Delta-matroids as subsystems of sequences of Higgs lifts: a way to think about delta-matroids from the perspective of matroids

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Joint work with

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These slides are available at

http://blogs.gwu.edu/jbonin/

A set system (or hypergraph) is a pair $S = (E, \mathcal{F})$ where E is a (finite) set and $\mathcal{F}$ is a set of subsets of E (the feasible sets).

E.g., $(E, \mathcal{B})$, where $\mathcal{B}$ is the set of bases of a matroid M on E.

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One of many basis-exchange properties of matroids:

> *for all $B_1, B_2 \in \mathcal{B}$ and $u \in B_1 - B_2$, there is a $v \in B_2 - B_1$ with $B_1 \triangle \{u, v\} \in \mathcal{B}$* (symmetric difference).

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For delta-matroids, replace set differences by symmetric differences.

A delta-matroid is a proper set system $D = (E, \mathcal{F})$ that satisfies delta-matroid symmetric exchange:

> *for all $X, Y \in \mathcal{F}$ and $u \in X \triangle Y$, there is a $v \in X \triangle Y$ with $X \triangle \{u, v\} \in \mathcal{F}$.* *(Bouchet, 1987)*

A delta-matroid is a proper set system $D = (E, \mathcal{F})$ such that

for all $X, Y \in \mathcal{F}$ and $u \in X \triangle Y$, there is a $v \in X \triangle Y$ with $X \triangle \{u, v\} \in \mathcal{F}$.

When $u \in X - Y$,

- if $v = u$, then $X \triangle \{u, v\} = X - u$,
- if $v \in X - (Y \cup u)$, then $X \triangle \{u, v\} = X - \{u, v\}$,
- if $v \in Y - X$, then $X \triangle \{u, v\} = (X - u) \cup v$.

When $u \in Y - X$,

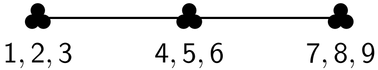
- if $v = u$, then $X \triangle \{u, v\} = X \cup u$,
- if $v \in Y - (X \cup u)$, then $X \triangle \{u, v\} = X \cup \{u, v\}$,
- if $v \in X - Y$, then $X \triangle \{u, v\} = (X - v) \cup u$.

For matroids Q and L on E, Q is a quotient of L, or L is a lift of Q, if $L = M \backslash A$ and $Q = M/A$ for some matroid M and $A \subseteq E(M)$.

E.g., extend the uniform matroid $U_{5,9}$ on $\{1, 2, \ldots, 9\}$ by three elements using the modular cuts generated by

- $\{1, 2, 3\}$ and $\{4, 5, 6\}$,
- $\{1, 2, 3\}$ and $\{7, 8, 9\}$,
- $\{4, 5, 6\}$ and $\{7, 8, 9\}$,

and then contract the added elements to get the quotient

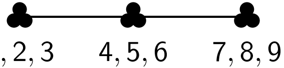


Background for our central example: Higgs lifts

For integers i with $0 \leq i \leq r(L) - r(Q)$, the function r_i given by

$$r_i(X) = \min\{r_Q(X) + i,\ r_L(X)\},$$

for $X \subseteq E$, is the rank function of a matroid $H^i_{Q,L}$ on E, called the i-th Higgs lift of Q toward L.



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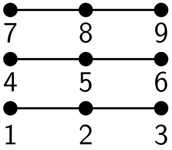

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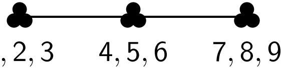

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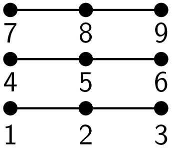

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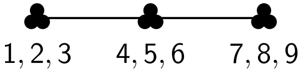

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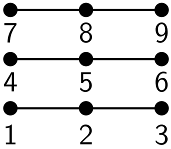

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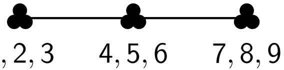
$H^i_{Q,L}$ is the freest quotient of L of rank $r(Q) + i$ having Q as a quotient.

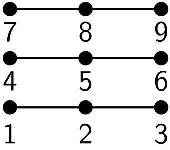
Its bases are the sets of size $r(Q) + i$ that span (contain a basis of) Q and are independent in (contained in a basis of) L.

An example of a delta-matroid

For all $X, Y \in \mathcal{F}$ and $u \in X \triangle Y$, there is a $v \in X \triangle Y$ with $X \triangle \{u, v\} \in \mathcal{F}$.



Q



$H^1_{Q,L}$

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Consider $(\{1, 2, \ldots, 9\}, \mathcal{F})$ where $\mathcal{F} = \mathcal{B}(Q) \cup \mathcal{B}(H^1_{Q,L}) \cup \mathcal{B}(L)$.

For $X = \{1, 4\}$, $Y = \{2, 3, 7, 8, 9\}$, and $u = 2$, we must take either $v = 1$, giving $\{2, 4\}$, or $v = u$, giving $\{1, 2, 4\}$.

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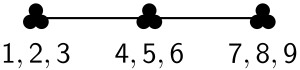

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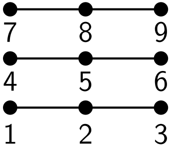

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For $X = \{1, 4\}$, $Y = \{1, 2, 3, 4, 5\}$, and $u = 2$, we must take $v = u$, giving $\{1, 2, 4\}$.

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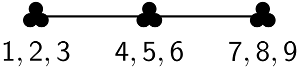

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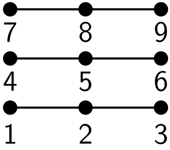

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For $X = \{1, 2, 4, 5, 6\}$, $Y = \{1, 4\}$, and $u = 2$, we must take either $v = 5$, giving $\{1, 4, 6\}$, or $v = 6$, giving $\{1, 4, 5\}$.

Theorem (Bonin, Chun, Noble, 2017)

Fix a matroid L on E and a quotient Q of L. Set $k = r(L) - r(Q)$. Let K be a subset of $\{0, 1, 2, \ldots, k\}$ for which $\{0, 1, 2, \ldots, k\} - K$ contains no two consecutive integers. Then $(E, \mathcal{F})$, where

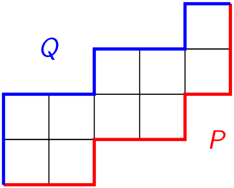
$$\mathcal{F} = \bigcup_{i \in K} \mathcal{B}(H^i_{Q,L}),$$

is a delta-matroid.

When $K = \{0, 1, 2, \ldots, k\}$, we get the full Higgs lift delta-matroid of the pair (Q, L).

Tardos studied that case, calling them generalized matroids (1985); Dupont, Fink, and Moci call them saturated delta-matroids (preprint).

Fix two lattice paths P and Q from $(0, 0)$ to a point (m, r), where P never rises above Q. Thus, P and Q bound a region $\mathcal{R}$ in $\mathbb{R}^2$.



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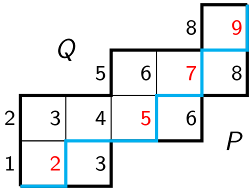

$P' = ENEENENEN$

$b(P') = \{2,5,7,9\}$

Let $\mathcal{P}$ be the set of lattice paths from $(0,0)$ to (m,r) that stay in $\mathcal{R}$. View paths as words in the alphabet $\{E,N\}$ (east, north).

For $P' \in \mathcal{P}$, let $b(P')$ be the set of positions in P' where N occurs.

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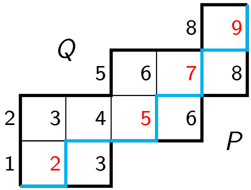

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For $P' \in \mathcal{P}$, let $b(P')$ be the set of positions in P' where N occurs.

Then $\{b(P') : P' \in \mathcal{P}\}$ is the set of bases of a transversal matroid.

Such matroids are lattice path matroids.

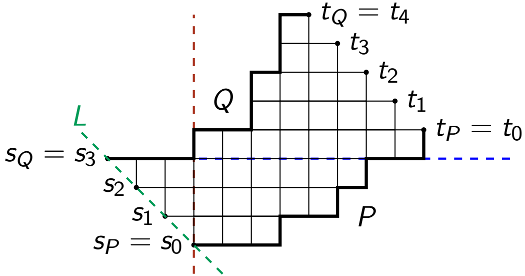
(Bonin, de Mier, Noy, 2003; orientations, Lawrence, 1984)

Some properties of lattice path matroids

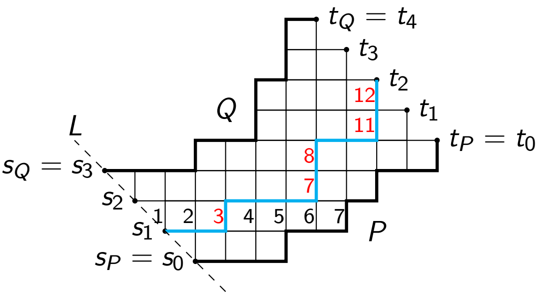
- Lattice path matroids form a minor-closed, dual-closed class, that is not well-quasi-ordered by minors.
- We know the excluded minors: there are five infinite families and four sporadic excluded minors.
- The Tutte polynomial of a lattice path matroid can be computed in polynomial time.
  This uses an interpretation of basis activities that has intrinsic significance for lattice paths.
- Other aspects that have been studied include: matroid base polytopes, h-vectors, toric ideals, Bergman complexes, the Rayleigh property, negative correlation, fast mixing, connections with decompositions of Grassmannians, the Merino-Welsh conjecture holds for these matroids, . . .

Fix lattice paths P, from s_P to t_P, and Q, from s_Q to t_Q, where

- each has n steps,
- the line L through s_P and s_Q has slope -1,
- t_Q is on or to the right of the vertical line through s_P,
- t_P is on or above the horizontal line through s_Q,
- and P never rises above Q.



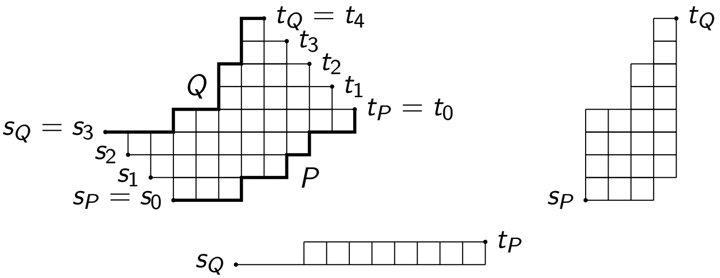
Paths P and Q, the line L, and the line through t_P and t_Q bound a region $\mathcal{R}$ in $\mathbb{R}^2$.



Label each north step in $\mathcal{R}$ by its distance from L, so the set of labels is $E = \{1, 2, \ldots, n\}$.

Let $\mathcal{P}$ be the set of lattice paths from an s_i to a t_j that stay in $\mathcal{R}$.

For a path $P' \in \mathcal{P}$, let $b(P')$ be the set of labels on its north steps.



The set $\{b(P') : P' \in \mathcal{P}$ from s_Q to $t_P\}$ is the set of bases of a lattice path matroid, $M(\mathcal{R}_{\min})$, on E.

Likewise, $\{b(P') : P' \in \mathcal{P}$ from s_P to $t_Q\}$ is the set of bases of a lattice path matroid, $M(\mathcal{R}_{\max})$, on E.

Theorem (Bonin, Chun, Noble, 2017)

1. *$M(\mathcal{R}_{\min})$ is a quotient of $M(\mathcal{R}_{\max})$, and*
2. *the map $P' \mapsto b(P')$ is a surjection from $\mathcal{P}$ onto the set of feasible sets of the full Higgs lift delta-matroid of the pair $(M(\mathcal{R}_{\min}), M(\mathcal{R}_{\max}))$.*

The resulting class of lattice path delta-matroids is minor-closed and dual-closed (to be defined soon).

Corollary

Let $j = r(M(\mathcal{R}_{\min}))$ and $k = r(M(\mathcal{R}_{\max}))$. Fix a subset K of $\{j, j+1, \dots, k\}$ for which $\{j, j+1, \dots, k\} - K$ contains no two consecutive integers. Then $\{b(P) : P \in \mathcal{P} \text{ and } |b(P)| \in K\}$ is the set of feasible sets of a delta-matroid.

Theorem (Bouchet)

For any delta-matroid D,

- *the maximal-sized feasible sets are the bases of a matroid, denoted D_{max},*
- *the minimal-sized feasible sets are the bases of a matroid, denoted D_{min},*
- *D_{min} is a quotient of D_{max}, and*
- *each feasible set contains a basis of D_{min} and is contained in a basis of D_{max}.*

Corollary

Each feasible set in a delta-matroid D is feasible in the full Higgs lift delta-matroid of the pair $(D_{\text{min}}, D_{\text{max}})$.

Next goal: an excluded-minor characterization of delta-matroids within the broader structure of set systems.

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An element in a proper set system $S = (E, \mathcal{F})$ is a loop if it is in no sets in $\mathcal{F}$; it is a coloop if it is in all sets in $\mathcal{F}$.

When e is not a loop, the contraction of e, written S/e, is

$$S/e = (E - e, \{F - e : e \in F \in \mathcal{F}\}).$$

When e is not a coloop, the deletion of e, written $S\backslash e$, is

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For loops or coloops, we set $S/e = S\backslash e$ (one is already defined).

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Sequences of deletions and contractions yields the minors of S.

The order of operations can matter. Minors are proper set systems.

For $A \subseteq E$, the twist of S on A, denoted $S * A$, is given by

$$S * A = (E, \{F \triangle A : F \in \mathcal{F}\}).$$

For $U_{2,4}$ on $\{1, 2, 3, 4\}$,

bases of $U_{2,4}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
$U_{2,4} * \{1,2\}$	$\emptyset$	$\{2,3\}$	$\{2,4\}$	$\{1,3\}$	$\{1,4\}$	$\{1,2,3,4\}$

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Minors and twists of delta-matroids are delta-matroids.

Note that $S/e = (S * e)\backslash e$ and $(S * A) * A = S$.

The dual S^* of S is $S * E$.

The excluded minors for delta-matroids, within set systems

Let $S_i = (\{1, 2, \ldots, i\}, \{\emptyset, \{1, 2, \ldots, i\}\})$. Let $\mathcal{S}$ be the set of all twists of $S_3, S_4, \ldots$.
Let T_1, T_2, T_3, T_4, on $\{a, b, c\}$, and T_5, T_6, T_7, T_8, on $\{a, b, c, d\}$, be given by:

T_1	$\emptyset$		$\{a, b\}$	$\{a, b, c\}$
T_2	$\emptyset$		$\{a, b\}$ $\{a, c\}$	$\{a, b, c\}$
T_3	$\emptyset$	$\{a\}$	$\{a, b\}$	$\{a, b, c\}$
T_4	$\emptyset$	$\{a\}$	$\{a, b\}$ $\{a, c\}$	$\{a, b, c\}$

T_5	$\emptyset$		$\{a, b\}$	$\{a, b, c, d\}$
T_6	$\emptyset$		$\{a, b\}$ $\{a, c\}$	$\{a, b, c, d\}$
T_7	$\emptyset$		$\{a, b\}$ $\{a, c\}$ $\{a, d\}$	$\{a, b, c, d\}$
T_8	$\emptyset$	$\{a\}$	$\{a, b\}$ $\{a, c\}$ $\{a, d\}$	$\{a, b, c, d\}$

Let $\mathcal{T}$ be the set of all 51 twists of $T_1, T_2, \ldots, T_8$.

No T_i is a delta-matroid since a is in all feasible sets except $\emptyset$.

Theorem (Bonin, Chun, Noble, 2017)

A proper set system is a delta-matroid if and only if it has no minor in $\mathcal{S} \cup \mathcal{T}$.

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A matroid is a delta-matroid in which all feasible sets have the same size, so:

Corollary

A proper set system $S = (E, \mathcal{F})$ is a matroid if and only if all sets in $\mathcal{F}$ have the same size, and S has no minor in

$$\{T_5 * \{a, c\},\ T_6 * \{a, d\}\} \cup \{S_{2k} * \{1, 2, \ldots, k\} : k \geq 2\}.$$

$T_5 * \{a, c\}$: $\{a, c\}$, $\{b, c\}$, $\{b, d\}$
$T_6 * \{a, d\}$: $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$

Given a set system $(E, \mathcal{F})$, set $f_{\min} = \min\{|X| : X \in \mathcal{F}\}$, $f_{\max} = \max\{|X| : X \in \mathcal{F}\}$, and $\mathcal{F}_i = \{X \in \mathcal{F} : |X| = i\}$ for $f_{\min} \le i \le f_{\max}$.

Apart from $D_{\min}$ and $D_{\max}$, $(E, \mathcal{F}_i)$ does not have to be a matroid.

A set system or delta-matroid $(E, \mathcal{F})$ is sparse paving if each proper set system $(E, \mathcal{F}_i)$ with $f_{\min} \le i \le f_{\max}$ is a sparse paving matroid.

Corollary

A sparse paving set system is a sparse paving delta-matroid if and only if it has no minor in

$$\{S_i : i \ge 3\} \cup \{T_2,\ T_2^*,\ T_3 * b,\ T_4 * b,\ T_4 * \{a, c\}\}.$$

Paving set systems and delta-matroids are defined similarly.

Corollary

A paving set system is a paving delta-matroid if and only if it has none of the following minors:

S_i *for* $i \geq 3$,

$T_1 * \{b, c\}, \; T_1^*$,

$T_2, \; T_2 * \{a, b\}, \; T_2 * \{b, c\}, \; T_2^*$,

$T_3 * b, \; T_3 * \{b, c\}$,

$T_4, \; T_4 * b, \; T_4 * \{a, c\}, \; T_4 * \{b, c\}$,

$T_6 * \{b, c, d\}$,

$T_7, \; T_7 * b, \; T_7 * \{b, c, d\}$,

$T_8, \; T_8 * \{b, c, d\}$.

A set system or delta-matroid $(E, \mathcal{F})$ is a quotient stack if for each i with $f_{\min} \leq i < f_{\max}$, the set system $(E, \mathcal{F}_i)$ is a matroid and is a quotient of $(E, \mathcal{F}_{i+1})$.

Corollary

A quotient stack set system is a quotient stack delta-matroid if and only if it does not have a minor in either $\{S_i : i \geq 3\}$ or

$$\{T_1,\ T_1^*,\ T_2,\ T_2^*,\ T_3,\ T_4,\ T_4^*,\ T_5,\ T_6,\ T_7,\ T_7^*,\ T_8,\ T_8^*\}.$$

Theorem (Bonin, Chun, Noble, 2017)

A delta-matroid is a full Higgs lift delta-matroid if and only if it has neither of the following as minors:

$$(\{a, b\}, \{\emptyset, a, \{a, b\}\}), \qquad (\{a, b\}, \{\emptyset, \{a, b\}\}).$$

More excluded-minor results

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What are the excluded minors for lattice path delta-matroids?

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Thank you for listening.