Matroidal maximum term rank

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joint work with András Frank

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Outline

Background

Maximum term rank problem

Brualdi's theorem

Covering supermodular functions by bipartite graphs

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Basic tools

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Network flow theory

- Degree constrained subgraph in a bipartite graph.

Matroid theory

- Cheapest rooted k-node-connected subgraph of a digraph.

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Submodular flows

(Edmonds, Giles, '77)

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Corresponding linear system is TDI $\Rightarrow$ weighted version is also tractable.

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Find a degree-prescribed simple bipartite graph with matching number $\geq \ell$.

Ryser, *The term rank of a matrix*, Canadian Journal of Mathematics (1958).

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Ryser, *The term rank of a matrix*, Canadian Journal of Mathematics (1958).

Problem

Find a degree-prescribed simple bipartite graph of minimum cost with matching number $\geq \ell$.

Pálvölgyi, *Partitioning to three matchings of given size is NP-complete for bipartite graphs*, Acta Universitatis Sapientiae, Informatica (2014).

$\Rightarrow$ **submodular flows cannot help!**

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Covering bi-set functions

(Frank, Jordán, '95)

- Degree sequences of k-edge-connected and k-node-connected digraphs.
- Covering a vertically convex polyomino by a minimum number of rectangles (Győri, '85)

B. and Frank, *Supermodularity in unweighted graph optimization I*, ***II*** and III, Mathematics of Operations Research (2018).

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Packing common bases

• Packing pairwise disjoint common bases of two matroids.

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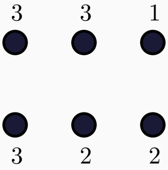
Maximum term rank problem

Degree sequences of simple bipartite graphs

Problem

Given a node set $V = S \cup T$ and a degree-specification $m_V = (m_S, m_T)$, find a simple bipartite graph $G = (S, T; E)$ fitting m_V.



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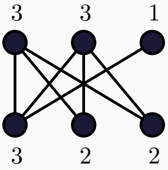

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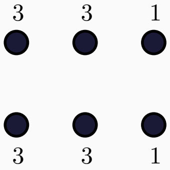

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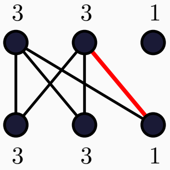

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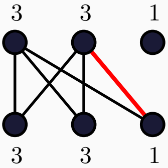

Theorem (Gale '57, Ryser '57)

There exists a simple bipartite graph fitting (m_S, m_T) if and only if

$$m_S(X) + m_T(Y) - |X||Y| \leq \gamma$$

for every $X \subseteq S$, $Y \subseteq T$, where $\gamma = m_S(S) = m_T(T)$.

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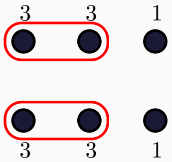

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Degree sequences of simple bipartite graphs

Remarks

- upper and lower bounds on the degrees
- edges can be chosen from a given graph

Maximum term rank

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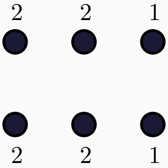
Given a node set $V = S \cup T$, a degree-specification $m_V = (m_S, m_T)$ and $\ell \in \mathbb{Z}_+$, find a simple bipartite graph $G = (S, T; E)$ fitting m_V with matching number $\nu(G) \geq \ell$.

Let $\ell = 3$:



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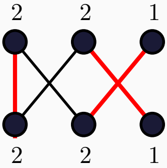

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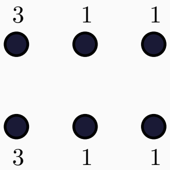

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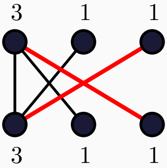

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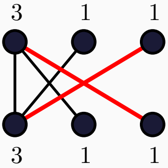

Theorem (Ryser '58)

Assume that there exists a simple bipartite graph fitting (m_S, m_T). There exists one with matching number $\nu(G) \geq \ell$ if and only if

$$m_S(X) + m_T(Y) - |X||Y| + (\ell - |X| - |Y|) \leq \gamma$$

for every $X \subseteq S$, $Y \subseteq T$, where $\gamma = m_S(S) = m_T(T)$.

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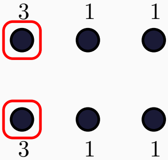

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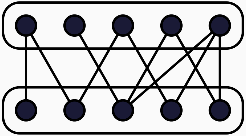
Brualdi's theorem

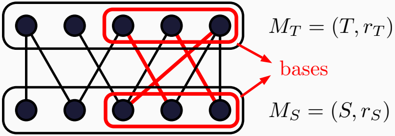
Matroidal matching



$M_T = (T, r_T)$

$M_S = (S, r_S)$

Matroidal matching



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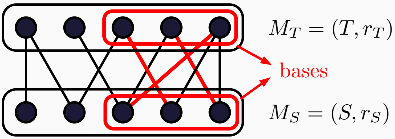

Theorem (Brualdi '70)

Let $G = (S, T; E)$ be a bipartite graph with matroids $M_S = (S, r_S)$ and $M_T = (T, r_T)$ for which $r_S(S) = r_T(T) = \ell$. There is a matching covering a basis of M_S and a basis of M_T if and only if

$$r_S(X) + r_T(Y) \geq \ell$$

whenever $X \cup T$ hits every edge of G ($X \subseteq S,\ T \subseteq T$).

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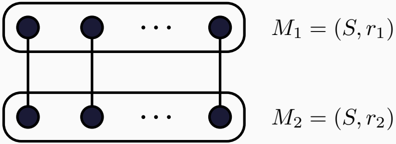

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Covering supermodular functions by bipartite graphs

Common generalization?

Rephrasing the problems

Maximum term rank. Find a degree-prescribed simple bipartite graph G such that

$$|\Gamma_G(Y)| \geq |Y| - \ell \qquad \text{for every } Y \subseteq T.$$

Brualdi. Given a bipartite graph with matroids on its color classes, there exists a matching covering bases of the matroids if and only if

$$r_S(\Gamma_G(Y)) \geq t_T(Y) \qquad \text{for every } Y \subseteq T,$$

where t_T denotes the co-rank function of M_T.

Abstract framework

Problem

Given $V = S \cup T$, a degree-specification $m_V = (m_S, m_T)$ and a (positively intersecting) supermodular function p_T on T, find a **simple** bipartite graph $G = (S, T; E)$ fitting m_V that covers p_T, that is,

$$|\Gamma_G(Y)| \geq p_T(Y)$$

for every $Y \subseteq T$.

Abstract framework

Problem

Given $V = S \cup T$, a degree-specification $m_V = (m_S, m_T)$ and a (positively intersecting) supermodular function p_T on T, find a **simple** bipartite graph $G = (S, T; E)$ fitting m_V that covers p_T, that is,

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Problem$^+$

Given a bipartite graph $H_0 = (S, T; F_0)$, a degree-specification $m_V = (m_S, m_T)$, a (positively intersecting) supermodular function p_T on T and a matroid $M_S = (S, r_S)$, find a bipartite graph $G = (S, T; E)$ fitting m_V so that $H_0 + G$ is **simple** and M_S-covers p_T, that is,

$$r_S(\Gamma_{H_0+G}(Y)) \geq p_T(Y)$$

for every $Y \subseteq T$.

Main result

Theorem (B. and Frank '18)

Given a bipartite graph $H_0 = (S, T; F_0)$, a degree-specification $m_V = (m_S, m_T)$, a positively intersecting supermodular function p_T on T and a matroid $M_S = (S, r_S)$. There exists a bipartite graph $G = (S, T; E)$ fitting m_V for which $G^+ = G + H_0$ is simple and M_S-covers p_T if and only if

$$m_S(X) + m_T(Y) - d_{\overline{H_0}}(X, Y) + \sum_{i=1}^{q} [p_T(T_i) - r_S(X \cup \Gamma_{H_0}(T_i))] \leq \gamma$$

whenever $Y \subseteq T$, $X \subseteq S$, and $\mathcal{T} = \{T_1, \ldots, T_q\}$ is a subpartition of $T - Y$.

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Remarks

- Proof uses the Frank-Jordán theorem on covering supermodular bi-set functions.
- Lower and upper bounds on the degrees are also tractable.
- In the intersecting case, efficient algorithm using a result of Frank and Tardos.

Consequence

Theorem (B. and Frank '18)

Given a bipartite graph $H_0 = (S, T; F_0)$, a degree-specification $m_V = (m_S, m_T)$, matroids $M_S = (S, r_S)$ and $M_T = (T, r_T)$ with $r_S(S) = r_T(T) = \ell$. There exists a bipartite graph $G = (S, T; E)$ fitting m_V for which $G^+ = G + H_0$ is simple and includes a matching covering a basis of M_S and a basis of M_T if and only if $\overline{H_0}$ has a subgraph fitting m_V and

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{\overline{H_0}}(X, Y) + \ell - r_S(X') - r_T(Y') \leq \gamma$$

whenever $X \subseteq X' \subseteq S$, $Y \subseteq Y' \subseteq T$, and $X' \cup Y'$ hits all the edges of H_0.

- If M_S, M_T are the ℓ-uniform matroids $\Rightarrow$ Maximum term rank.
- If $H_0 = \emptyset$ and $m_S \equiv m_T \equiv 0 \Rightarrow$ Brualdi's theorem.

Further results

- Characterization of the degree sequences of wooded hypergraphs with prescribed edge sizes (and further constraints).
- Packing branchings of given sizes (and further constraints).
- Degree sequences of highly connected **simple** directed graphs.
- ...