_1 \sqcup P_2$ into 2 chains, there is a (simple) bijection $\varphi : LinExt(P) \to \mathcal{L}^{int}(\mathcal{A}, \mathcal{K}_P)$.

Case Study: Ferrers' Posets

Let $F_{2,n}$ denote the poset associated to a $2 \times n$ rectangular Ferrers' diagram. Recall that

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \mathrm{LinExt}(F_{2,n})$$

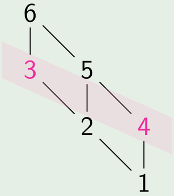
Theorem (GDB, 2018)

We have $C_n = LinExt(F_{2,n}) = w_0 + w_1 + \cdots + w_{n-1}$ where the w_k are Narayana numbers

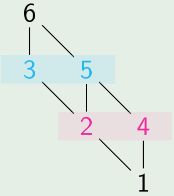
$$w_k = N(n, k+1) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

In this bijection, intersections of 2 hyperplanes in $\mathcal{L}^{\text{int}}(\mathcal{A}, \mathcal{K}_P)$ correspond to valleys DU-adjacent pairs of the Dyck path.

Example



$\leftrightarrow$ $UUDDUD$



$\leftrightarrow$ $UDUDUD$

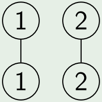
Disjoint Union of Chains

Suppose $P = \mathbf{a}_1 \sqcup \mathbf{a}_2 \sqcup \cdots \sqcup \mathbf{a}_\ell$ is a disjoint union of ℓ chains with cardinalities a_i. Then the Dilworth decomposition is unique (up to labelling chains).

Example

Let $P = \mathbf{a}_1 \sqcup \mathbf{a}_2$ where $a_1 = a_2 = 2$.

$P =$



Linear extensions of P: 1122, 1212, 2112,1221,2121,2211

The linear extensions are permutations of a multiset $\{1^{a_1}, 2^{a_2}, \ldots, \ell^{a_\ell}\}$.

Example

The permutations of $M = \{1^2, 2^2\}$ are

$$\begin{pmatrix}1 & 1 & 2 & 2\\1 & 1 & 2 & 2\end{pmatrix}, \begin{pmatrix}1 & 1 & 2 & 2\\1 & 2 & 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 1 & 2 & 2\\2 & 1 & 1 & 2\end{pmatrix}$$
$$\begin{pmatrix}1 & 1 & 2 & 2\\1 & 2 & 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 1 & 2 & 2\\2 & 1 & 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 1 & 2 & 2\\2 & 2 & 1 & 1\end{pmatrix}$$

The bottom row of each of these permutations corresponds to a linear extension of P: 1122, 1212, 2112,1221,2121,2211.

Question

Is there some statistic on multisets that will help us describe $w_0, w_1, \ldots$ for a cone defined by a disjoint union of chains?

Foata's Intercalation Product

Example

Let $\sigma = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 \end{pmatrix}$ and let $\rho = \begin{pmatrix} 1 & 2 & 2 & 3 & 4 \\ 2 & 2 & 4 & 3 & 1 \end{pmatrix}$. To compute $\sigma \intercal \rho$, we first juxtapose σ and ρ. This gives

$$\left(\begin{array}{cccc|ccccc} 1 & 1 & 2 & 3 & 1 & 2 & 2 & 3 & 4 \\ 1 & 2 & 3 & 1 & 2 & 2 & 4 & 3 & 1 \end{array}\right).$$

Then we *stably sort* columns in nondecreasing order

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 1 & 2 & 2 & 3 & 2 & 4 & 1 & 3 & 1 \end{pmatrix}.$$

Decomposition into Primes: Existence and Uniqueness

Theorem (Foata, 1969)

Every multiset permutation has a decomposition into a product of prime cycles. That is, for a multiset permutation σ there exist $t \geq 0$ prime cycles $\sigma_1, \ldots, \sigma_t$ such that

$$\sigma = \sigma_1 \intercal \sigma_2 \cdots \intercal \sigma_t.$$

Theorem (Foata, 1969)

The cycle decomposition of a multiset permutation is unique up to interchanging pairs of adjacent, disjoint prime cycles.

Intercalation Statistic

The intercalation product gives a map $f : \mathrm{LinExt}(P) \to \mathcal{L}^{\mathrm{int}}(\mathcal{A}, \mathcal{K}_P)$ in which $\sigma \in \mathrm{LinExt}(P)$ is sent to $x \in \mathcal{L}^{\mathrm{int}}(\mathcal{A}, \mathcal{K}_P)$ which has blocks corresponding to cycles of σ.

Example ($M = \{1^2, 2^2\}$)

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \intercal \begin{pmatrix} 1 \\ 1 \end{pmatrix} \intercal \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \intercal \begin{pmatrix} 2 \\ 2 \end{pmatrix} \intercal \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then $f\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \end{pmatrix}$ is $13|2|4$.

Intercalation and Characteristic Polynomials

For a multiset permutation σ, let $\text{fcyc}(\sigma)$ denote number of cycles in the decomposition of σ into prime cycles.

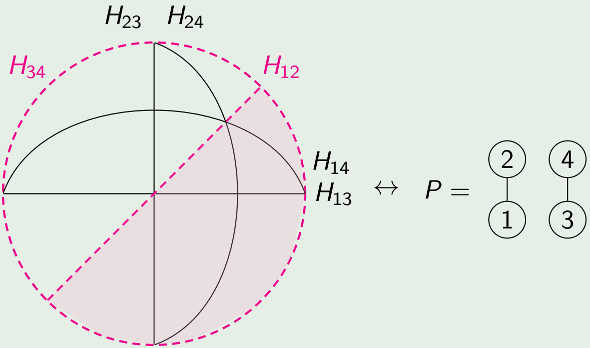
Theorem (GDB, 2018)

Let $\mathbf{a} \vDash n$, $P = \mathbf{a_1} \sqcup \cdots \sqcup \mathbf{a}_\ell$ *and* $M = \{1^{a_1}, 2^{a_2}, \ldots, \ell^{a_\ell}\}$. *Then*

$$\chi^{int}_{A_{n-1}}(\mathcal{K}_P, t) = \sum_{\substack{\text{multiset} \\ \text{permutations} \\ \omega \text{ of } M}} (-t)^{fcyc(\omega)}.$$

Example (A_{4-1})

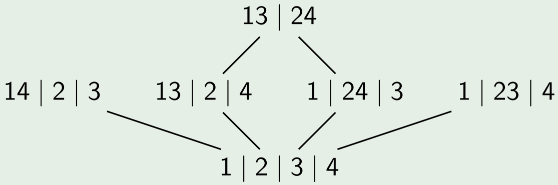
Consider the cone defined by



Let's compute the characteristic polynomial in two ways.

Example (Method 1)

The intersection meet semilattice of $\mathcal{K}_P$ is



We have $\chi^{\text{int}}_{A_{n-1}}(\mathcal{K}_P, t) = t^4 - 4t^3 + t^2$.

Example (Method 2)

For $M = \{1^2, 2^2\}$, we have

$$\begin{pmatrix}1&1&2&2\\1&1&2&2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \top \begin{pmatrix}1\\1\end{pmatrix} \top \begin{pmatrix}2\\2\end{pmatrix} \top \begin{pmatrix}2\\2\end{pmatrix}$$

$$\begin{pmatrix}1&1&2&2\\1&2&1&2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \top \begin{pmatrix}1&2\\2&1\end{pmatrix} \top \begin{pmatrix}2\\2\end{pmatrix}$$

$$\begin{pmatrix}1&1&2&2\\2&1&1&2\end{pmatrix} = \begin{pmatrix}1&2\\2&1\end{pmatrix} \top \begin{pmatrix}1\\1\end{pmatrix} \top \begin{pmatrix}2\\2\end{pmatrix}$$

$$\begin{pmatrix}1&1&2&2\\1&2&2&1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \top \begin{pmatrix}2\\2\end{pmatrix} \top \begin{pmatrix}1&2\\2&1\end{pmatrix}$$

$$\begin{pmatrix}1&1&2&2\\2&1&2&1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} \top \begin{pmatrix}1&2\\2&1\end{pmatrix} \top \begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}1&1&2&2\\2&2&1&1\end{pmatrix} = \begin{pmatrix}1&2\\2&1\end{pmatrix} \top \begin{pmatrix}1&2\\2&1\end{pmatrix}$$

Example (Method 2)

For $M = \{1^2, 2^2\}$, we have

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \top \begin{pmatrix} 1 \\ 1 \end{pmatrix} \top \begin{pmatrix} 2 \\ 2 \end{pmatrix} \top \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad t^4$$

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \top \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \top \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad -t^3$$

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \top \begin{pmatrix} 1 \\ 1 \end{pmatrix} \top \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad -t^3$$

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \top \begin{pmatrix} 2 \\ 2 \end{pmatrix} \top \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad -t^3$$

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \top \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \top \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad -t^3$$

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \top \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad t^2$$

Example

Adding up terms gives

$$\chi^{\text{int}}_{A_{n-1}}(\mathcal{K}_P, t) = t^4 - 4t^3 + t^2.$$

Compare this to what we got from the intersection poset:

$$\chi^{\text{int}}_{A_{n-1}}(\mathcal{K}_P, t) = t^4 - 4t^3 + t^2.$$

They are the same!

Future Work

Goal: Extend Foata's theory of multisets to arbitrary posets using a choice of Dilworth decomposition.

- I have a rough idea of what the map might look like.
- I'm working on refining that idea.

Thank you!

Selected References

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