Representable Orientable Matroids that are not Real-Representable

Rutger Campbell

"The fundamental question of completely characterizing [real-representable matroids] is left unsolved."

Whitney, 1935

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Real-representability is not polynomially-certifiable with rank evaluations.

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Theorem (Lee, Scobee):
If an orientable matroid is ternary, then it is real-representable.

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If an orientable matroid is representable over some field, then it is real-representable.

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Theorem:

For each odd prime power $q > 3$, there exists an orientable matroid that is $GF(q)$-representable but not real-representable.

Facts about Fields

$GF(q)^* \cong \mathbb{Z}_{q-1}$

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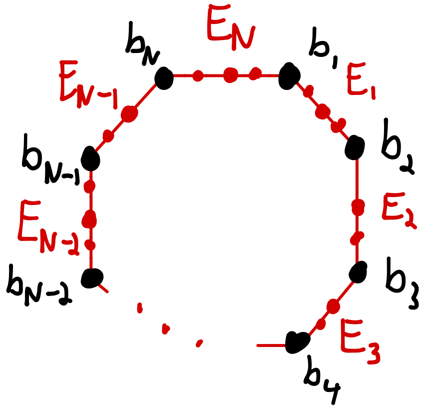
$$\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & -\alpha_N \\ -\alpha_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\alpha_2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -\alpha_{N-1} & 1 \end{vmatrix} = 0 \iff \alpha_1 \alpha_2 \cdots \alpha_N = 1$$

Swirl-like Matroids

M:



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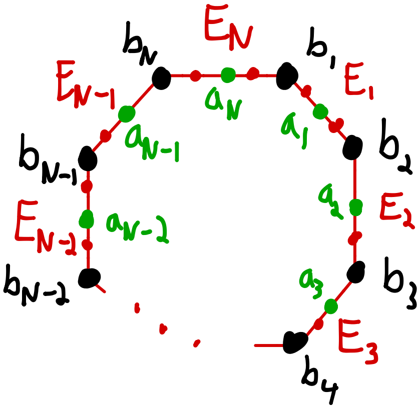

$T = \{a_1, a_2, \dots, a_N\}$

A transversal of M is a set $T = \{a_1, a_2, \dots, a_N\}$ where $a_i \in E_i$; this is either a circuit hyperplane or a basis.

Let $\mathcal{C}(M)$ be the dependent transversals of M.

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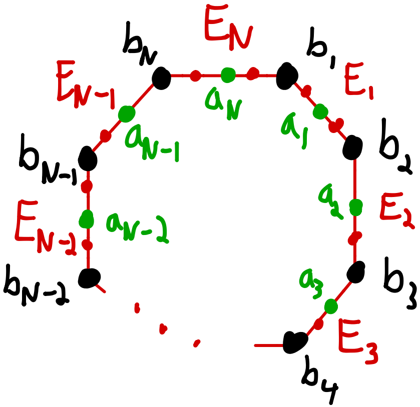

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Remark:
A swirl-like matroid M is uniquely determined by its vertices, edges and $\mathcal{C}(M)$.

Representations of Swirl-like Matroids

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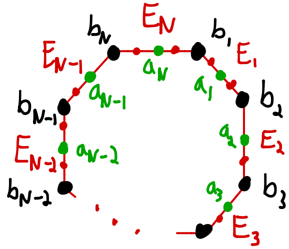

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For a representation $f: E \to \mathbb{F}^N$, of a swirl-like matroid $M=(E,r)$, we may assume that: $f(b_i)=e_i$ for $b_i \in B$, and

$f(a) = e_i - \alpha_a e_{i+1}$ for some $\alpha_a \in \mathbb{F}$ for $a \in E_i$

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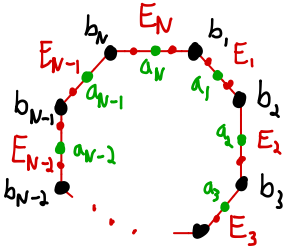

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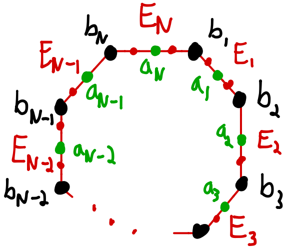

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$$T=\{a_1, a_2, \dots, a_N\} \in \mathcal{C}(M) \iff \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & -\alpha_N \\ -\alpha_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\alpha_2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -\alpha_{N-1} & 1 \end{vmatrix} = 0 \iff \alpha_1\alpha_2\cdots\alpha_N = 1$$

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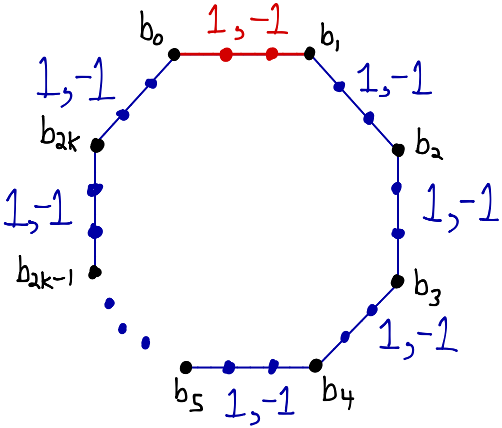

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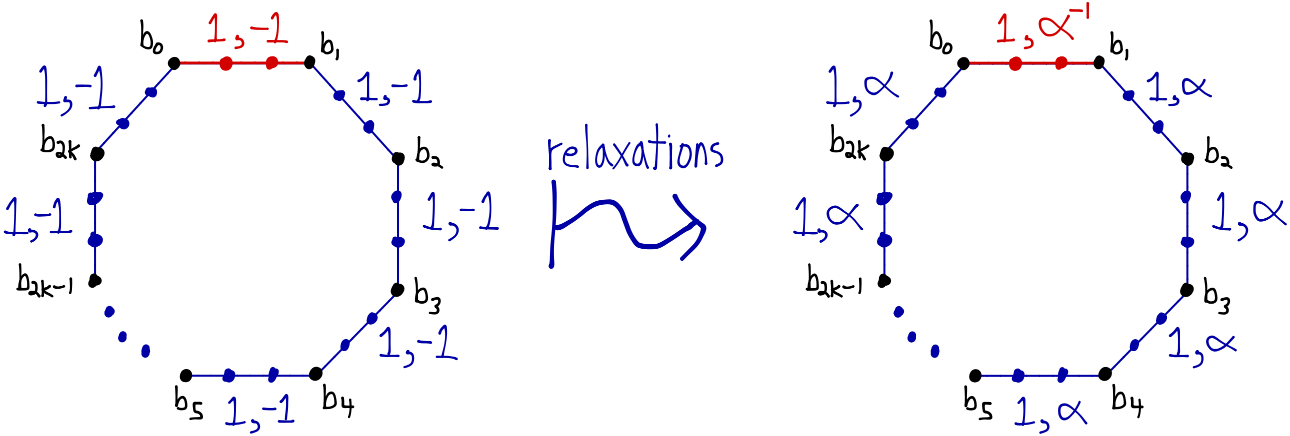
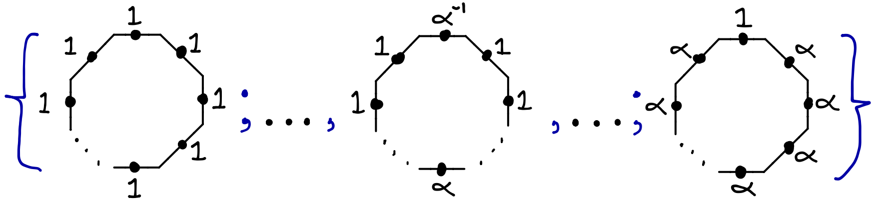


$\mathcal{C}(M) =$



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For each prime power q with $q-1$ composite, there exists an orientable matroid that is $GF(q)$-representable but not real-representable.

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Problem:

If an orientable matroid is $GF(2^n)$-representable for some 2^n with 2^n-1 prime, then must it be real-representable?

Theorem (C, Geelen):
There exists an orientable matroid that is complex-representable but not real-representable.

Worse News:

Corollary (analog of result by Mayhew, Newman, Whittle):
Each real-representable matroid is a minor of an excluded-minor for real-representability that is complex-representable and orientable.

Corollary (analog of result by Mayhew, Newman, Whittle):
There is no sentence in the monadic second-order language MS_0 that characterizes real-representability for complex-representable orientable matroids.

Corollary (analog of result by Seymour):
Real-representability is not polynomially-certifiable with rank evaluations for complex-representable orientable matroids.

Whittle's Conjecture (revised):

If an orientable matroid is representable over some field, then it is complex-representable.

Thank You!