Topological hyperfields and hyperfield Grassmannians

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Main Points

- Joint work with Laura Anderson
- Topological hyperfields
  - Examples
  - Hyperfield Grassmannians (with topology)
    - Cohomology (Steifel-Whitney classes)
    - Realization spaces as point inverses of continuous maps

Hyperfield (definition)

Slogan: A hyperfield is a field, but with multivalued addition:

Definition

A hyperfield $(F, \cdot, \boxplus, 1, 0)$ is

- set F
- elements $0, 1 \in F$ with $0 \neq 1$.
- $\cdot : F \times F \to F$
- $\boxplus : F \times F \to \{\text{non-empty subsets of } F\}$

so that

- $F^\times = (F - \{0\}, \cdot, 1)$ is an abelian group.
- $x \cdot 0 = 0 = 0 \cdot x$
- $x \cdot (y \boxplus z) = (x \cdot y) \boxplus (x \cdot z)$
- $(F, \boxplus, 0)$ is an abelian hypergroup
    - $0 \boxplus x = x$
    - $\forall x, \exists! -x,$ so that $0 \in x \boxplus -x$
    - associative, commutative

Topological hyperfield

Definition (AD)

A *topological hyperfield* is a hyperfield $F = (F, \cdot, \boxplus, 1, 0)$ with a topology on F so that

- $F^\times$ is a topological group (i.e. multiplication and inversion are continuous)
- $F^\times$ is open (i.e. $\{0\}$ is closed).

AD wrote *"The reader may think it is odd that we do not require addition to be continuous. We do too!"*

Definition

A *morphism* $f : F \to F'$ of topological hyperfields is

- continuous
- hyperfield homomorphism
  $f(0) = 0, f(1) = 1, f(xy) = f(x)f(y), f(x \boxplus y) \subset f(x) \boxplus f(y)$

Poset topology

Definition

Let P be a poset. $U \subset P$ is an *upper set* if $x \in U, y \geq x \implies y \in U$

Definition

The *poset topology* on a poset P is the topology where the open sets are the upper sets.

The largest vertex map from the order complex is continuous!

$$\|P\| \to P$$

Theorem (McCord 66)

This is a weak homotopy equivalence (isomorphism on homotopy groups)

Viro's dequantization; graph of $z = (x^5 + y^5)^{1/5}$