Geometric bijections for regular matroids, zonotopes, and Ehrhart theory

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Chip-firing

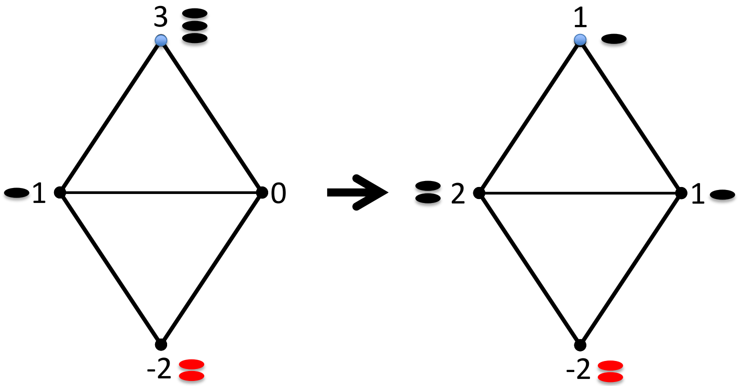
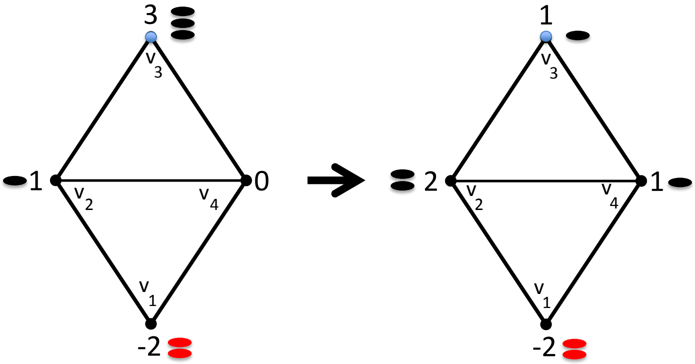


Figure: A chip-firing move

The graph Laplacian and chip-firing

- The graph Laplacian is $Q = D - A$ where D is a diagonal matrix with $D_{i,i} = \deg(v_i)$ and A is the adjacency matrix.
- Alternately, $Q = B * B^T$, where B is the oriented incidence matrix of the graph. Thus Q is the boundary operator composed with the coboundary operator.
- If we represent a chip configuration by a vector $\vec{x}$, then firing v_i gives the new vector $\vec{x} - Qe_i$.



$$\begin{bmatrix} -2 \\ 1 \\ 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

Figure: Chip-firing and the Laplacian matrix

The Jacobian of G

- We define $Pic^d(G)$ be the set of chips configurations with d chips modulo chip-firing.
- The *Jacobian* of G is $Jac(G) = Pic^0(G)$.
- Algebraically, this group is the torsion part of the $\mathbb{Z}$-cokernel of the Laplacian.

By Kirchoff's matrix-tree theorem plus linear algebra,

$$|Jac(G)| = \#\text{ of spanning trees.}$$

History

Chip-firing was independently introduced in several different communities.

- Poset Theory: '72 Mosesian
- Discrete Probability: '75 Engel
- Statistical Physics: '87 Bak-Tang-Weisenfeld
- Coxeter Groups: '87 Mozes
- Arithmetic Geometry: '70 Raynaud
- Graph Theory: '91 Björner-Lovász-Shor

G-parking functions and spanning trees

- The G-parking functions (also known as reduced divisors, dual critical configurations, etc.) give distinguished representatives for the chip-firing equivalences classes.
- There exist several explicit bijections between the G-parking functions and spanning trees, e.g.
  1. A bijection due to Cori and LeBorgne maps # chips to external activity.
  2. A bijection due to Perkinson, Yang, and Yu maps # chips to tree inversion number.

Regular Matroids

- Merino introduced the notion of a chip-firing group ($Jac(B)$) of a totally unimodular matrix, which generalizes the chip-firing group of a graph.
- Wagner showed that this group is a regular matroid invariant for $M(B)$.
- $|Jac(M)| = \#$ of bases of M
- Our goal: We would like to produce a collection of bijections between the elements of the Jacobian of a regular matroids and the bases of the matroid.
- Nb: Any such bijection must avoid reference to vertices. Our approach is to instead work with orientations.

Orientations and chip configurations

- Given an orientation $\mathcal{O}$, we can associated a chip configuration

$$D_{\mathcal{O}} = \sum_{v \in V(G)} \mathrm{indegree}_{\mathcal{O}}(v).$$

- There is an equivalence relation on orientations which captures chip-firing.

Cycle and cut reversals

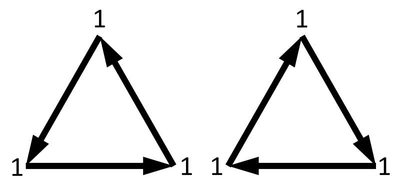


Figure: A directed cycle reversal

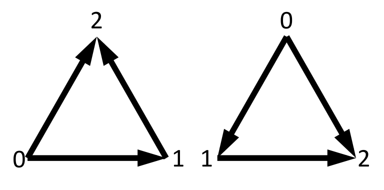


Figure: A directed cut reversal

Cut-cycle reversal classes

- Given two orientations $\mathcal{O}$ and $\mathcal{O}'$, the associated chip configurations $D_{\mathcal{O}}$ and $D_{\mathcal{O}'}$ are related by chip-firing moves if and only if they are related by directed cut reversals and directed cycle reversals.

Theorem (Gioan, An-Baker-Kuperberg-Shokrieh)

Given D with $|E(G)|$ chips, $D \sim D_{\mathcal{O}}$ for some orientation $\mathcal{O}$.

- This implies the important fact:
  The set of orientations modulo directed cut reversals and cycle reversals is canonically in bijection with $Pic^m(G)$, where $m = E(G)$.

Representatives for orientation classes

- Fix a total order on the edges and a reference orientation $(<, \mathcal{O}_{ref})$.
- We will call an orientation *cycle-minimal* if the minimum edge in each directed cycle is oriented in agreement with the reference orientation. We define *cut-minimal* orientations similarly.

Theorem (folklore?)

Each orientation $\mathcal{O}$ is equivalent by cut and cycle reversals to a unique cut-cycle minimal orientation.

Fundamental cuts and cycles

- Let T be a spanning tree and $e \in E(G)$.
  1. If $e \in T$, we can associate a fundamental cut $Cu(e, T)$, which is the unique cut in $E(G) \setminus \{T \setminus e\}$.
  2. If $e \notin T$, we can associate a fundamental cycle $Cy(e, T)$, which is the unique cycle in $T \cup e$.

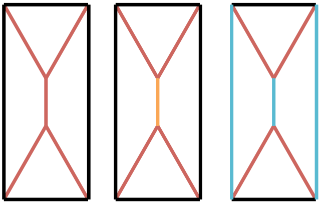


Figure: A fundamental cut $Cu(e, T)$

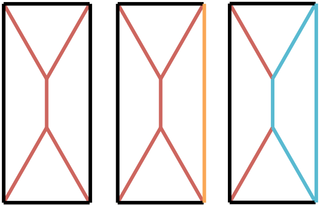


Figure: A fundamental cycle $Cy(e, T)$.

Our bijection

Map from spanning trees to orientations

- Fix a total order on the edges and a reference orientation $(<, \mathcal{O}_{ref})$
- Let T be a spanning tree and $e \in E(G)$.
  1. If $e \in T$, orient e in agreement with the minimum edge in $Cu(e, T)$.
  2. If $e \notin T$, orient e in agreement with the minimum edge in $Cy(e, T)$.
- We claim that this map is a bijection between spanning trees and cycle-cut minimal orientations.
- Remark: Our bijection works more generally if we replace the total order on the edges with a generic weight vector. Moreover, we can choose our data for cuts and cycles separately.

The Tutte polynomial and Ehrhart theory for Zonotopes

A *zonotope* is a Minkowski sum of line segments, i.e. a projection of a hypercube.

- The graphical zonotope is

$$Z_G = \sum_{(i,j)\in E(G)} [e_i, e_j]$$

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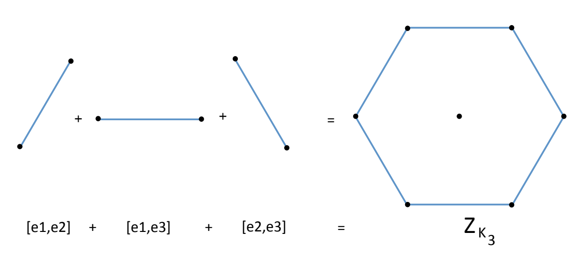


The Ehrhart polynomial

The Ehrhart polynomial of a graphical zonotope

- Given an integer polytope P, the Ehrhart polynomial is such that $E_P(q) = \#$ lattice points in qP.
- Given a graph (more generally a matroid) the Tutte polynomial is a bivariate universal deletion contraction invariant.

Theorem (Stanley)

$$E_{Z_G}(q) = q^{n-1} T_G(1 + \frac{1}{q}, 1)$$

Proof sketch: Look at a tiling of the zonotope by parallelepipeds and take a half open decomposition.

New derivation of the Ehrhart polynomial of Z_G

1. The lattice points of $qZ_G \leftrightarrow$ indegree sequences of orientations of qG.
2. Indegree sequences of orientations of $qG \leftrightarrow$ cycle reversal classes of orientations of qG.
3. Cycle reversal classes of orientations of $qG \leftrightarrow$ Cycle minimal orientations of qG.
4. Cycle minimal orientations of $qG \leftrightarrow$ a class of $q-$weighted partial orientations introduced and enumerated by Sam Hopkins and myself.

Fourientation expansion of the Tutte polynomial [B.-Hopkins-Traldi 2015]

There exists a $2^{|E(G)|}$-to-1 surjection $\varphi\colon \mathcal{O}^4(G) \to \mathcal{S}(G)$, such that

$$(k_1 + m)^{n-\kappa}(k_2 + l)^g T_G\left(\frac{k_1 x + k_2 w + m\hat{x} + l\hat{w}}{k_1 + m}, \frac{k_2 y + k_1 z + l\hat{y} + m\hat{z}}{k_2 + l}\right)$$

$$= \sum_{O \in \mathcal{O}^4(G)} k_1^{|O^o \cap \varphi(O)|} k_2^{|O^o \setminus \varphi(O)|} l^{|O^u|} m^{|O^b|} x^{|I^+(O)|} w^{|I^-(O)|} \hat{x}^{|I^b(O)|} \hat{w}^{|I^u(O)|} y^{|L^+(O)|} z^{|L^-(O)|} \hat{y}^{|L^u(O)|} \hat{z}^{|L^b(O)|}.$$

New derivation of Ehrhart polynomial of Z_G

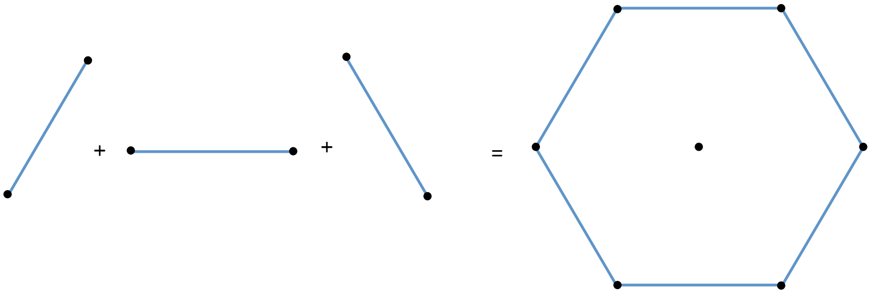
- The cycle-minimal orientations of qG are in enumerated by a single variable specialization of the previous formula.

$$(k_1, k_2, l, m, w, \hat{w}, x, \hat{x}, y, \hat{y}, z, \hat{z}) = (1, 1, 0, q, 1, 1, 1, 1, 1, 1, 0, 0).$$

- Side remark on Ehrhart reciprocity: The interior lattice points in qZ_G correspond strongly connected cycle-minimal orientations of qG. Similarly, this is another single variable specialization of the previous formula.

$$(k_1, k_2, l, m, w, \hat{w}, x, \hat{x}, y, \hat{y}, z, \hat{z}) = (1, 1, 0, q, 0, 1, 0, 1, 1, 1, 0, 0).$$

Therefore # interior lattice points in $qZ_G = q^{n-1} T_G(1 - \frac{1}{q}, 1)$.
This gives a direct verification of Ehrhart reciprocity.



$$[e1,e2] + [e1,e3] + [e2,e3] = Z_{K_3}$$

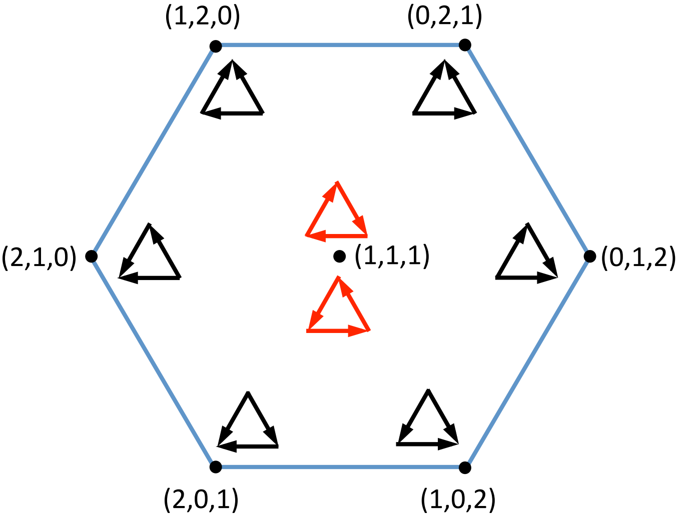


Figure: Cycle reversal classes and lattice points in Z_{K_3}

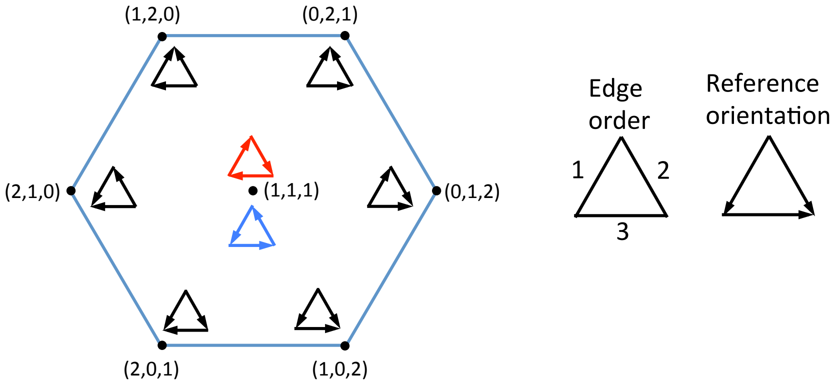


Figure: Cycle-minimal orientations and lattice points in Z_{K_3}

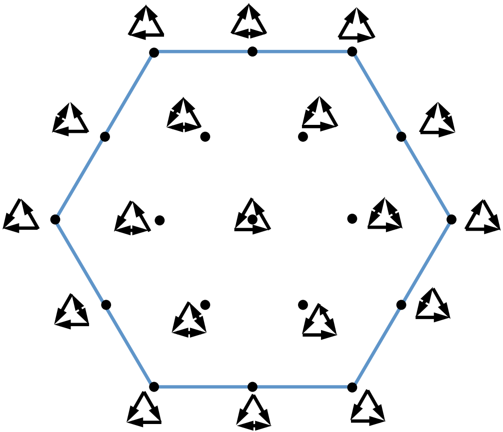


Figure: Cycle-minimal orientations and lattice points in $2Z_{K_3}$

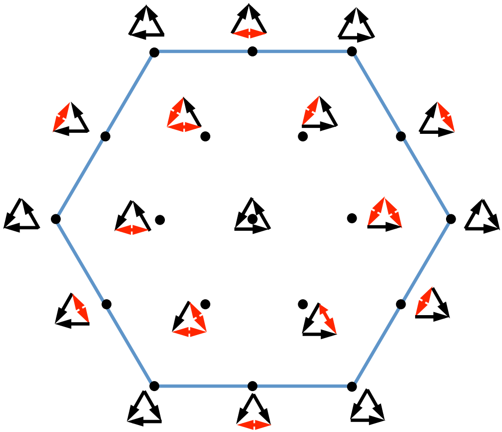


Figure: Cycle-minimal orientations and lattice points in $2Z_{K_3}$. The bioriented edges are labeled red.

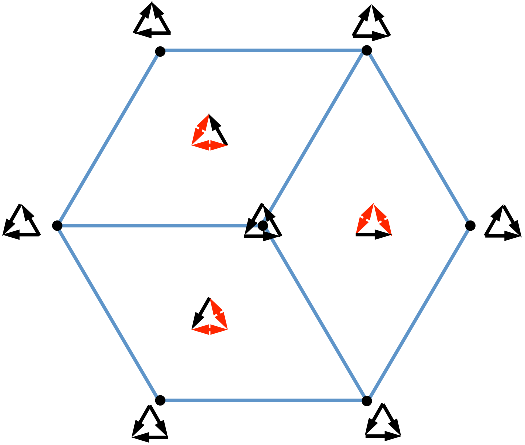


Figure: The limit of cycle minimal orientations of qK_3 as $q \to \infty$.

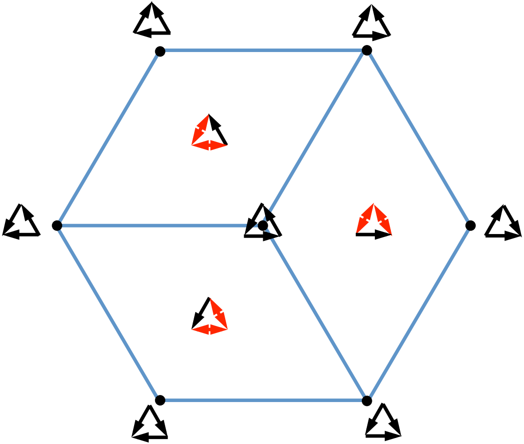


Figure: The limit of cycle minimal orientations of qK_3 as $q \to \infty$.

The edges in the complement of each spanning tree are oriented precisely as in our bijection!

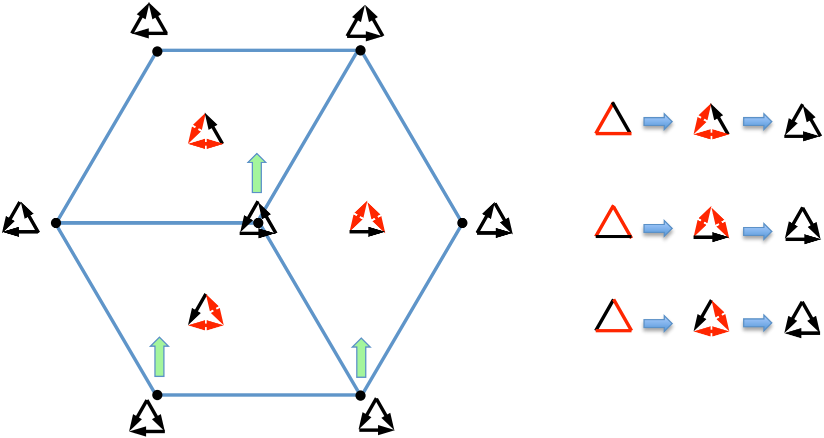


Figure: The other half of our bijection comes from a shifting map!

Torsors

- The set of orientation classes is not a group.
- Let G be a group acting on X. If G acts freely and transitively on X we say X is a torsor for G.

Theorem

[B.] (for graphs) and [B. Baker, Yuen] (for regular matroids)
This set of orientation classes is canonically a torsor for the Jacobian, which acts via edge reversals.

- By fixing an orientation we obtain a bijection between the Jacobian and the orientation classes and, by composition with our bijection, the spanning trees of G.

Summary

We have four different equinumerous sets of objects counted by $T_M(1,1)$:

1. Bases
2. Circuit-cocircuit minimal orientations
3. Circuit-cocircuit reversal classes
4. Jacobian

- We give a bijection between bases and circuit-cocircuit minimal orientations
- The cut-cycle minimal orientations give distinguished representatives for the circuit-cocircuit reversal classes.
- The circuit-cocircuit reversal classes are a canonical torsor for the Jacobian.
- Thus we obtain a bijection between the bases and the Jacobian.

Merci!