Tverberg's theorems with Altered Nerves

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Joint work with T. Hogan, D. Oliveros, D. Yang

LE MENU

Tverberg's theorem: good things happen with a lot of points in R^d

Our Results

A few key ideas and ingredients

When sufficient elements exist, suddenly structure appears

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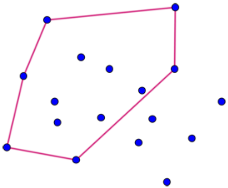
1. **"the more the merrier" principle:** Every sufficiently large system contains a large well-organized subsystem. Complete disorder is impossible!!!
2. **Ramsey's theorem:** One will always find monochromatic cliques in any edge coloring of a **sufficiently large** complete graph!!!
3. **Erdös Szekeres:** Every sufficiently large set of points in general position contains a subset of k points in convex position.

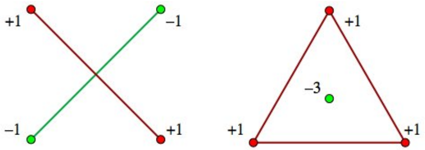


Johann Radon & Helge Tverberg:

SUFFICIENTLY LARGE SETS OF POINTS CAN ALWAYS BE PARTITIONED IN SPECIAL WAYS...

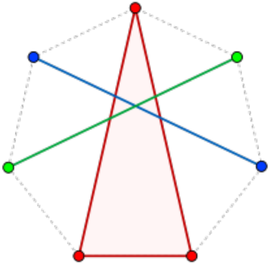
RADON's THEOREM (1920)

If $X \subset \mathbb{R}^d$ contains sufficiently many points (at least $d+2$ points!!!), then X can be partitioned into two disjoint subsets X_1, X_2 such that $conv(X_1) \cap conv(X_2) \neq \emptyset$



Theorem (H. Tverberg, 1966)

Let $X = \{a_1, \ldots, a_n\}$ be points in $\mathbb{R}^d$. If the number of points satisfies $n > (d+1)(r-1)$, then they can be partitioned into r disjoint parts $A_1, \ldots, A_r$ in such a way that the r convex hulls $\operatorname{conv} A_1, \ldots, \operatorname{conv} A_r$ have a point in common.



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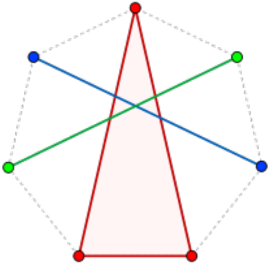
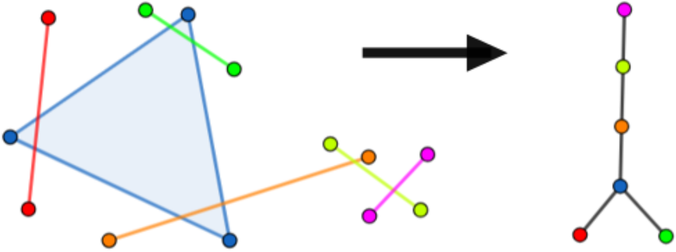


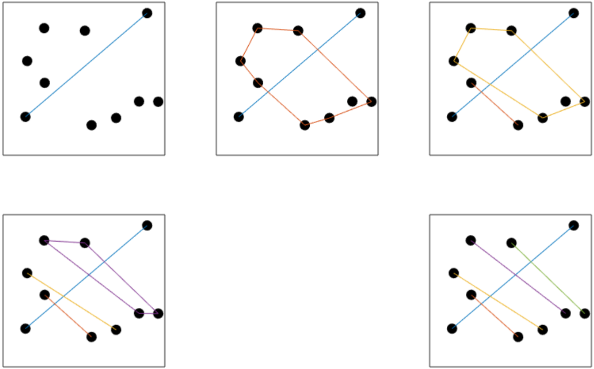
Remark This constant is best possible.

Partitions of Point Set and Nerves

- Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a family of convex sets in $\mathbb{R}^d$. The *nerve* $\mathcal{N}(\mathcal{F})$ of $\mathcal{F}$ is the simplicial complex with vertex set $[m] := \{1, 2 \ldots, m\}$ whose faces are $I \subset [m]$ such that $\cap_{i \in I} F_i \neq \emptyset$.



- Given a collection of points $S \subset \mathbb{R}^d$ and an m-partition into m color classes $\mathcal{P} = S_1, \ldots, S_m$ of S, *the nerve of the partition*, $\mathcal{N}(\mathcal{P})$ is the nerve complex $\mathcal{N}(\{\text{conv}(S_1), \ldots, \text{conv}(S_n)\})$,



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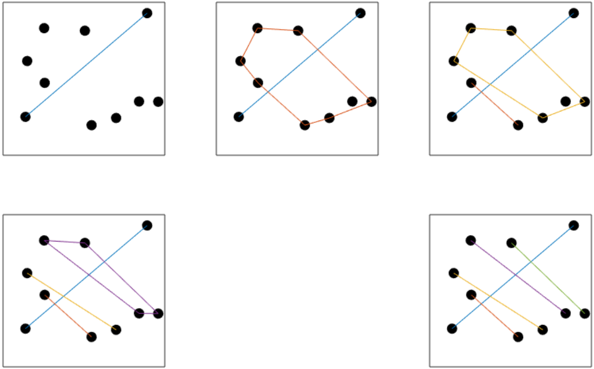
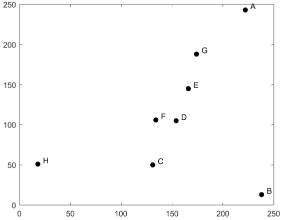


- Question: For a given point set, as we run over all partitions, what are the possible induced simplicial complexes? What happens when we have A LOT OF POINTS?

The point set A(222,243), B(238,13), C(131,50), D(154,105), E(166,145), F(134,106), G(174,188), H(18,51). cannot be partition to have a 4-path tree as a nerve!!!



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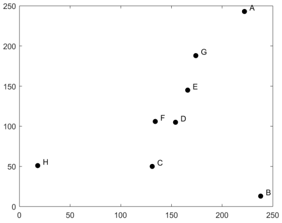

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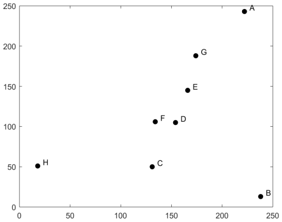

But once you have at least 9 points you will be able to find a partition that gives a 4-path tree nerve!!!

Theorem (Tverberg's theorem rephrased 1)

The $(m-1)$-simplex is a partition induced for every point configuration with sufficiently many points!!

- **Definition:** A simplicial complex K is *d-Tverberg* if there exists a constant $\mathrm{Tv}(K,d)$ such that K is partition induced on all point sets $S \subset \mathbb{R}^d$ in general position with $|S| > \mathrm{Tv}(K,d)$. The minimal such constant $\mathrm{Tv}(K,d)$ is called the *Tverberg number for K in dimension d*.

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- Question: Which simplicial d complexes are d-Tverberg complexes?

OUR MAIN RESULTS

Not all complexes are Tverberg

Theorem

The following complex is NOT a 2-Tverberg complex.

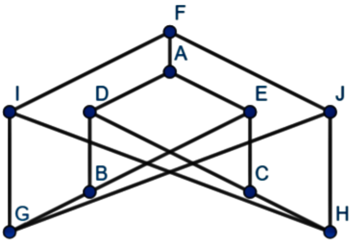

Figure: A 2-partition induced one-dimensional complex that is not 2-Tverberg

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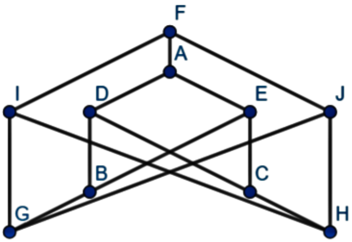

Figure: A 2-partition induced one-dimensional complex that is not 2-Tverberg

This 1-dimensional simplicial complex is partition induced on SOME planar point sets, but not for points in convex position, regardless of how many points we use!!

A Tverberg theorem with nerves are TREES or CYCLES

Theorem

All trees and cycles are d-Tverberg complexes for all $d \geq 2$.

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(A) *Every tree T_n on n nodes, is a d-Tverberg complex for $d \geq 2$. The Tverberg number $\mathrm{Tv}(T_n, d)$ exists and it is at most $R_{d+1}((d+1)(n-1)+1)$. More strongly, $\mathrm{Tv}(T_n, 2)$ is at most $\binom{4n-4}{2n-2} + 1$.*

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(B) *Every n-cycle C_n with $n \geq 4$ is a d-Tverberg complex for $d \geq 2$. The Tverberg number exists and $\mathrm{Tv}(C_n, d)$ is at most $nd + n + 4d$.*

Some improvements on the bounds for the Tverberg numbers of trees

Theorem

If a tree T_n is a caterpillar tree with n nodes, then T_n is d-Tverberg complex for all d, and its d-Tverberg number $\mathrm{Tv}(T_n, d)$ *is no more than* $(d+1)(n-1)+1$.

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Theorem

(A) *The 2-Tverberg numbers $\mathrm{Tv}(S_n, 2)$ for a star tree with n nodes equals $2n$.*

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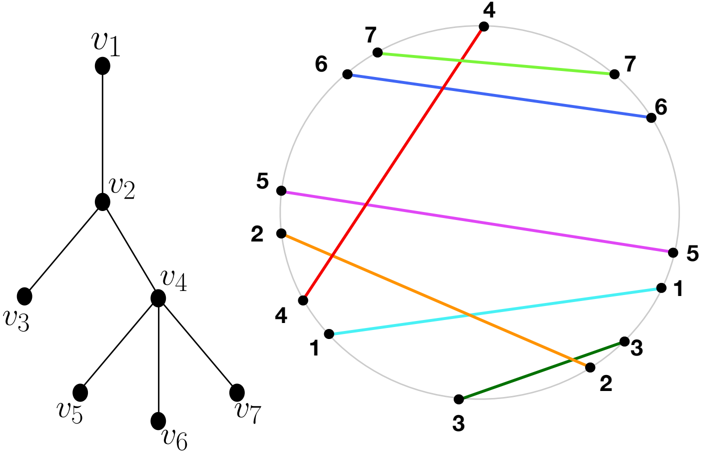
Theorem

(A) *The 2-Tverberg numbers $\mathrm{Tv}(S_n, 2)$ for a star tree with n nodes equals $2n$.*

(B) *The 2-Tverberg numbers of the path and cycle with four nodes are $\mathrm{Tv}(P_4, 2) = 9$ and $11 \leq \mathrm{Tv}(C_4, 2) \leq 13$.*

NICE IDEAS BEHIND

Trees are induced by partitions of "large" point sets in CONVEX POSITION



Extending the Partition

We can always extend the partition (or coloring) of convex position points to the rest of the points in $\bar{S}$, which may not be in the convex polytope

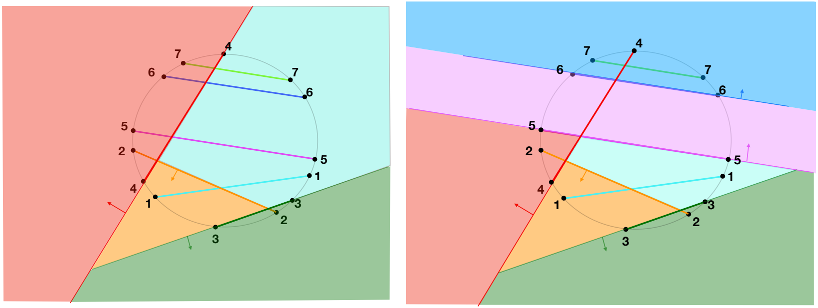
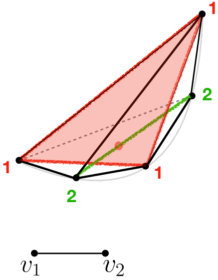
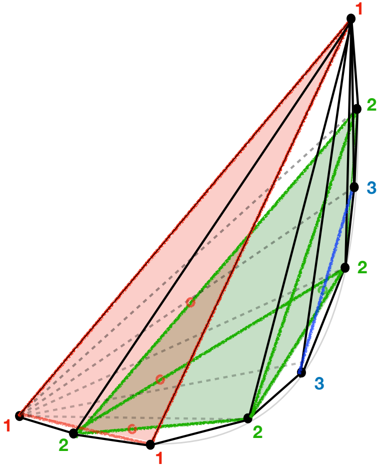


Figure: The extension of the partition obtained in Figure 2. The left figure, is the extension up to $n = 4$, the central figure is the extension up to $n = 6$

Using Matroids 1: A Multi-dimensional Erdős-Szekeres

- **Theorem:** [Grünbaum, Cordovil and Duchet]: There exist a number $N = N(k, d)$ such that every set of at least N points in general position in $\mathbb{R}^d$ contains the vertices of an ordered cyclic d-polytope. $C_m(d)$.





Using Matroids 2: Using OM classification in small rank

- **lemma** Let S_1,S_2 be two point sets in $\mathbb{R}^d$ with the same oriented matroid, and let σ be a bijection from S_1 to S_2 that preserves the orientation of any $(d+1)$-tuple in S_1. Then any partition $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ of S_1 and the corresponding partition of S_2 via σ, denoted $\sigma\mathcal{P} = \{\sigma(P_1), \sigma(P_2), \ldots, \sigma(P_n)\}$, have the same intersection graph $\mathbb{N}^1(\mathcal{P})$.
- **Moral:** It suffices to check one representative configuration of points from each oriented matroid type, reducing calculations to finitely many cases!

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- **Moral:** It suffices to check one representative configuration of points from each oriented matroid type, reducing calculations to finitely many cases!
- For small complexes, we can use exhaustive computer enumeration of all possible partitions, over all possible oriented matroids of point sets with fewer than ten points in rank 3. Luckily, they were classified by Aichholzer et al.

Lemma But the chirotope-preserving bijections do not preserve the higher-dimensional skeleton of the nerve of a partition!

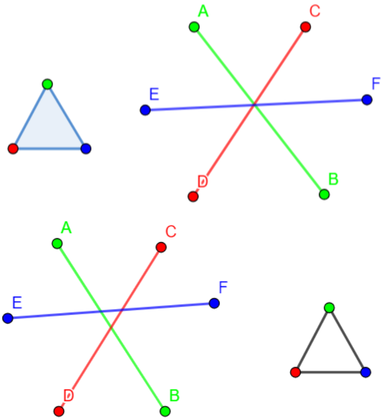

Figure: Only the 1-skeleton of the nerve is preserved by order-preserving bijection.

NOTE: We can still make use of Lemma oriented matroids because in our cases, the nerve complex equal their 1-skeleton!

Open Questions

1. OPEN PROBLEM: What is the exact value of $\mathrm{Tv}(T_n, d)$ where T_n is a tree with n nodes? Is $(d+1)(n-1)+1$ the correct value? What about the case of $d = 2$?
2. OPEN PROBLEM: What is the computational complexity of determining if a point configuration can partition induce a given complex?
3. OPEN PROBLEM: What is the computational complexity of computing the Tverberg numbers of a given Tverberg complex, such as a tree?
4. OPEN PROBLEM: Is there a complex K which is not d-Tverberg for any d?
5. OPEN PROBLEM: Is there a complex K and $i, j \in \mathbb{N}$, $i < j$ so that K is i-Tverberg but not j-Tverberg?

THANK YOU!
MERCI!
GRACIAS!