Polyhedral representations of oriented matroids

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The Bergman fan of a matroid

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Definition (Sturmfels, Ardila, Klivans)

The **Bergman fan** $\mathcal{B}(M)$ is the set of all $\omega \in \mathbf{R}^E/\mathbf{R} \cdot \mathbf{1}$ such that ω encodes a chain of flats of M. For example, if $E = \{1, 2, 3, 4, 5\}$,

Flag of flats:

	1	2	3	4	5
F_1 :	0	0	0	0	$*$
F_2 :	0	0	0	$*$	$*$
F_3 :	0	$*$	$*$	$*$	$*$

$$\Longrightarrow \quad \omega = (0, 1, 1, 2, 3) \in \mathcal{B}(M)$$

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$\mathcal{B}(M)$ is a union of cones in the inner normal fan of the polytope:

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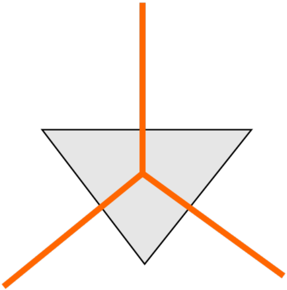

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The **real Bergman fan** Σ_M of an oriented matroid is the set of all $\omega \in \mathbf{R}^E$ such that ω is a "signed" flag of conformal covectors:

Flag of covectors:

	1	2	3	4	5
X_3 :	$+$	$-$	$+$	$+$	0
X_2 :	$+$	$-$	$+$	0	0
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$$\implies \quad \omega = (3, -2, 2, 1, 0) \in \Sigma_M$$

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Σ_M is a union of cones in the outer normal fan of the polytope

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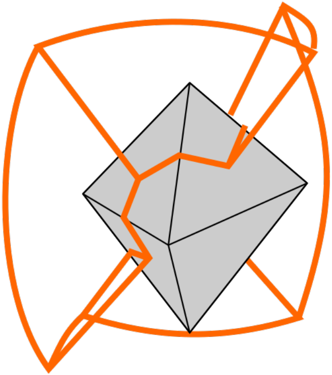

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- Let's extend the chirotope χ of M by allowing for signs in the arguments:

$$\chi(s_1 e_1, s_2 e_2, \dots, s_r e_r) := s_1 s_2 \cdots s_r \chi(e_1, e_2, \dots, e_r)$$

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- Remarkable fact: Let $\mathcal{F} = \{\emptyset = X_0 \lneq X_1 \lneq \cdots \lneq X_r\}$ be a maximal flag of conformal covectors. For $i = 1, 2, \ldots, r$, let

$$b_i \in X_i \setminus X_{i-1}$$
$$s_i = X_i(b_i)$$

Then $\chi(s_1 b_1, \ldots, s_r b_r)$ depends only $\mathcal{F}$!

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- For example, if $r = 3$ and $E = \{1, 2, 3, 4, 5, 6\}$ and we have the following flag $\mathcal{F}$ of conformal covectors:

$$\mathcal{F} = \begin{array}{lcccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ X_3: & + & + & - & + & - & + \\ X_2: & + & + & - & + & 0 & 0 \\ X_1: & + & + & 0 & 0 & 0 & 0 \end{array}$$

Then, writing bars for the signs, we have, for example,

$$\chi(1, \overline{3}, \overline{5}) = \chi(2, 4, 6) = \chi(1, \overline{3}, 6) = \chi(2, \overline{3}, \overline{5}).$$

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- Hence χ defines a map

$$\chi : \{\text{Maximal flags of conformal covectors}\} \to \{-1, 1\}$$

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- For such $\mathcal{F} = \{\emptyset = X_0 \lneq X_1 \lneq \cdots \lneq X_r\}$, define the map

$$\begin{aligned} \sigma_{\mathcal{F}} : \Delta_{[r]} &\to \Delta_M \\ \mathbf{e}_i &\mapsto \mathbf{e}_{X_i} \end{aligned}$$

where $\mathbf{e}_X \in \{-1, 0, 1\}^E$ represents the sign vector X.

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Theorem

The element

$$\sum_{\mathcal{F}} \chi(\mathcal{F})[\sigma_{\mathcal{F}}] \in H_{r-1}(\Delta_M; \mathbf{Z})$$

is a generator for the top homology group of Δ_M. Here the sum is over all maximal flags of conformal covectors of M.

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- Let $M + f$ the result of adjoining a coloop f to M. Consider the composite map

$$S^{r-1} \longrightarrow \Delta_{\tilde{M}} \longrightarrow \mathrm{bd}(\mathcal{Z}_{M+f}) \longrightarrow S^{r-1}$$

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  - The third map is the radial projection map.

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- This map is furthermore orientation preserving, has degree 1, and is injective on each simplex of $\Delta_{\tilde{M}}$. Therefore this map is a homeomorphism.

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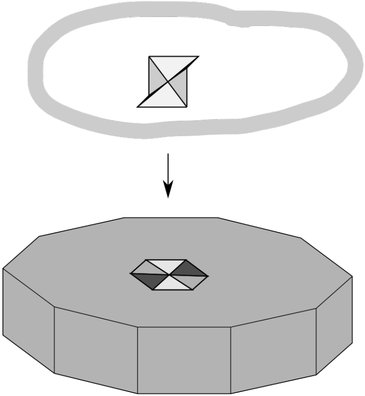
- From here one can verify Bohne-Dress by noting that the corresponding piecewise linear map $\Delta_{\tilde{M}} \to \mathrm{bd}(\mathcal{Z}_M)$ sends tiles of a zonotopal tiling onto the two copies of $\mathcal{Z}_M$ realized as facets of $\mathcal{Z}_{M+f}$:



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- Suppose $\pi : \mathbf{R}^E \to \mathbf{R}^r$ is a linear map which restricts to a homeomorphism $\pi : \Sigma_M \to \mathbf{R}^r$. Define the *pseudozonotope* $\mathcal{Z}_{\pi,M}$ to be the image $\pi(\Sigma_M \cap [-1,1]^E)$.

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Theorem (McMullen's formula for pseudozonotopes)

The volume of $\mathcal{Z}_{\pi,M}$ *is given by*

$$\mathrm{vol}(\mathcal{Z}_{\pi,M}) = 2^r \left| \sum_{B \in \binom{E}{r}} \chi(b_1, \ldots, b_r) \det(\pi(\mathbf{e}_{b_1}), \ldots, \pi(\mathbf{e}_{b_r})) \right|.$$

where the sum is over all r*-element subsets* $B = \{b_1, \ldots, b_r\}$ *of* E*.*

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- Note that some of these terms can be negative!

End.

Thanks for coming!