Dyck Paths and Positroids from Unit Interval Orders

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Some history

- Unit interval orders originated in the study of *psychological preferences*, in the work of Norbert Wiener (1914).
- W. E. Armstrong (1939), among others, later explored them in greater detail. For a flavor of this field, Armstrong describes one possible assumption about preferences as follows:

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- Robert Luce (1956) studied unit interval orders to axiomatize a class of utilites in the theory of preferences.
- Unit interval orders show up in many circles: psychology, economics, game theory, mathematics . . .

For today:

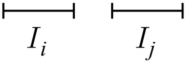
- Get to know unit interval orders (UIO).
- Introduce Matroids (and Positroids) and their connection to UIOs.
- Main results and final thoughts.

Unit Interval Orders

Definition

A poset P is a *unit interval order* if it can be represented by a collection of intervals $[q_i, q_i + 1]$ for $q_i \in \mathbb{R}$ such that for distinct $i, j \in P$,

$i <_P j \quad \longleftrightarrow$



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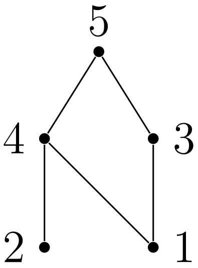
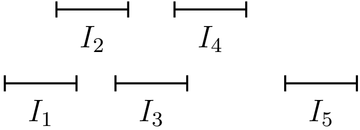
Example



$\longmapsto$

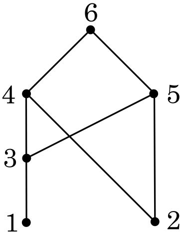


Antiadjacency Matrices of Labeled Posets

Definition (Antiadjacency Matrix)

If P is a poset on n elements, then the *antiadjacency matrix* of P is the $n \times n$ binary matrix $A = (a_{i,j})$ with $a_{i,j} = 0$ if and only if $i <_P j$.

Example



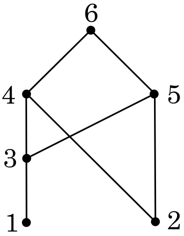
$$\longmapsto \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Antiadjacency Matrices of Labeled Posets

Proposition (Skandera–Reed, 2003)

A unit interval order has an altitude preserving labeling if and only if its antiadjacency matrix has the 0's and 1's separated by a Dyck path supported on the main diagonal.

Example



$$\longmapsto \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Dyck Matrices

Definition (Dyck Matrix)

A binary square matrix is called a *Dyck matrix* if the 0's and 1's are separated by a Dyck path supported on the main diagonal. Denote the set of Dyck matrices of size n as $\mathcal{D}_n$.

Example

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

A 6×6 Dyck matrix

Propositions

- *(Stanley, 1999)* *Every Dyck matrix is totally positive (all minors ≥ 0).*
- *(Freund–Wine, 1957; Dean–Keller, 1968)*

  $|\mathcal{D}_n| = \frac{1}{n+1}\binom{2n}{n}$, *the n-th Catalan number.*

Matroids and Positroids

Definition (Matroid)

Let E be a finite set, and let $\mathcal{B}$ be a nonempty collection of subsets, called *bases*, of E. The pair $\mathcal{M} = (E, \mathcal{B})$ is a *matroid* if they satisfy the following axioms:

(B1) $\mathcal{B} \neq \emptyset$.

(B2) For all $A, B \in \mathcal{B}$ and $a \in A \setminus B$, there exists $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$.

Definition (Positroid)

A representable matroid $\mathcal{M}$ on $[n]$ of rank d, associated with matrix A, is a *positroid* when A is totally nonnegative (all maximal minors are nonnegative).

Positroids

Example

Recall the 3×6 real matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

- All maximal minors are nonnegative, thus A is totally nonnegative.
- The matroid $\mathcal{M} = ([6], \mathcal{B})$ represented by A is a positroid.

Induced Positroids

Lemma (Postnikov 2007)

For an $n \times n$ real matrix $A = (a_{i,j})$, consider the $n \times 2n$ matrix $B = \phi(A)$, where

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,n} \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \overset{\phi}{\mapsto} \begin{pmatrix} 1 & \dots & 0 & 0 & \pm a_{n,1} & \dots & \pm a_{n,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & -a_{2,1} & \dots & -a_{2,n} \\ 0 & \dots & 0 & 1 & a_{1,1} & \dots & a_{1,n} \end{pmatrix}.$$

Then minors of A correspond to maximal minors of $\phi(A)$.

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Then minors of A correspond to maximal minors of $\phi(A)$.

This allows us to associate each Dyck matrix to a positroid.

Induced Positroids

Example

By Lemma, the Dyck matrix A induces the positroid represented by $\phi(A)$:

$$\underset{A}{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \overset{\phi}{\mapsto} \underset{\phi(A)}{\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}}$$

$$\mathcal{B} = \begin{Bmatrix} 123, & 124, & 125, & 134, & 135, & 234, \\ 235, & 236, & 246, & 256, & 346, & 356 \end{Bmatrix}.$$

Unit Interval Positroid

Definition (Unit Interval Positroid)

For an $n \times n$ Dyck matrix D, the positroid on $[2n]$ represented by $\phi(D)$ is called a *unit interval positroid.*

Theorem (C.–Gotti 2017)

Every n-element unit interval order induces a unit interval positroid on the ground set $[2n]$. This implies there are $\frac{1}{n+1}\binom{2n}{n}$ unit interval positroids on $[2n]$.

Positroid Representations

Theorem (Postnikov 2007)

Positroids can be represented by several classes of combinatorial objects, with bijections existing among them:

- *Grassmann necklaces*
- *Decorated permutations*
- *Le-diagrams*
- *Plabic graphs*

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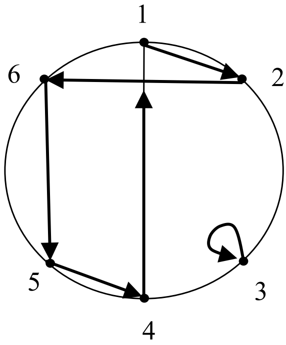
Decorated Permutation

Definition (Decorated Permutation)

A *decorated permutation* is an element $\pi \in S_n$ whose fixed points j are marked either "clockwise"(denoted by $\pi(j) = \underline{j}$) or "counterclockwise" (denoted by $\pi(j) = \overline{j}$).

Example

$$26\bar{3}145 = (12654)(\bar{3})$$



Decorated Permutations of Unit Interval Positroids

Theorem (C.–Gotti 2017)

- *The decorated permutation representation of a unit interval positroid encodes a Dyck path as a full-length cycle.*
- *The decorated permutation can be recovered by a special labeling of the Dyck path on the associated antiadjacency matrix.*

Decorated Permutation from Dyck Matrix

Example

The decorated permutation π associated to the positroid represented by the 5×5 Dyck matrix D

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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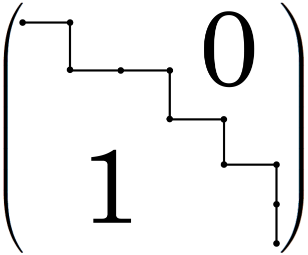

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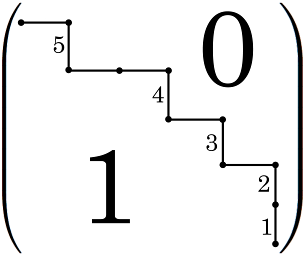

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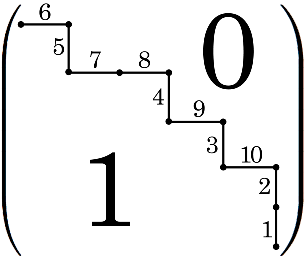

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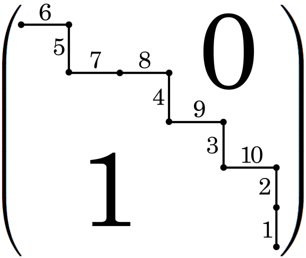


can be read bottom to top from the Dyck path of D, obtaining

$$\pi = (1, 2, 10, 3, 9, 4, 8, 7, 5, 6).$$

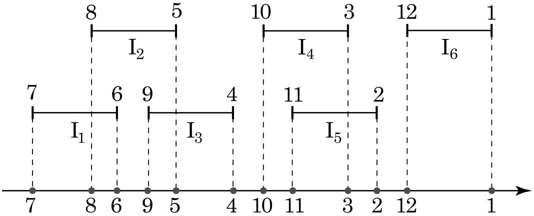
Decorated Permutation from Interval Representation

Theorem (C–Gotti)

Given a unit interval representation, label the left and right endpoints of $[q_i, q_i + 1]$ by $n + i$ and $n + 1 - i$, respectively. Then the decorated permutation representation of the associated unit interval positroid is the cycle given by reading the labels from right to left.

Example

The decorated permutation $(1, 12, 2, 3, 11, 10, 4, 5, 9, 6, 8, 7)$ is obtained by reading the labels from right to left.



Merci!

Reference: A. Chavez and F. Gotti, *Dyck Paths and Positroids from Unit Interval Orders*, J. Comb. Theory A Ser. A, **154** (2018), 507–532.

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