Approximating clutters with matroids

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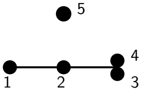
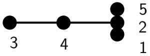
Example: let $\Lambda = \{1\,2\,3, 1\,2\,4, 3\,4\,5\}$

- Both $\{1\,2\,3, 1\,2\,4, 3\,4\}$ and $\{1\,3\,4, 2\,3\,4, 3\,4\,5, 1\,2, 1\,5, 2\,5\}$ are are clutters of circuits of a matroid:





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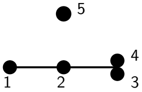
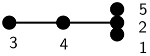
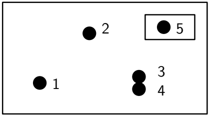
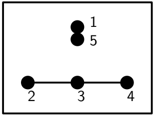




- Both $\{1\,2\,3, 1\,2\,4\}$ and $\{1\,2\,3, 1\,2\,4, 3\,4\,5, 1\,3\,4, 2\,3\,5, 2\,4\,5\}$ are clutters of bases of a matroid:





Previous efforts in linking clutters and matroids

- Vaderlind, 1986: *Clutters and semimatroids*
- Dress and Wenzel, 1990: *Matroidizing set systems: a new approach to matroid theory*
- Cordovil, Fukuda, and Moreira, 1991: *Clutters and matroids*
- Traldi, 1997-2003:*Clutters and circuits I, II, III*
- Blasiak, Rowe, Traldi, and Yacobi, 2005: *Several definitions of matroids*
- Martini and Wenzel, 2005: *Symmetrization of closure operators and visibility*

Definitions: clutters

Denote by $\mathrm{Clutt}(\Omega)$ the set of all clutters on a finite set Ω

For $\Lambda \in \mathrm{Clutt}(\Omega)$, let

$$\Lambda^+ = \{B \subseteq \Omega : B \supseteq A \text{ for some } A \in \Lambda\}$$
$$\Lambda^- = \{B \subseteq \Omega : B \subseteq A \text{ for some } A \in \Lambda\}$$

Hence

$$\Lambda = \mathrm{minimal}(\Lambda^+) = \mathrm{maximal}(\Lambda^-)$$

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Ex: for a matroid M, $\mathcal{C}(M)^+ \to$ dependent sets
$\mathcal{B}(M)^- \to$ independent sets

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Ex: for a matroid M, $\mathcal{C}(M)^{+} \to$ dependent sets
$\mathcal{B}(M)^{-} \to$ independent sets

Define the following two partial orders on $\mathrm{Clutt}(\Omega)$

$$\begin{aligned} \Lambda_1 \leqslant^{+} \Lambda_2 &\iff \Lambda_1^{+} \subseteq \Lambda_2^{+} \\ &\iff \forall\ A \in \Lambda_1\ \exists\ A' \in \Lambda_2\ s.t.\ A \supseteq A' \\ \Lambda_1 \leqslant^{-} \Lambda_2 &\iff \Lambda_1^{-} \subseteq \Lambda_2^{-} \\ &\iff \forall\ A \in \Lambda_1\ \exists\ A' \in \Lambda_2\ s.t.\ A \subseteq A' \end{aligned}$$

Definitions: clutters

The clutter Λ^c is $\{\Omega \setminus A : A \in \Lambda\}$

The blocker of a clutter is

$$b(\Lambda) = \text{minimal}\{B : B \cap A \neq \emptyset \text{ for all } A \in \Lambda\}$$

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Lem $\Lambda_1 \leqslant^+ \Lambda_2 \Leftrightarrow \Lambda_1^c \leqslant^- \Lambda_2^c$
$\Lambda_1 \leqslant^+ \Lambda_2 \Leftrightarrow b(\Lambda_2) \leqslant^+ b(\Lambda_1)$
$\Lambda_1 \leqslant^- \Lambda_2 \Leftrightarrow b(\Lambda_2^c)^c \leqslant^- b(\Lambda_1^c)^c$

Definitions: interpretations

For matroids:

Λ	Λ^c	$b(\Lambda)$
$\mathcal{B}(M)$	$\mathcal{B}(M^*)$	$\mathcal{C}(M^*)$
$\mathcal{C}(M)$	$\mathcal{H}(M^*)$	$\mathcal{B}(M^*)$
$\mathcal{H}(M)$	$\mathcal{C}(M^*)$	$-$

$\mathcal{C}(M_1) \leqslant^+ \mathcal{C}(M_2) \Leftrightarrow M_1$ is above M_2 in the *weak* order

$\mathcal{B}(M_1) \leqslant^- \mathcal{B}(M_2) \Leftrightarrow M_1$ is below M_2 in the *weak* order

$\mathcal{B}(M_1) \leqslant^+ \mathcal{B}(M_2) \Leftrightarrow M_1^*$ is below M_2^* in the *weak* order

Answering our initial question

Given $\Lambda \in \text{Clutt}(\Omega)$:

- We want it close to a matroid clutter. Do we choose circuit clutters or basis clutters? (Or hyperplane clutters, or ...) Let's say we choose $\Sigma \subseteq \text{Clutt}(\Omega)$

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- Now, which order do we use to compare? $\leqslant^+$ or $\leqslant^-$? Let's say we take $\textsc{order}$
- And, with respect to $\textsc{order}$, we seek clutters from Σ that are above or below our clutter Λ? Let's say we take $\textsc{side}$

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- Now, which order do we use to compare? $\leqslant^+$ or $\leqslant^-$? Let's say we take ORDER
- And, with respect to ORDER, we seek clutters from Σ that are above or below our clutter Λ? Let's say we take SIDE

Thm (Informal)
For any choice of ORDER and SIDE, there is a family of clutters $\mathcal{F} \subset \text{Clutt}(\Omega)$ such that:
If $\mathcal{F} \subseteq \Sigma$, then there exist $\Lambda_1, \ldots, \Lambda_s$ in Σ that are closest to Λ with respect to ORDER and SIDE. (The optimal completions)
Moreover, Λ can be recovered from $\Lambda_1, \ldots, \Lambda_s$.

(The decomposition)

Decomposition theorems: in general

(Martí-Farré, dM 17)

Thm Let $\Lambda \in \mathrm{Clutt}(\Omega)$ and $\Sigma \subseteq \mathrm{Clutt}(\Omega)$
If for all $S = \{x_1, \ldots, x_r\} \subseteq \Omega$ the clutter $\Lambda_S = \{\{x_1\}, \ldots, \{x_r\}\}$ belongs to Σ, then

(1) there exists some $\Lambda' \in \Sigma$ such that $\Lambda \leqslant^+ \Lambda'$

(2) if $\Lambda_1, \ldots, \Lambda_s \in \Sigma$ are the minimal clutters in (1) then $\Lambda = \mathrm{minimal}\left(\Lambda_1^+ \cap \cdots \cap \Lambda_s^+\right)$

Decomposition theorems: in general

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Thm

$\boxed{\leqslant^+, \text{ upper}}$ $\mathcal{F}_u^+ = \{\{\{x_1\}, \dots, \{x_r\}\} : x_1, \dots, x_r \in \Omega\}$
$\Lambda = \text{minimal}\left(\Lambda_1^+ \cap \dots \cap \Lambda_s^+\right)$

$\boxed{\leqslant^+, \text{ lower}}$ $\mathcal{F}_\ell^+ = \{\{x_1 \dots x_r\} : x_1, \dots, x_r \in \Omega\}$
$\Lambda = \text{minimal}\left(\Lambda_1^+ \cup \dots \cup \Lambda_s^+\right)$

$\boxed{\leqslant^-, \text{ upper}}$ $\mathcal{F}_u^- = \{\{\Omega \setminus x_1, \dots, \Omega \setminus x_r\} : x_1, \dots, x_r \in \Omega\}$
$\Lambda = \text{maximal}\left(\Lambda_1^- \cap \dots \cap \Lambda_s^-\right)$

$\boxed{\leqslant^-, \text{ lower}}$ $\mathcal{F}_\ell^- = \{\{x_1 \dots x_r\} : x_1, \dots, x_r \in \Omega\}$
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$\boxed{\leqslant^{+}, \text{ upper}}$ $\mathcal{F}_u^{+} = \{\{\{x_1\}, \ldots, \{x_r\}\} : x_1, \ldots, x_r \in \Omega\}$
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$\boxed{\leqslant^{+}, \text{ lower}}$ $\mathcal{F}_\ell^{+} = \{\{x_1 \ldots x_r\} : x_1, \ldots, x_r \in \Omega\}$
$\Lambda = \text{minimal}\left(\Lambda_1^{+} \cup \cdots \cup \Lambda_s^{+}\right)$

$\boxed{\leqslant^{-}, \text{ upper}}$ $\mathcal{F}_u^{-} = \{\{\Omega \setminus x_1, \ldots, \Omega \setminus x_r\} : x_1, \ldots, x_r \in \Omega\}$
$\Lambda = \text{maximal}\left(\Lambda_1^{-} \cap \cdots \cap \Lambda_s^{-}\right)$

$\boxed{\leqslant^{-}, \text{ lower}}$ $\mathcal{F}_\ell^{-} = \{\{x_1 \ldots x_r\} : x_1, \ldots, x_r \in \Omega\}$
$\Lambda = \text{maximal}\left(\Lambda_1^{-} \cup \cdots \cup \Lambda_s^{-}\right)$

Lem All clutters above are clutters of bases, and all are clutters of circuits except $\mathcal{F}_u^{-}$
(and of bases/circuits of graphic/representable/transversal matroids)

Decomposition theorems: matroids

So, given Λ we can consider approximations for any combination of

$\textsc{circuits}$ / $\textsc{bases}$; $\leqslant^{+}$ / $\leqslant^{-}$; $\textsc{upper}$ / $\textsc{lower}$

except: $\textsc{circuits}$, $\leqslant^{-}$, $\textsc{upper}$

Now, can we effectively find them?

Decomposition theorems: matroids

So, given Λ we can consider approximations for any combination of

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Now, can we effectively find them?

By combining blockers and complements, it is enough to solve one case in each group

($\textsc{circuits}$, $\leqslant^+$, $\textsc{upper}$)	($\textsc{circuits}$, $\leqslant^+$, $\textsc{lower}$)
($\textsc{bases}$, $\leqslant^+$, $\textsc{lower}$)	($\textsc{bases}$, $\leqslant^+$, $\textsc{upper}$)
($\textsc{bases}$, $\leqslant^-$, $\textsc{lower}$)	($\textsc{bases}$, $\leqslant^-$, $\textsc{upper}$)

($\textsc{circuits}$, $\leqslant^-$, $\textsc{lower}$)

Finding the completions

Martí-Farré 14: algorithm for $(\textsc{circuits}, \leqslant^{+}, \textsc{upper})$

Martí-Farré, dM 17: algorithms for $(\textsc{circuits}, \leqslant^{+}, \textsc{lower})$ and $(\textsc{circuits}, \leqslant^{-}, \textsc{lower})$

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Idea: For distinct $A_1, A_2 \in \Lambda$, define

$$I_\Lambda(A_1 \cup A_2) = \bigcap_{X \in \Lambda, X \subseteq A_1 \cup A_2} X$$

Then:

$$\Lambda \text{ is the clutter of circuits of some matroid}$$

$$\Updownarrow$$

$$I_\Lambda(A_1 \cup A_2) = \emptyset \text{ for all } A_1 \neq A_2 \in \Lambda$$

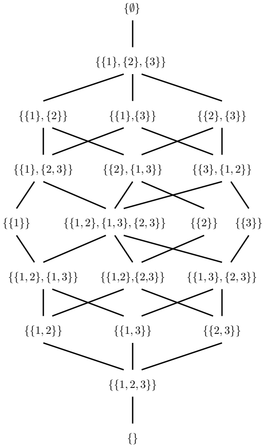
The lattice structure

Recall the operations from the theorems:

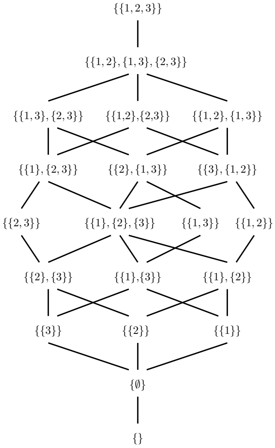
$$\begin{aligned} \Lambda_1 \sqcap^+ \Lambda_2 &= \min\left(\Lambda_1^+ \cap \Lambda_2^+\right) \\ \Lambda_1 \sqcup^+ \Lambda_2 &= \min\left(\Lambda_1^+ \cup \Lambda_2^+\right) \\ \Lambda_1 \sqcap^- \Lambda_2 &= \max\left(\Lambda_1^- \cap \Lambda_2^-\right) \\ \Lambda_1 \sqcup^- \Lambda_2 &= \max\left(\Lambda_1^- \cup \Lambda_r^-\right) \end{aligned}$$

Obs $(\mathrm{Clutt}(\Omega), \leqslant^+, \sqcup^+, \sqcap^+)$ and $(\mathrm{Clutt}(\Omega), \leqslant^-, \sqcup^-, \sqcap^-)$ are distributive lattices

The lattice structure



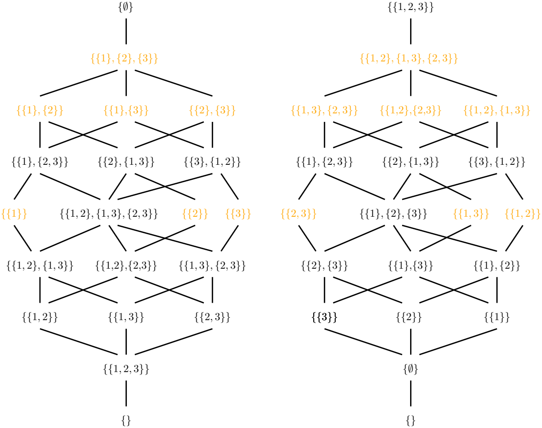
(Clutt({1, 2, 3}), $\leq^+$)



(Clutt({1, 2, 3}), $\leq^-$)

The lattice structure

The "special families" are nothing but join/meet-irreducibles in this lattices!

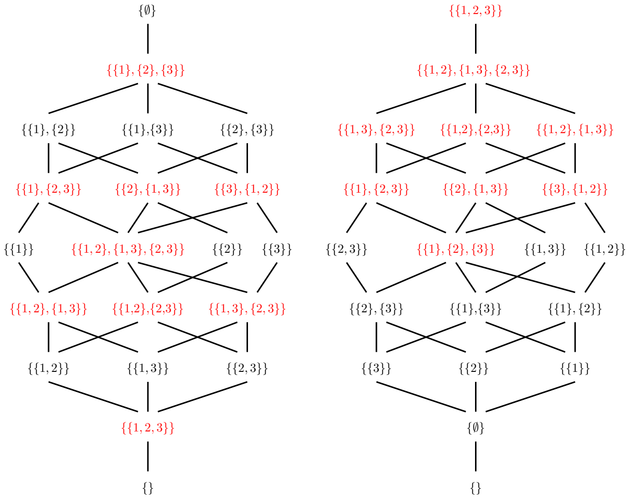


$(\mathrm{Clutt}(\{1,2,3\}), \leq^{+})$ $(\mathrm{Clutt}(\{1,2,3\}), \leq^{-})$

Fixing the ground set

Let

$$\mathrm{Clutt}_0(\Omega) = \{\Lambda \in \mathrm{Clutt}(\Omega) : \bigcup_{A\in\Lambda} A = \Omega\}$$



Fixing the ground set

(Martí-Farré, dM, Ruiz 18+)

$(\mathrm{Clutt}_0(\Omega), \leqslant^-, \sqcup^-, \sqcap^-)$ is a lattice, but $(\mathrm{Clutt}_0(\Omega), \leqslant^+)$ is not

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- To decompose $\Lambda \in \mathrm{Clutt}_0(\Omega)$ with members of a family $\Sigma \subseteq \mathrm{Clutt}_0(\Omega)$ with respect to the order $\leqslant^-$, one only needs to check if Σ_0 contains the corresponding meet- or join-irreducibles

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- To decompose $\Lambda \in \mathrm{Clutt}_0(\Omega)$ with members of a family $\Sigma \subseteq \mathrm{Clutt}_0(\Omega)$ with respect to the order $\leqslant^+$, the family Σ_0 must contain
  - The clutter $\{1\,2\,\ldots\,n\}$ and all clutters of the form $\{\alpha_1, \alpha_2, \ldots, \alpha_m, \beta\,\beta_1, \ldots, \beta\,\beta_{n-m-1}\}$ (upper case)
  - The clutter $\{1, 2, \ldots, n\}$ and all clutters of the form $\{\alpha_1\,\alpha_2 \ldots\, \alpha_m\,\beta, \alpha_1\,\alpha_2 \ldots\, \alpha_m\,\beta_1\, \ldots \beta_{n-m-1}\}$ (lower case)

A few of the questions we'd like to solve

- Why in some cases not all optimal completions are needed in the decomposition formula? If only two matroids are needed in the decomposition, does this gives an interesting class of quasi-matroidal clutters?
- When computing completions, can we work directly with respect to bases? how do we restrict to some class of matroids?
- How do deletion and contraction (for clutters) behave in this framework?