From a Combinatorial Input to a Polyhedral Realization

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Michael Cuntz, Hannover, Germany
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CIRM, Marseille–Luminy, 2018

Hurwitz's Regular Map of Genus 7

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[ 1, 2, 3],[ 1, 3, 4],[ 1, 4, 5],[ 1, 5, 6],[ 1, 6, 7],[ 1, 7, 8],[ 1, 8, 2],
[ 2, 8, 9],[ 2,10, 3],[ 3,11, 4],[12, 5, 4],[13, 6, 5],[14, 7, 6],[ 8, 7,15],
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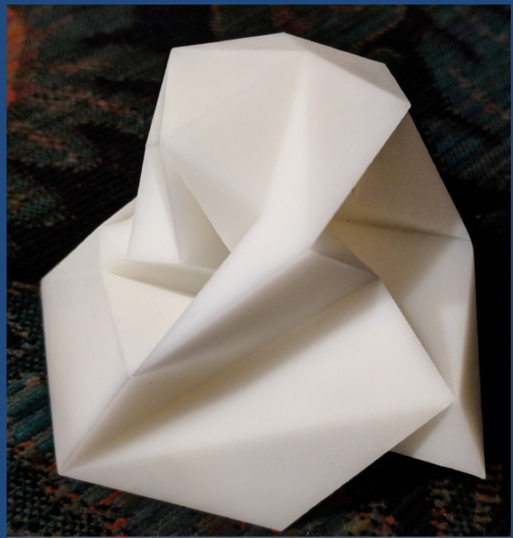


Combinatorial Input → Topological Realization → Polyhedral Realization

From a Combinatorial Input to a Polyhedral Realization

3 – Spheres

Combinatorial spheres

Topological spheres

Geometric polyhedral spheres

From a Combinatorial Input to a Polyhedral Realization

Point Line Configurations

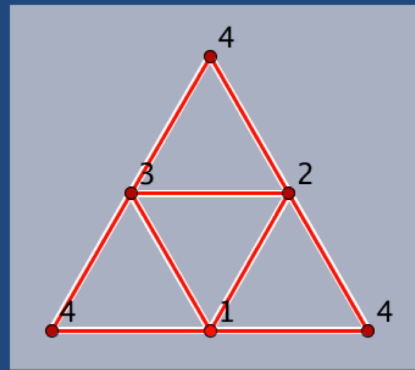
Combinatorial

Topological pseudoline configurations, rank 3 oriented matroids

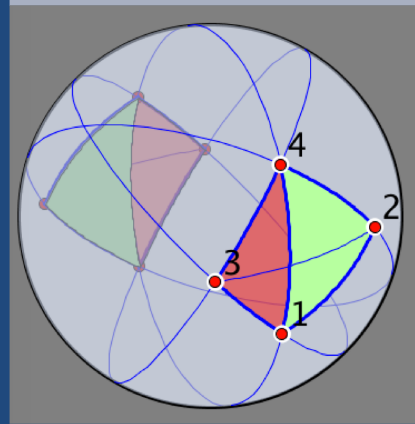
Geometric

From a Combinatorial Input to a Polyhedral Realization

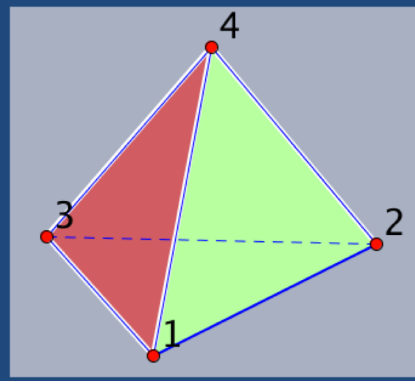
2-manifolds



Combinatorial 2-manifolds



Topological visualizations



Geometric realizations

with flat cells and

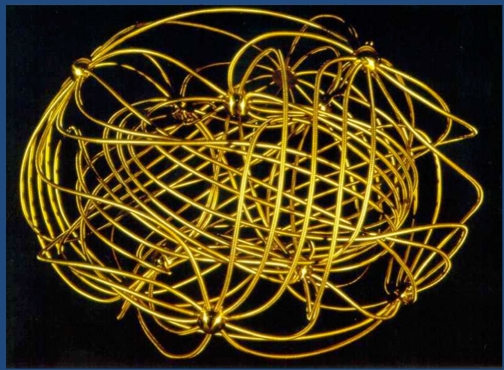
without self-intersections

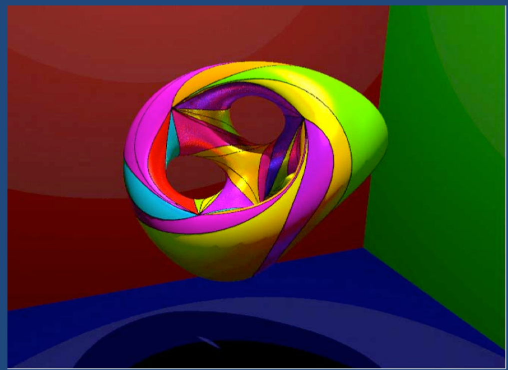
Example: 2-dimensional manifold,

genus 6, 44 triangles, 12 points

A first example of a non-realizable 2-manifold.

A. Guedes de Oliveira, J.B., 2000



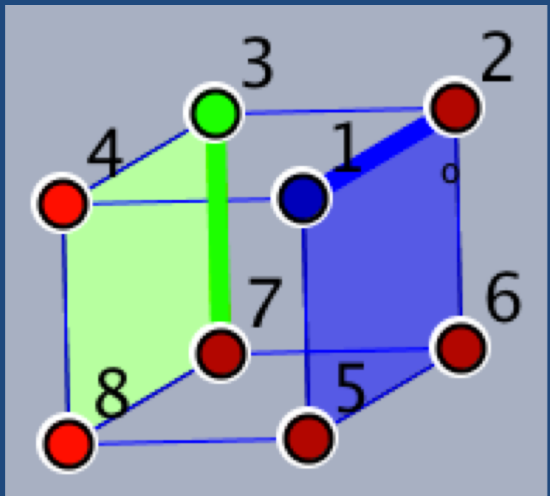


topological visualizations , J.B. & Carlo Sequin, Berkeley

All other 58 examples of this type are not realizable,

L. Schewe, 2006

A **regular map** is a decomposition of a two dimensional manifold into topological disks, such that every flag can be transformed into any other flag by a symmetry of the decomposition.



Flag 1: ({1}, {1,2}, {1,2,6,5})

Flag 2: ({3}, {3,7}, {3,7,8,4})

Some Historical Remarks

F. Klein,
Über die Transformation siebenter Ordnung der elliptischen Funktionen.
Math. Ann. 14 (1879), 428-471.

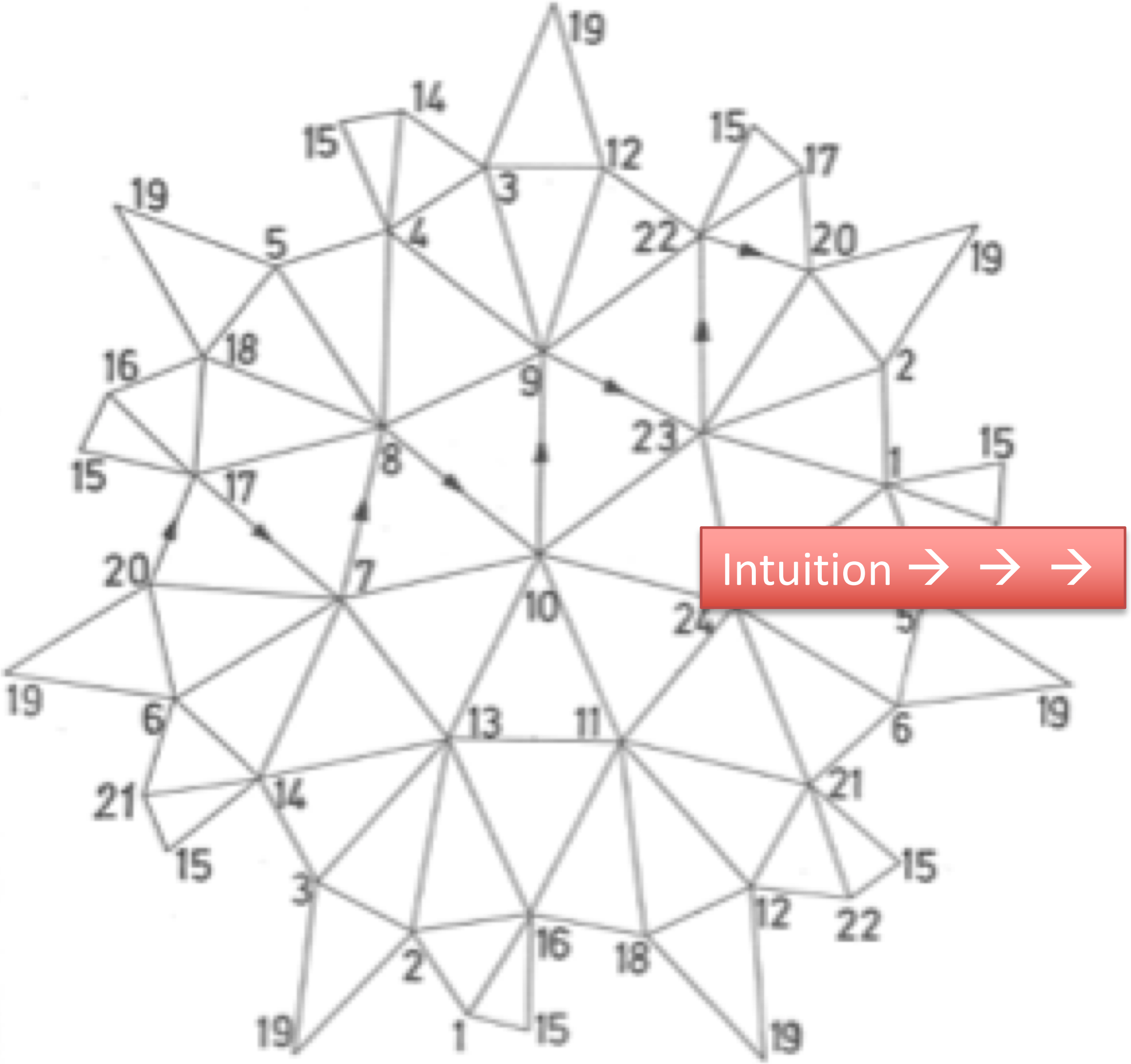
W. Dyck,
Notiz über eine reguläre Riemannsche Fläche vom Geschlecht 3
und die zugehörige Normalkurve 4. Ordnung.
Math. Ann. 17 (1880), 510 – 516.

A. Hurwitz,
Über algebraische Gebilde mit eindeutigen Transformationen in sich,
Math. Annalen 41(1893), 403 -442.

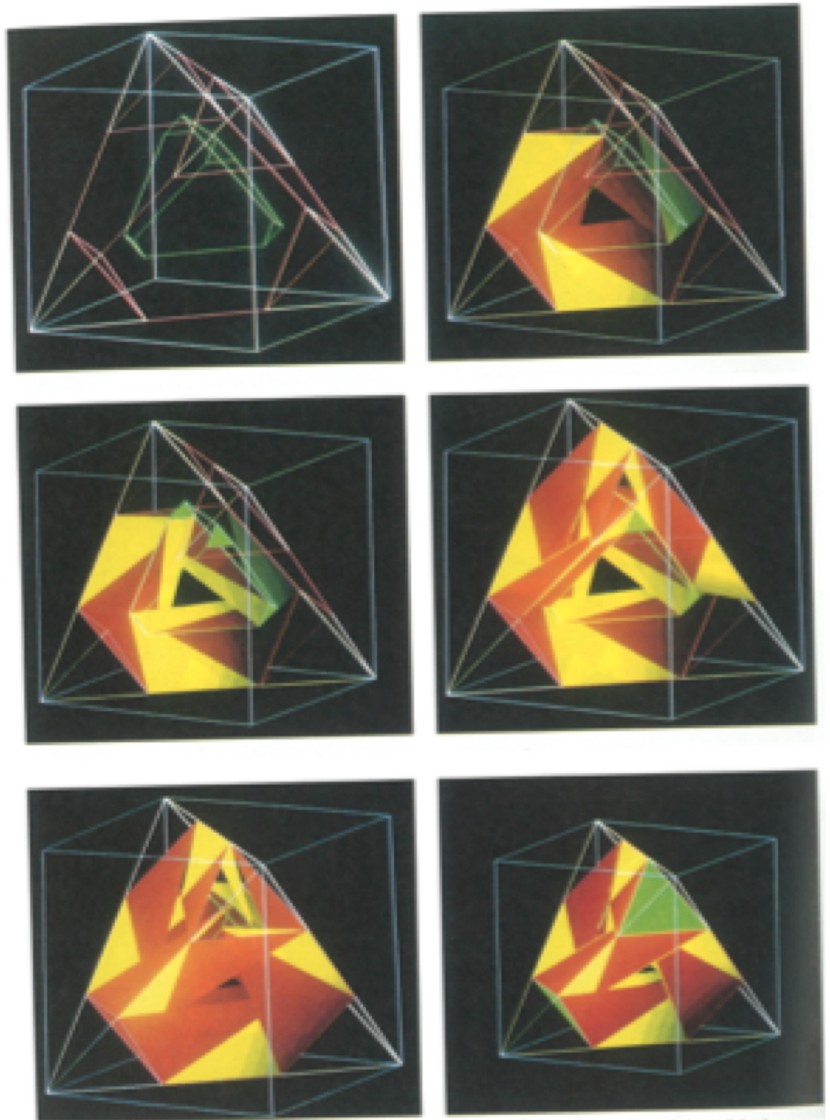
E. Schulte and J. M. Wills,
Geometric realizations for Dyck's regular map on a surface of genus 3.
Discr. Comp. Geom. 1 (1986), 141 – 153.

Some Historical Remarks

Polyhedral regular map {3,7}_8
of Felix Klein
Jörg M. Wills and Egon Schulte



Intuition → → →



Felix Klein's map $\{3,7\}_8$ of genus 3. Successive steps to understand the shape of the polyhedron

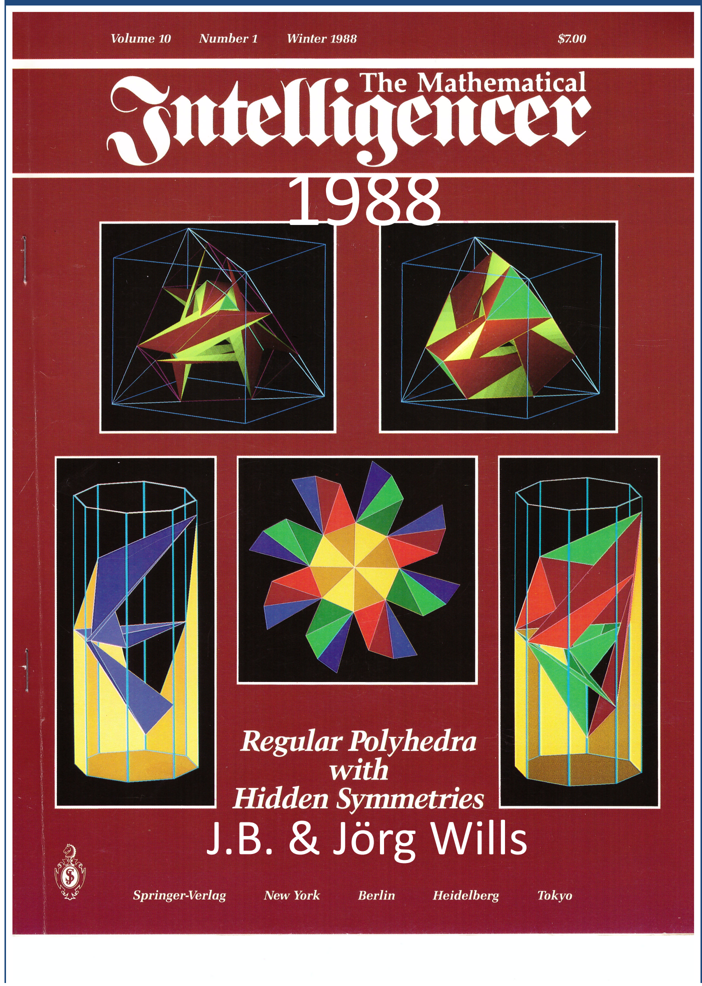
Some Historical Remarks

A. Hurwitz 1893

(A. Macbeath 1965)

F. Klein 1884

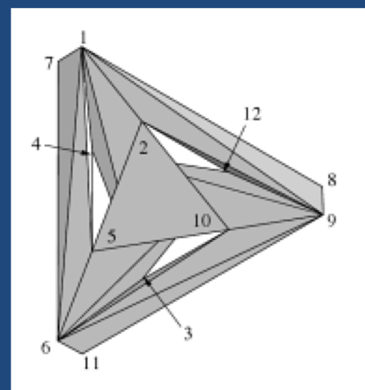
W. Dyck 1880



Polyhedral regular map {3,7}_8
of Felix Klein
Jörg M. Wills and Egon Schulte

First polyhedral regular map
of Walther Dyck {3,8}_6
J.B.

Symmetrical polyhedral
version of Walther Dyck‘s
regular map {3,8}_8
U. Brehm



Regular Maps

Combinatorial

M. Conder, P. Dobcsányi,

“Determination of all regular maps of small genus“, 2001

R3.2 : Dyck’s regular map
Type {3,8}_6 Order 192 mV = 1 mF = 1
Defining relations for automorphism group: [T^2, R^-3, (R * S)^2, (R * T)^2, (S * T)^2, S^8, (S * R^-1 * S)^3]

R3.1: Felix Klein’s map
Type {3,7}_8 Order 336 mV = 1 mF = 1
Defining relations for automorphism group: [T^2, R^-3, (R * S)^2, (R * T)^2, (S * T)^2, S^-7, (R * S^-2)^4]

R7.1: Hurwitz‘s regular map
Type {3,7}_18 Order 1008 mV = 1 mF = 1
Defining relations for automorphism group: [
T^2, R^-3, (R * S)^2, (R * T)^2, (S * T)^2, S^-7,
S^-2 * R * S^-3 * R * S^-2 * R^-1 * S^2 * R^-1 *
S^2 * R^-1 * S^-2 * R * S^-1]

Regular Maps

Topological visualization, computer graphics

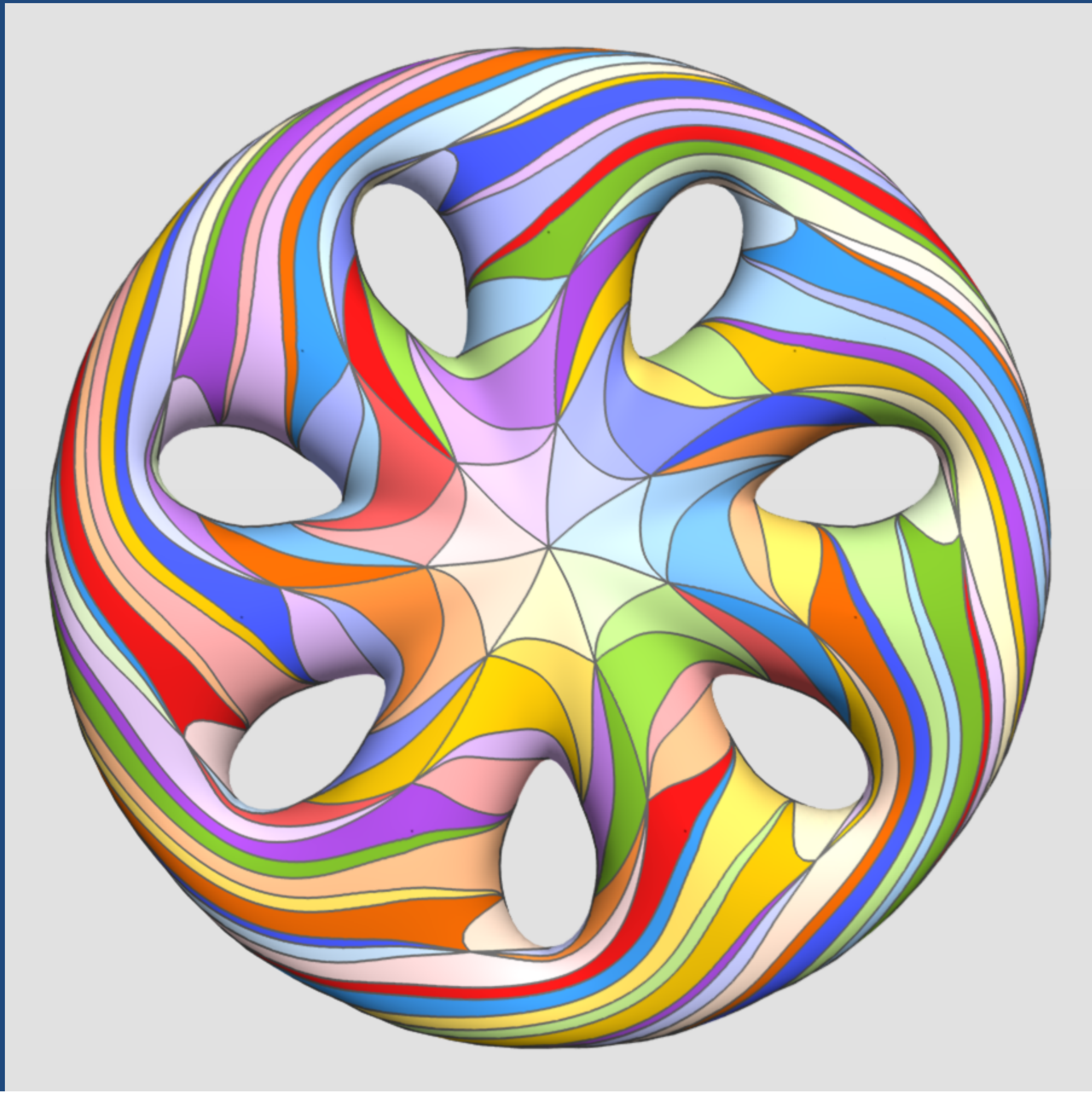
Jarke J. van Wijk, Visualizations of regular maps, 2009.

Carlo H, Sequin, My Search for Symmetrical Embeddings of Regular Maps.

Faniry Razafindrazaka & Konrad Polthier, FU Berlin
Regular Surfaces and Regular Maps

Some Historical Remarks

Jarke J. van Wijk (2014), TU Eindhoven
Topological visualization of Hurwitz´s regular map



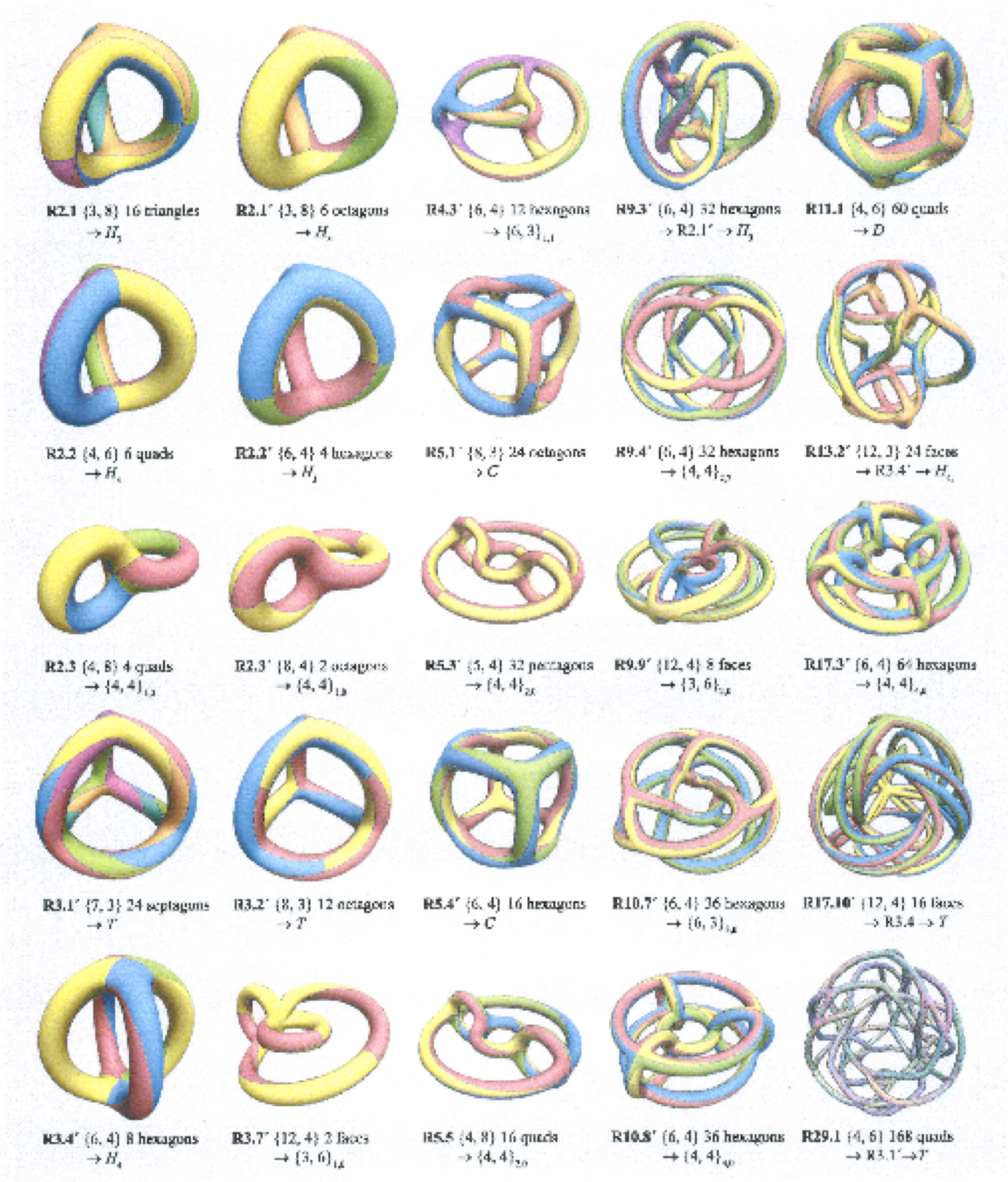
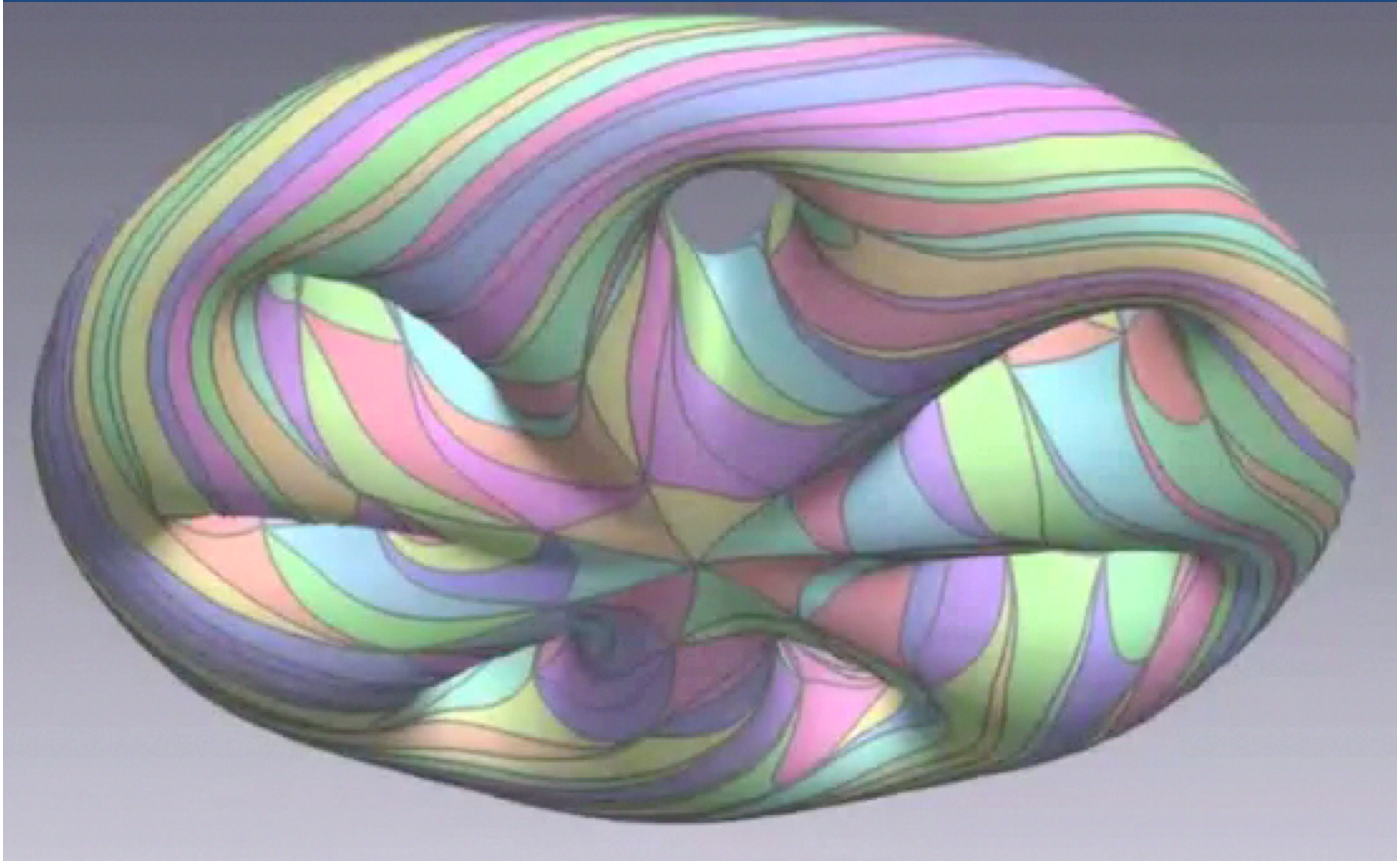


Figure 14: *Space models of regular maps. Rg.i are the labels of Conder and Dobcsányi, where g is the genus; {p, q} is the Schläfli symbol: faces have p sides and q of them meet in a vertex. The second line describes the mapping used to produce the space model. Here T, C, D, and H_n refer to a Tetrahedron, Cube, Dodecahedron and Hosohedron with n bases, and $\{p, q\}_{i,j}$ refers to a torus.*

Computer graphics
Visualizations of regular maps
by Jarke van Wijk,

Jarke J. van Wijk (2014), TU Eindhoven
Topological visualization of Hurwitz´s regular map



Start of investigation

Hurwitz‘s Regular Map {3,7}_18

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[ 2, 8, 9],[ 2,10, 3],[ 3,11, 4],[12, 5, 4],[13, 6, 5],[14, 7, 6],[ 8, 7,15],
[ 2, 9,23],[ 3,10,24],[11,25, 4],[12,26, 5],[13,27, 6],[14,28, 7],[ 8,15,29],
[ 2,16,10],[ 3,17,11],[12, 4,18],[13, 5,19],[20,14, 6],[21,15, 7],[22, 9, 8],
[ 2, ... 8,29],
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[22, ... 57,15],
[46, ... ,11],[49,26,12],[50,27,13],[44,28,14],[45,29,15],
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[37,23,60], ... ,[64,41,27],[58,42,28],[59,43,29],
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[44,30,38], ... ,[48,34,42],[49,35,43],[50,36,37],
[30,56,43], ... ,[40,34,53],[54,41,35],[55,42,36],
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[44,51,65], ... ,[48,55,69],[49,56,70],[50,57,71],
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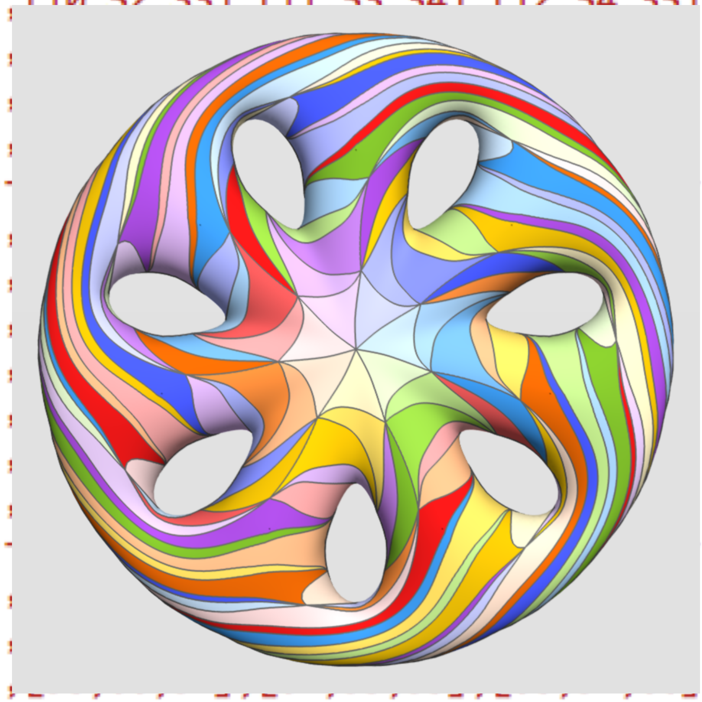
[64,65,51], ... ,[55,61,69],[56,62,70],[63,71,57],
[58,65,66],[67,59,66],[67,68,60],[68,69,61],[69,70,62],[63,70,71],[64,71,65],
[66,65,72],[67,66,72],[67,72,68],[68,72,69],[69,72,70],[71,70,72]][65,71,72],
```

Symmetry of order 1008 =
7x144=7x12x12

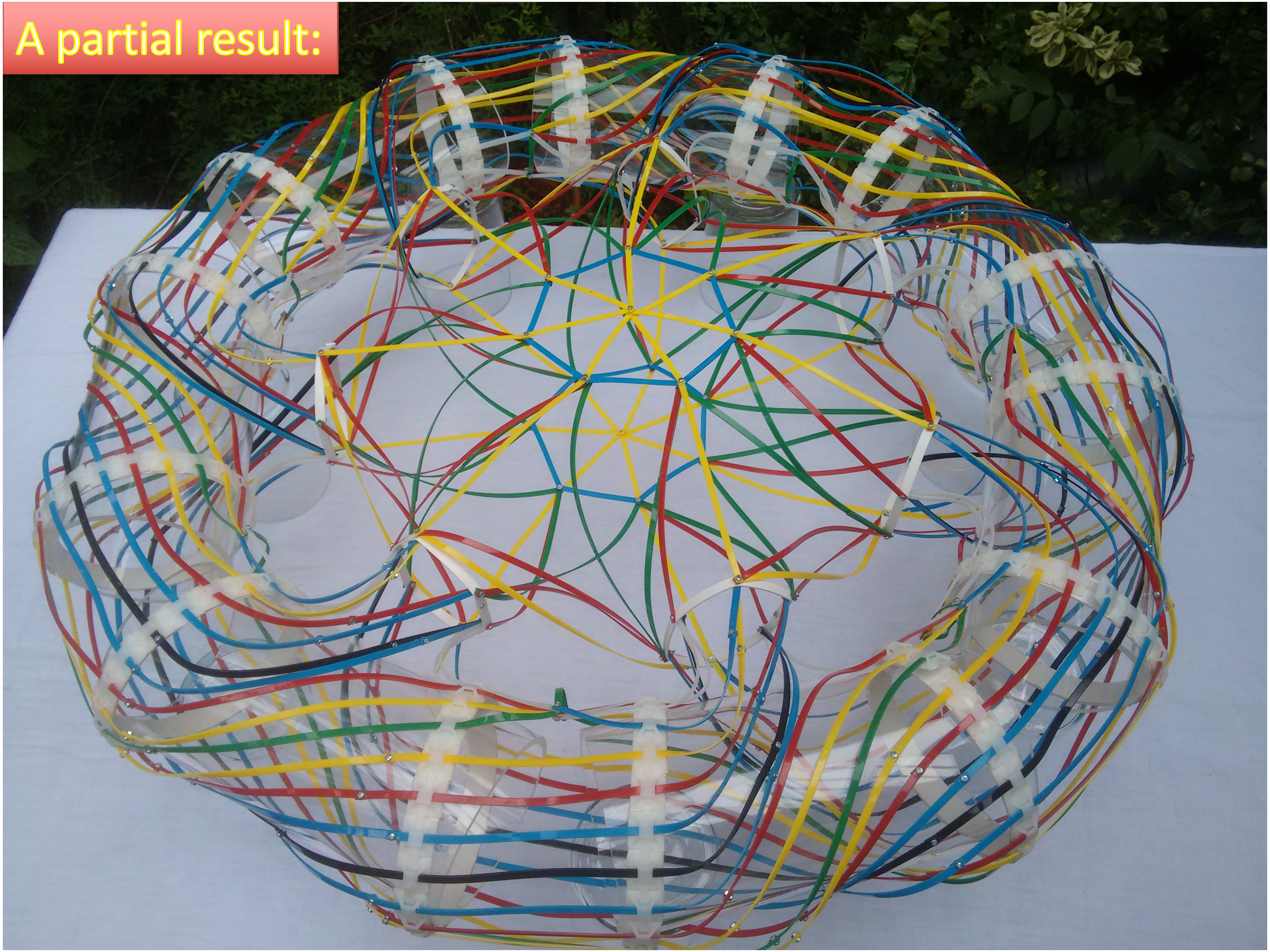


Start of investigation

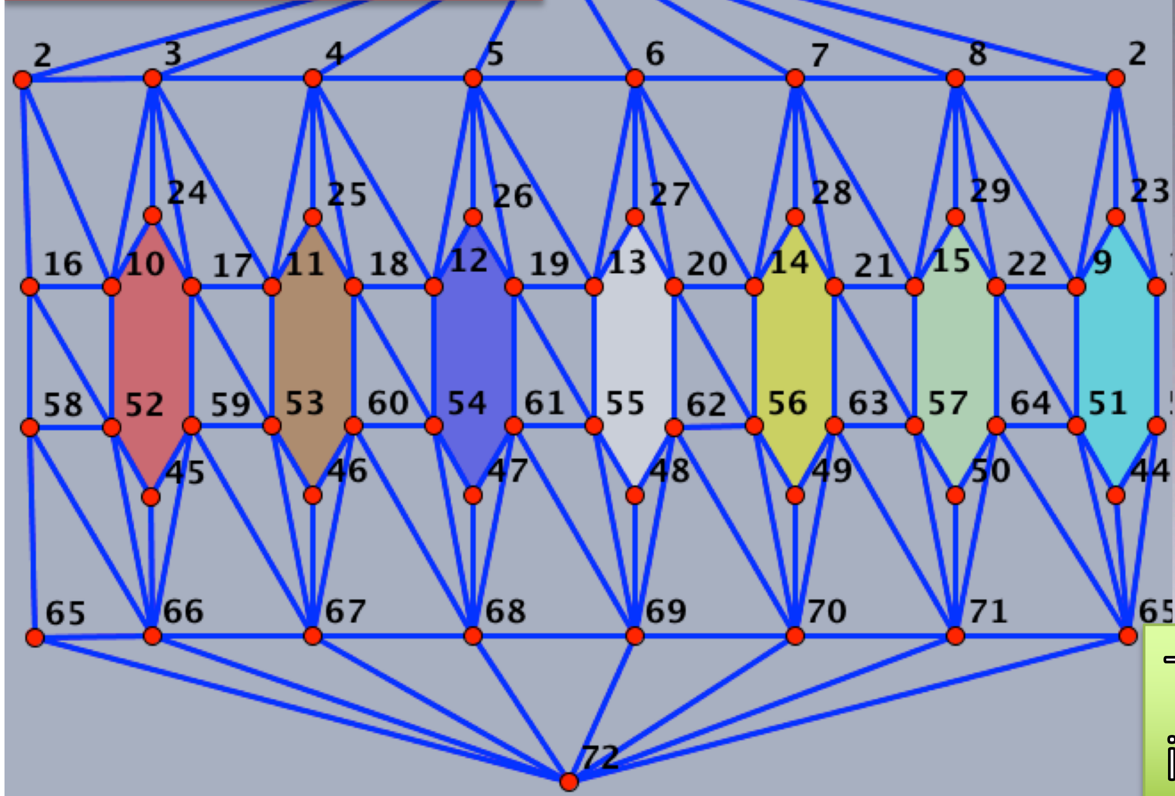
Start of investigation

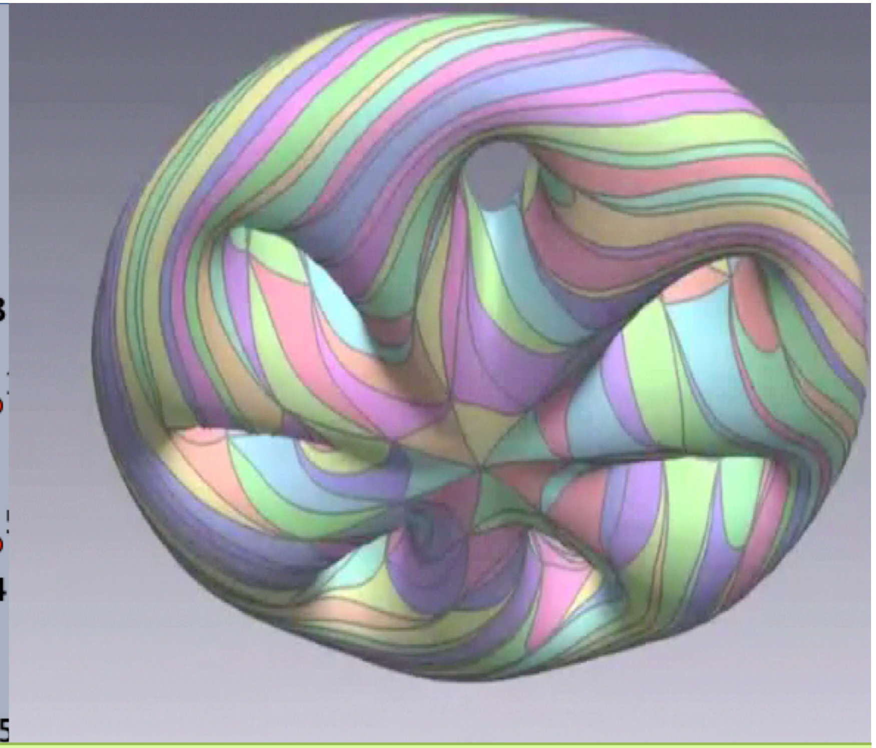
Start of investigation

A partial result:

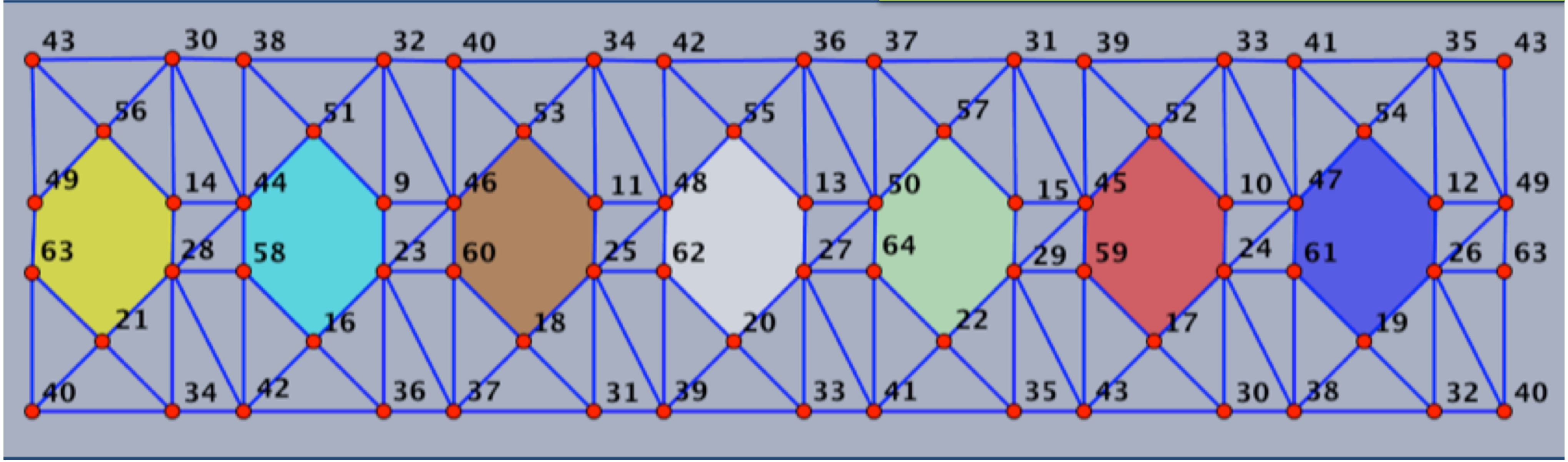


A partial result:





The genus 7 surface splitted into a 2-sphere and a torus

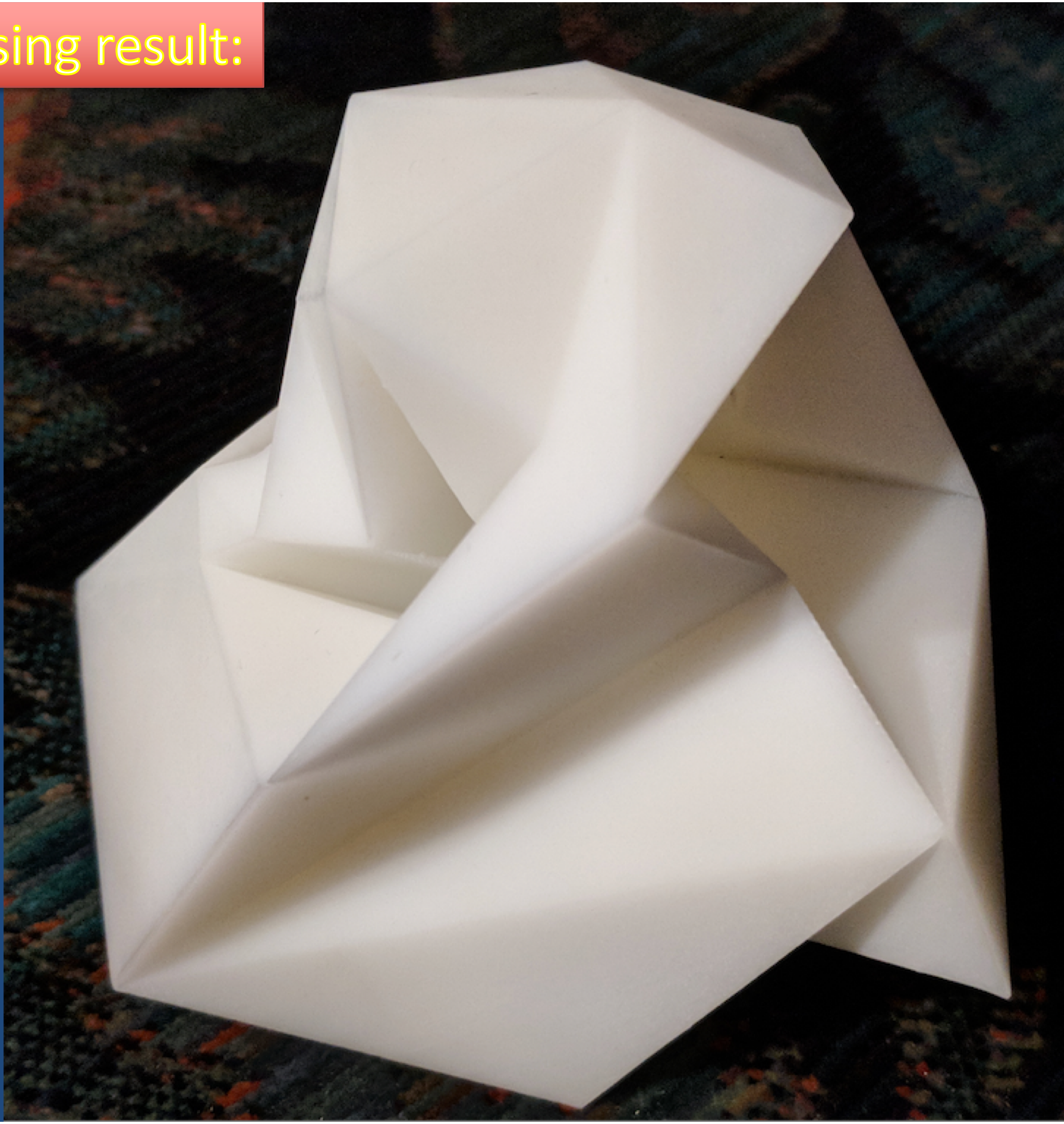


Start of investigation

Summary of my talk in Berlin 2017

Investigation with Michael Cuntz, Hannover
resulted in a very strong conjecture that
Hurwitz´s regular map {3,7} has no polyhedral realization

A surprising result:



Methods for Geometric Realization Problems

use topological models

use oriented matroids

use symmetries

use try and error methods

use related realizations

use partial realizations

use a dynamical geometric software like Cinderella

use the software Magma or GAP for symmetry investigations

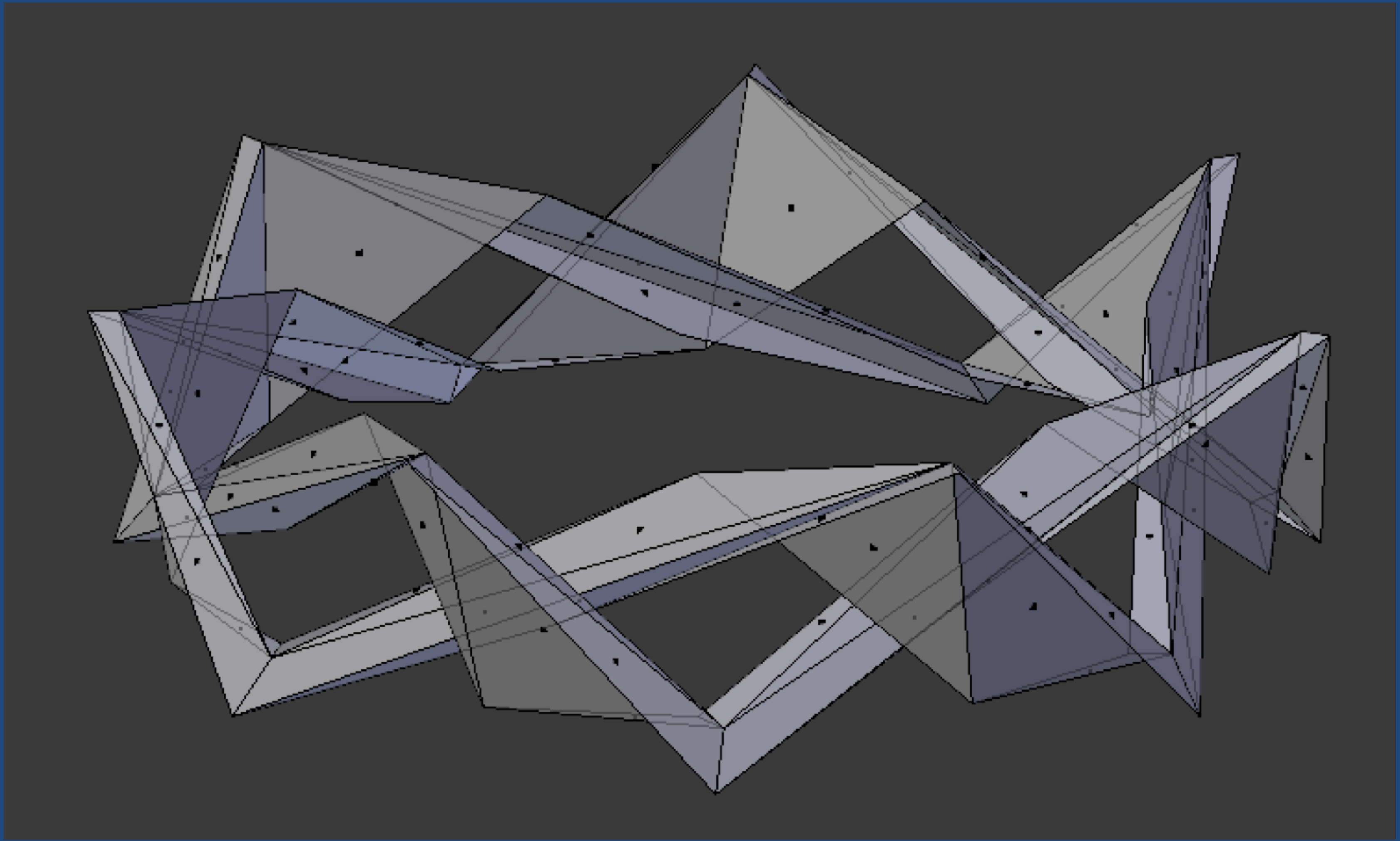
use models instead or in addition to computer graphics

use functional programming for combinatorial support

use intuition

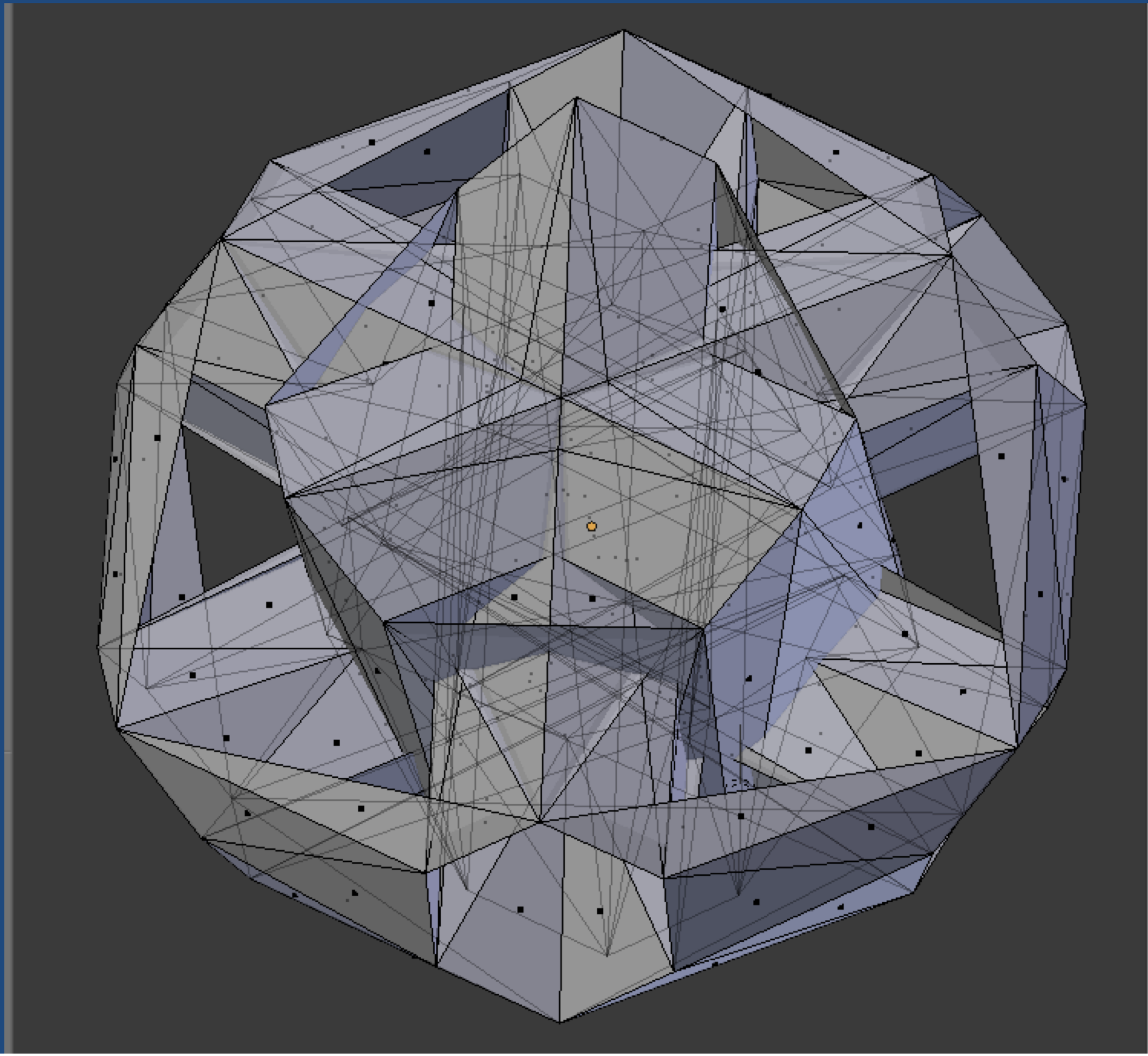
don´t be frustrated when you find no solution.

Search for symmetric realizations



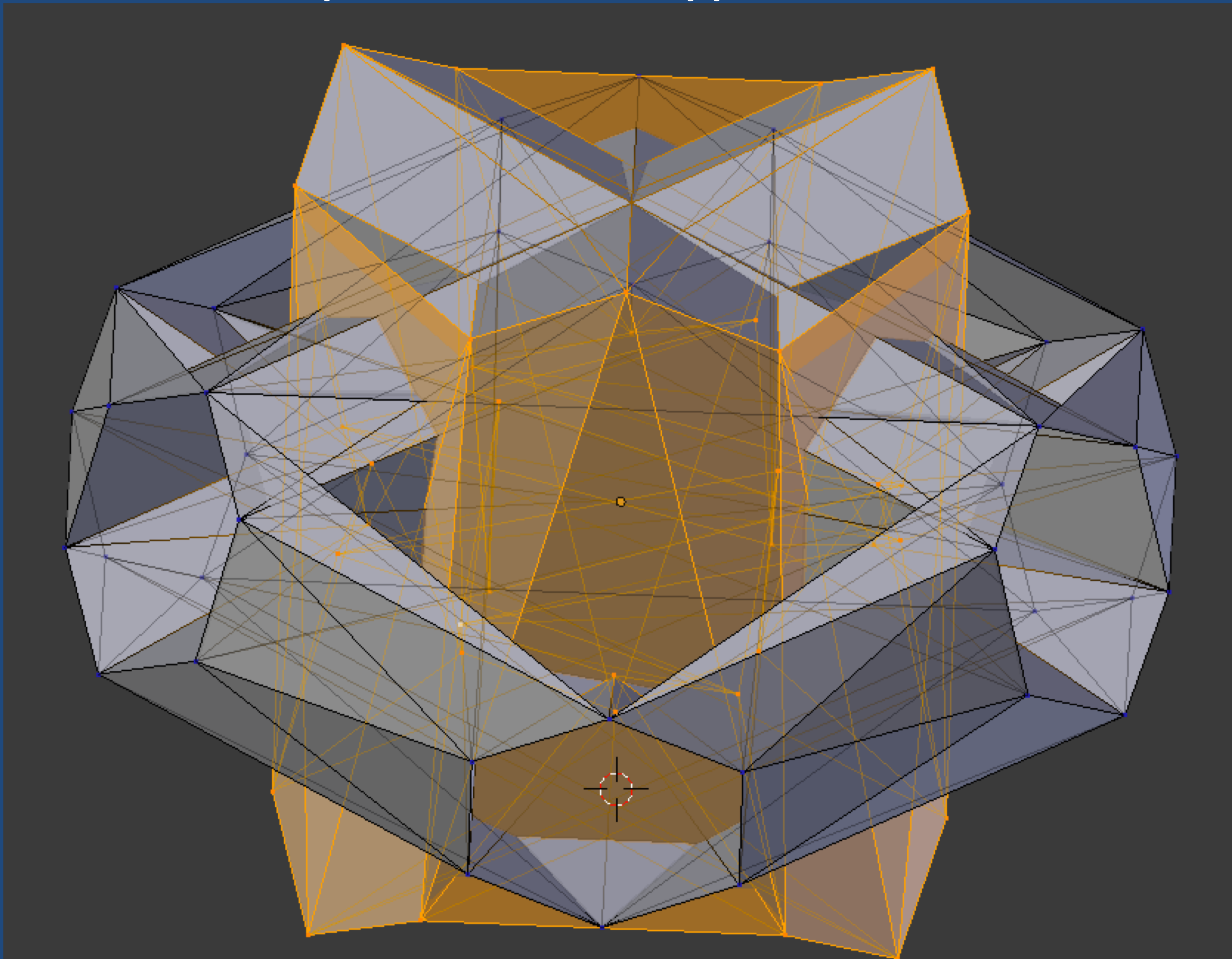
Search for symmetric realizations

Kepler-Poinsot type models



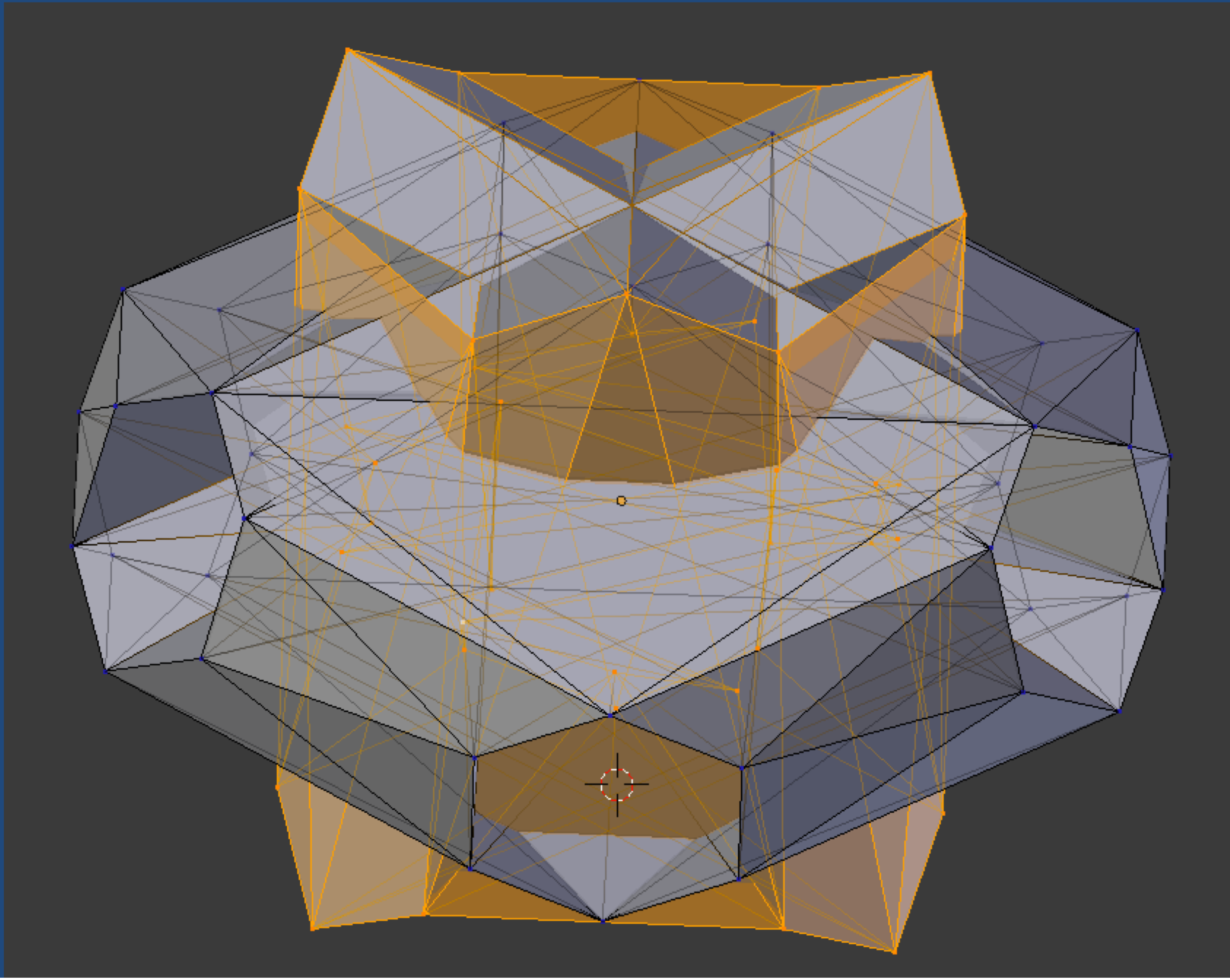
Search for symmetric realizations

Kepler-Poinsot type models

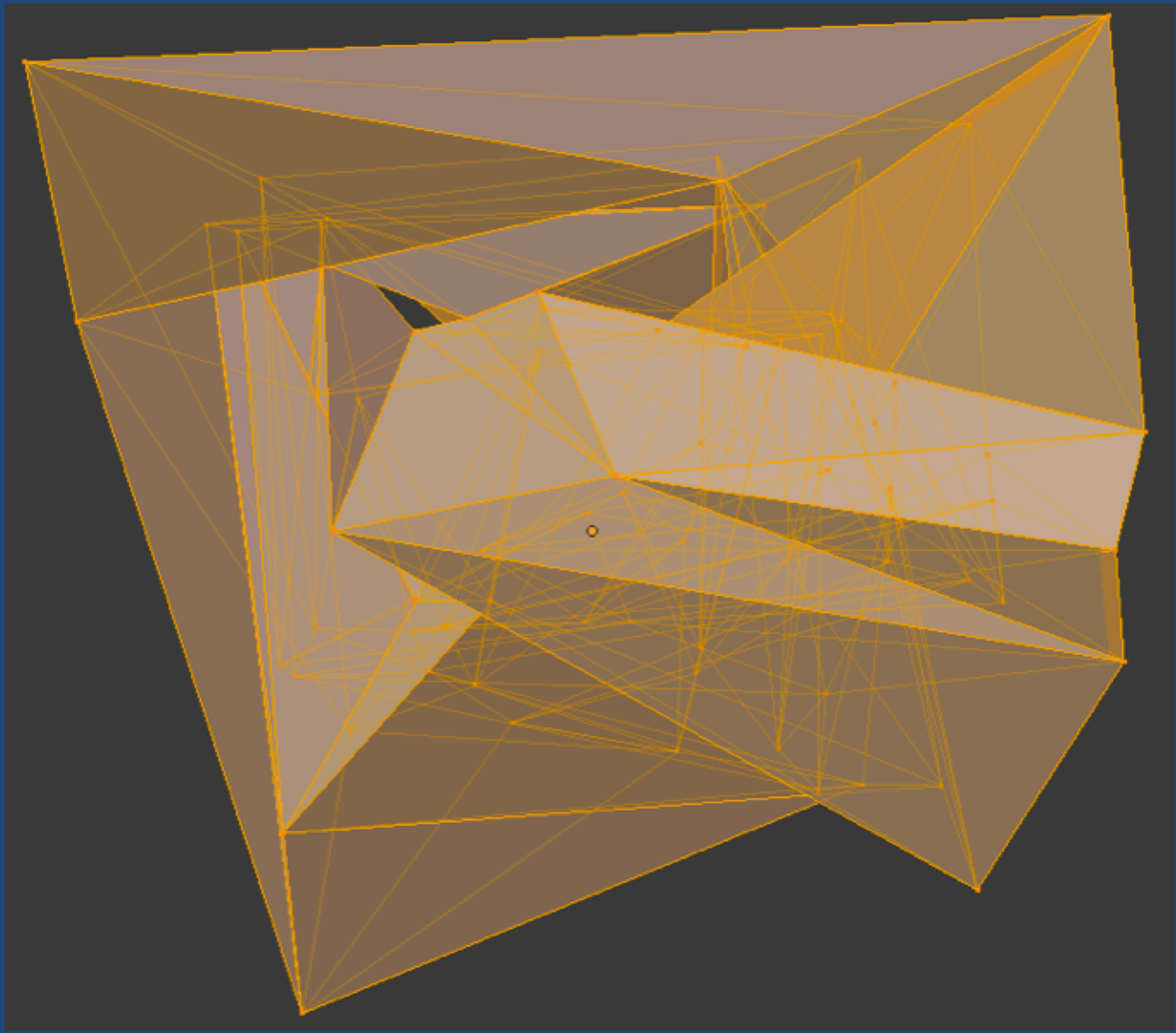


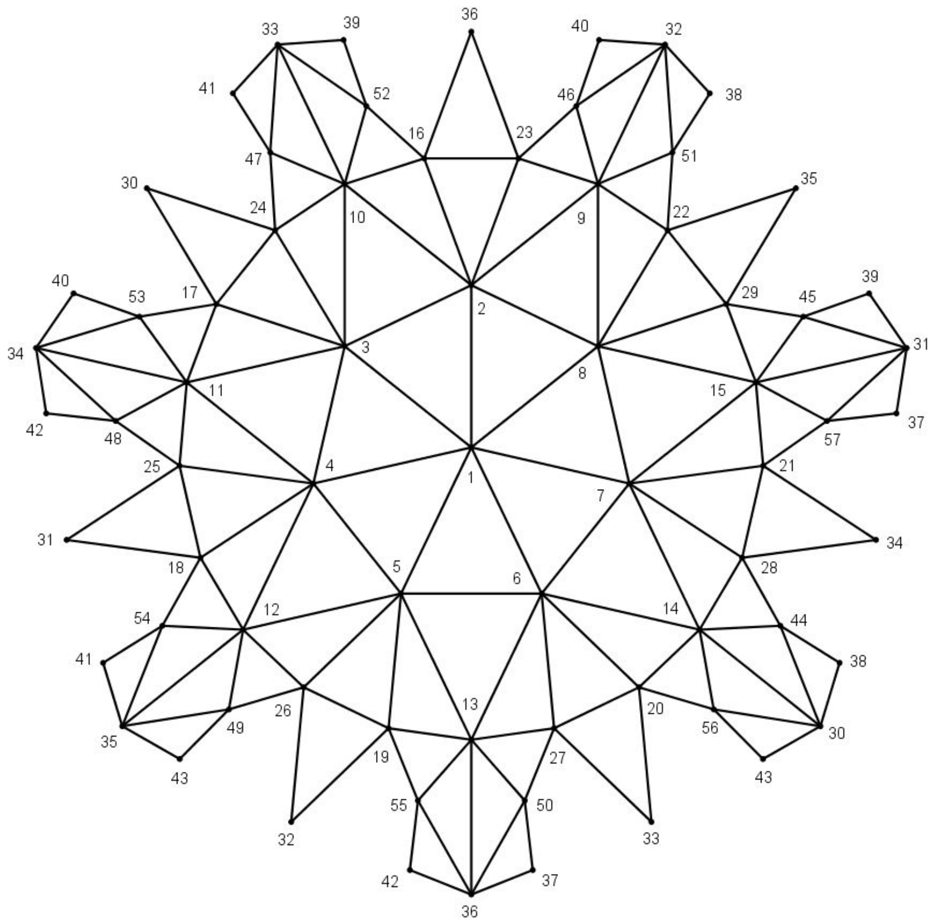
Search for symmetric realizations

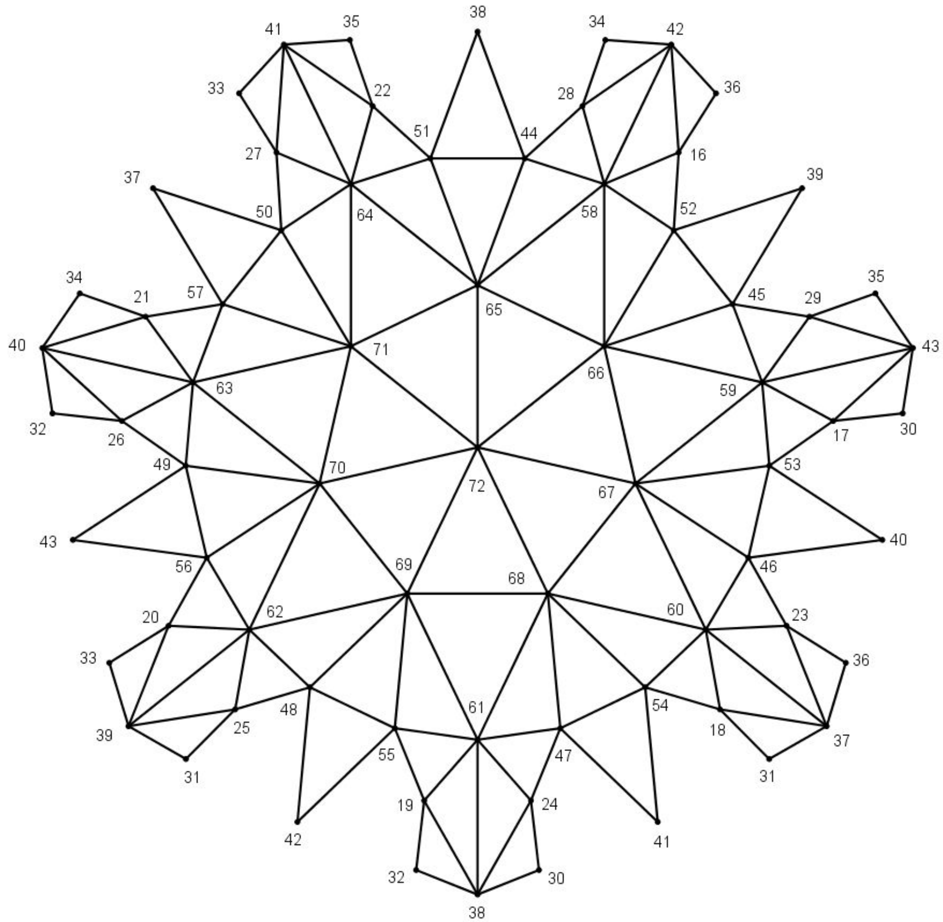
Kepler-Poinsot type models

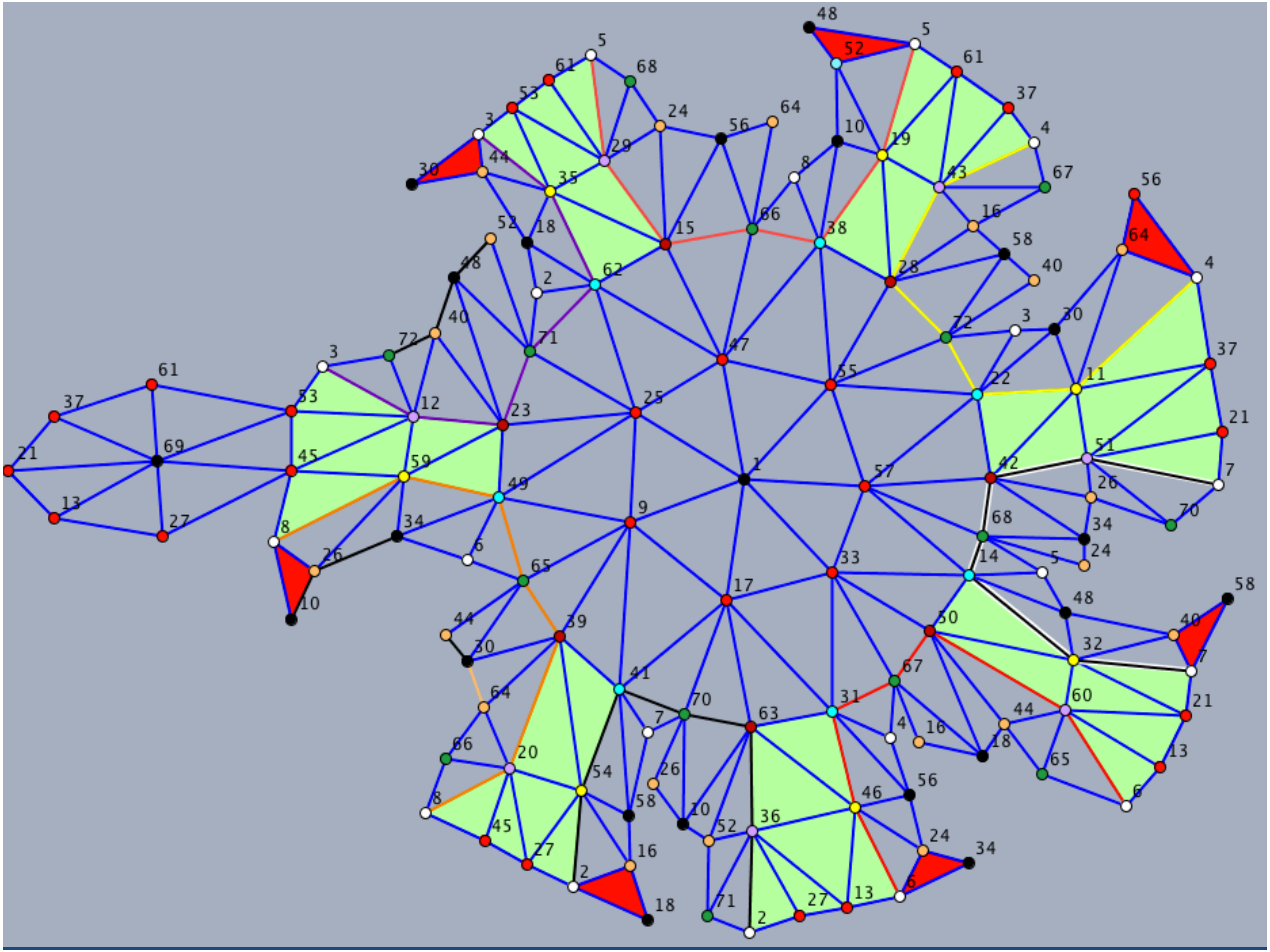


Who is familiar with the powerful Blender software ?





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