Sufficient condition for almost-irreducibility

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Motivation

Conjecture

(Recski 1982) The union of graphic matroids is graphic or nonbinary.

Binary: can be represented by a matrix over $GF(2)$
Graphic: $\exists G$ graph on the edge set E, that the independent sets are the circuit free subsets
Union: $M_1(E, I_1) \vee M_2(E, I_2)$ is the matroid on E where the independents are the sets X which can be partitioned to $X = X_1 \cup X_2$, so that $X_1 \in I_1$, $X_2 \in I_2$

First approach I

Fix a graphic matroid and characterize those graphic matroids where the union is graphic.

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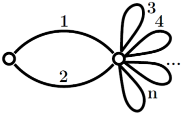
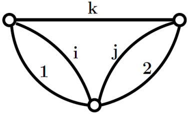
Theorem

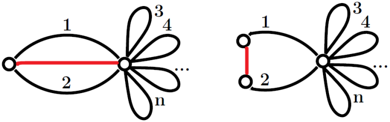
(Recski 1975) If A is the cycle matroid of the left graph, than the union $A \vee M$ is graphic if and only if M does not contain the cycle matroid of the right graph as a minor.





First approach II

(Cs 2012) Forbidden minor characterization of the next two cases: if the fixed matroid has 3 parallel edges or a 3 long circuit.



First approach III

Theorem

(Cs CG13) Suppose that G_1 consists of loops and a single circuit of length n ($n \geq 2$) and $M(G_2)$ is an arbitrary graphic matroid on the same ground set. The union $M_1 \vee M_2$ is graphic if and only if for the reduced pair M_1', M_2' every nonloop circuit C of M_1' contains a cut set in M_2' or $M_2' \setminus C$ is the free matroid.

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Theorem

(Cs 2013) Suppose that G_1 consists of loops and two points joined by n ($n \geq 2$) parallel edges and $M(G_2)$ is an arbitrary graphic matroid on the same ground set. The union $M_1 \vee M_2$ is graphic if and only if for the reduced pair M_1', M_2' every nonloop circuit C of M_1' contains a cut set in M_2' or $M_2' \setminus C$ is the free matroid or the elements of C are not in the same 2-connected component of G'.

In both cases: otherwise the union is not binary.

A sufficient, and a necessary condition (Cs 2015)

Theorem

Assume that M_2 is graphic. Then $M_1 \vee M_2$ is graphic if for every circuit C of length at least two in M_1 either $r_2(E - C) < r_2(E)$ or $r_2(E - C) = |E - C|$.

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Theorem

M_1 and M_2 graphic matroids. If there exists a disjoint pair X_1, X_2 with the following conditions, then $M_1 \vee M_2$ is not binary.

1. $i = 1, 2 : \exists C_i$ *of* M_i *in* X_i *so that* $|C_i| \geq 2$
2. $r_i(X_i) = r_i(X_1 \cup X_2)$ *for* $i \in \{1, 2\}$
3. $i = 1, 2 : \exists a \neq b \in C_1 \cup C_2$ *such that:*
   - $a \in C_i$, $b \in C_{3-i}$ *and a and b are in the same component in both matroids* **OR**
   - $a, b \in C_i$ *and* $\exists X'_{3-i} \subset X_{3-i}$ *so that in* M_{3-i}/X'_{3-i} *a and b are diagonals of* C_{3-i} *connecting distinct pairs of vertices*

New directions

Many different approaches, for example: handling union as homomorphism (also defined almost-graphicity: can be get from a graphic matroid with a homomorphism)

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Theorem

(Tutte 1959) A matroid is binary if and only if it has no $U_{2,4}$ minor. A matroid is graphic if and only if it has no $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$ or $M^*(K_{3,3})$ minor.

Theorem

(Bixby 1977) A binary matroid is graphic if and only if it has no series minor F_7, F_7^*, R_{10}, $M^*(K_5)$, or $M^*(K_{3,3}^i)$, for some $0 \geq i \geq 3$.

Almost-irreducibility

Almost-irreducible: If M is a series minor in the union $M_1 \vee M_2$, then M must be the series extension of a submatroid of M_1 or M_2.

Theorem

(Recski 1981) A graphic matroid is almost-irreducible if and only if it is non-separable and does not have a separating series class.

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Lemma

(1973 Lovász, Recski) Every arc of a θ-graph (or non-separable line) of M is critical (the sum of the ranks is the size).

Lemma

Every θ-graph M is almost-irreducible.

Generalizing Recski's result

The graphicity is needed in Recski's proof for two properties:

1. Every series class can be an arc of a θ-graph
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- (Cs CG18) The first property is true for almost all of Bixby's series minors (F_7, R_{10}, $M^*(K_5)$, $M^*(K_{3,3}^i) : i \in \{0, 1, 2, 3\}$), but is not true for F_7^*.
- (Cs CG18) The second property is true for all of Bixby's series minors (F_7, F_7^*, R_{10}, $M^*(K_5)$, $M^*(K_{3,3}^i) : i \in \{0, 1, 2, 3\}$)

Problem with the proof

- As mentioned F_7^* can not be good (misses property 1)
- I could not reproduce the proof for binary matroids (not necessary graphic)
- I found an error in the original proof (probably not fatal, but not easily repairable)

Cunningham's results

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Cunningham's proof uses Tutte's ideas (1965: for a circuit Y define the $Y-components$ and $Y-bridges$, etc.)
Later Duke gave an easier proof using Mason construction (1988).

Theorem

(Cunningham 1978) in the union $M_1 \vee M_2$ of binary matroids, every element which is not a loop in M_1 and not a loop in M_2 must be separating ($(M_1 \vee M_2) \setminus e$ is not connected).

Generalizing Cunningham's result

Conjecture

A binary matroid is almost-irreducible if and only if it is the series extension of an irreducible matroid.

If we could prove that contracting an element from a series class in the union is somehow equivalent with contracting something in an addend...

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Counterexample: $\{a, e\} \vee \{b, e\}$
Suppose that a and b are serial in the union: if an edge a is a loop in M_2 then a can be contracted in M_1; if a and b are parallel in both M_1 and M_2 then a can be contracted in M_1.

Something is missing in the middle

Theorem

(Cs CG18) Suppose that for every series class S in $M(= M_1 \vee M_2)$ S is critical and $M \setminus S$ is connected. Then S contains a separator of M_i for $i = 1$ or 2.

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Sketch of the proof:

- Indirectly: $\exists P_1, P_2 \subset E \setminus S$, $P_i \in M_i$ but $P_i \notin M_i/S$
- $\forall e_k \in P_1 \cup P_2$ get the series class S_k of e in M
- $\bigcup_k S_k$ is disjoint from S
- any spanning set X of $\bigcup_i S_i$ in M must be critical
- $r(X \cup S) \leq |X \cup S| - 2$ contradicting that S is a series class

Sufficient condition for almost-irreducibility

Corollary

Suppose that M is a series extension of a binary irreducible matroid. If every series class is critical in M then it is almost-irreducible.

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Bixby's minors?

Lemma

If every element of M is an arc of a proper θ-graph, and N is a series extension of M then every series class of N is critical.

Lemma

If N is connected and a series class S of N is an arc of a proper θ-graph then S is non-separating.

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If N is connected and a series class S of N is an arc of a proper θ-graph then S is non-separating.

Corollary

The series extensions of F_7, R_{10}, $M^*(K_5)$ and $M^*(K^i_{3,3}) : i \in \{0, 1, 2, 3\}$ are almost-irreducible.

F_7^* remains a problem?

Lemma

(Cunningham 1978) Let Y be a circuit in $M = M_1 \vee M_2$ and $A, B \subset Y$ cricital sets in M. If $A \cup B \neq Y$ then $A \cap B$ must be critical also.

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Lemma

Every element of F_7^* is critical.

Corollary

The series extensions of F_7^* are almost-irreducible.

Revisiting motivating conjecture

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Another consequence:
The union of regular matroids is regular or nonbinary.

Open questions

Is the series extension of any irreducible matroid is almost-irreducible? (S must be critical if $M \setminus S$ is connected?)

Is the minor irreducibility similar to almost-irreducibility?

What about the non-binary matroids. Is there a meaningful irreducibility for them?