The Geometry of Geometries

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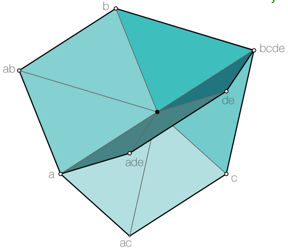
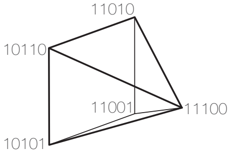




Geometry and Combinatorics. Two visionary remarks.

example is so beautiful that we decided to publish it independently of the applications. We believe that combinatorial methods will play an increasing role in the future of geometry and topology.
We consider the Grassmann manifold G^k_{n-k} of all $(n-k)$-dimensional

Gelfand–Goresky–MacPherson–Serganova, 1987

of dedication and lasting achievements, we were struck by one remark, which to our minds was later to prove prophetic: "We combinatorialists have much to gain from the study of algebraic geometry, if not by its direct applications to our field, at least by the analogies between the two subjects."

R. C. Bose (quoted by Kelly–Rota, 1973)

The one thing I want to say today:

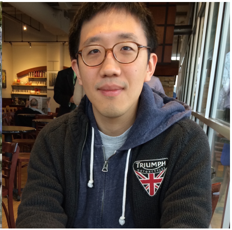
Matroids are geometric.

I will mostly talk about other people's work.

If I have time, I'll discuss some of my joint work with
Carly Klivans (06), Carolina Benedetti + Jeff Doker (10)
Marcelo Aguiar (08-17), Graham Denham + June Huh (17-18).



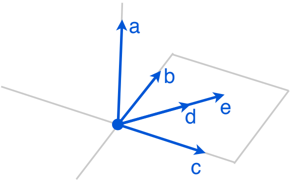


Matroids

Goal: Capture the combinatorial essence of independence.

E= set of vectors spanning $\mathbb{R}^d$.
$\mathcal{B}$ = collection of subsets of E which are bases of $\mathbb{R}^d$.



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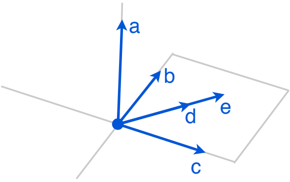

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Properties:
(B1) $\mathcal{B} \neq \emptyset$
(B2) If $A, B \in \mathcal{B}$ and $a \in A - B$, then there exists $b \in B - A$ such that $(A - a) \cup b \in \mathcal{B}$.

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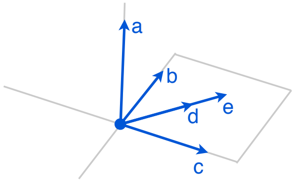

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Definition. (Nakasawa, Whitney, 35)
A set E and a collection $\mathcal{B}$ of subsets of E are a **matroid** if they satisfies properties (B1) and (B2).

Many points of view.

1. Bases

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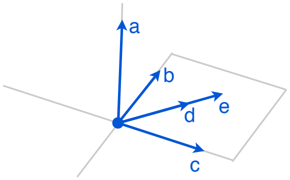

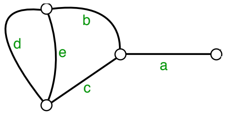

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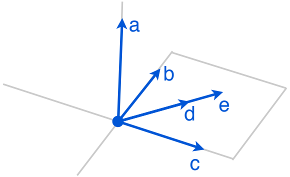

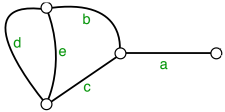

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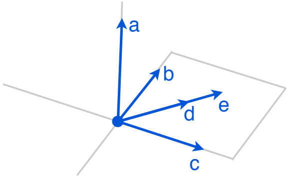
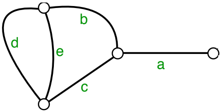
3. Circuits (minimal dependences.)

$\mathcal{C} = \{de, bcd, bce\}$





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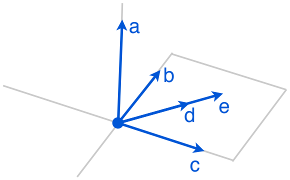

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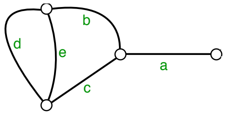

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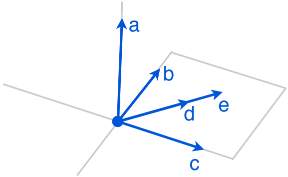

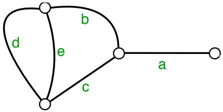

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4. Flats (spanned sets.)
$\mathcal{F} = \{abcde$
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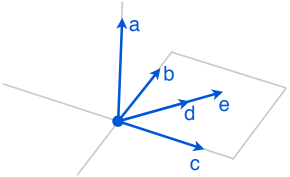
Many points of view.

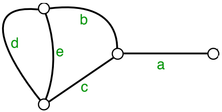
1. Bases (polytope)

2. Independents (simplicial complex)

3. (Broken) Circuits (monomial ideal)

4. Flats (lattice)





It is as if one were to condense all trends of present day mathematics onto a single finite structure, a feat that anyone would a priori deem impossible, were it not for the fact that matroids do exist.

Gian-Carlo Rota

The characteristic polynomial

The **characteristic polynomial** of M is

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Circuits (monomial ideals):

$$Hilb(\mathbb{R}[x_1, \ldots, x_n]/BC_{<}(M)) = \left(\frac{t}{t-1}\right)^r \chi_M\left(\frac{t-1}{t}\right)$$

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- $(\mathbb{F} = \mathbb{C})$ $\quad V(\mathcal{A})$ has Betti numbers = coefficients of $\chi_M(q)$.

The characteristic polynomial

Two enumerative invariants of matroids:

- f-vector: coefficients of $\chi_M(q)/q$
- h-vector: coefficients of $\chi_M(q+1)/(q+1)$

They have enum+alg+geom+top+prob interpretations.

Are matroids geometric? Take 1.

(linear matroids) vs. (all matroids):

- Almost any matroid we think of is linear (geometric).
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- Is there a “missing axiom" for linear matroids?
  No. (Mayhew, Newman, Whittle, 14)
- This is a feature, not a flaw!

Are matroids geometric? Take 2.

My main point today.
Matroids are natural geometric objects.

Three manifestations:

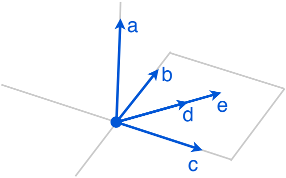
1. the matroid polytope,
2. the Bergman fan,
3. the conormal fan.

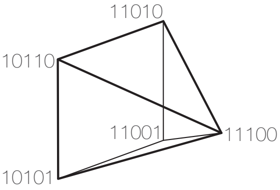
Model 1: Matroid polytopes

Def (Edmonds 70)
The **matroid polytope** of a matroid M on E is

$$P_M = \text{conv}\{e_B : B \text{ is a basis of } M\} \subset \mathbb{R}^E$$

where e_B is the $0-1$ indicator vector of B.





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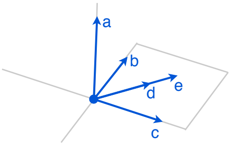

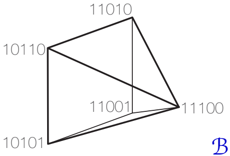

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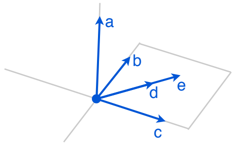

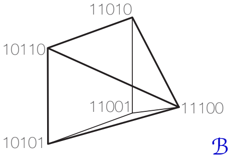

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Matroid polytopes in “nature":

1. Optimization. (Edmonds 70) For a cost function $c : E \to \mathbb{R}$, find the bases $\{b_1, \ldots, b_r\}$ of minimal cost $c(b_1) + \cdots + c(b_r)$.

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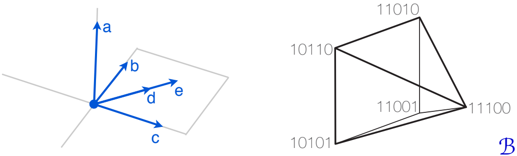

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2. Algebraic geometry. (Gel'fand–Goresky–MacPherson–Serganova 87) Understand torus orbits in the Grassmannian.

A “Zome tool" characterization of matroids

Theorem. (GGMS 87) A collection $\mathcal{B}$ of r-subsets of $[n]$ is a matroid if and only if every edge of the polytope

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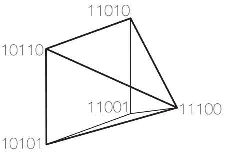
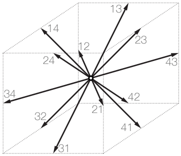
is a translate of vectors $e_i - e_j$ for some i, j.

Def. A **matroid** is a 0-1 polytope with edge directions $e_i - e_j$.





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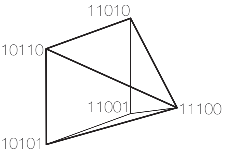

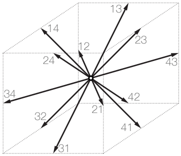

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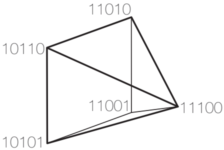
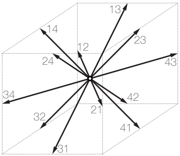
From this geometric viewpoint, all matroids are equally natural.
Matroids provide the correct level of generality!

Applications.





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$\longmapsto$ theory of matroid subdivisions (Derksen-Fink 10)

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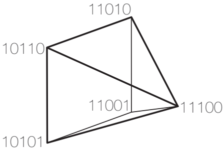

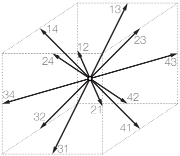

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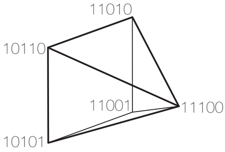

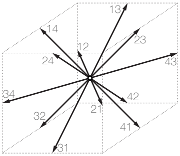

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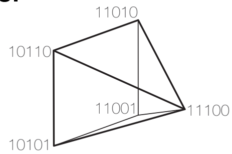
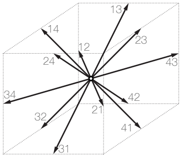
3. (Joni-Rota 78) Hopf algebra of matroids via $\oplus$, $/$, $\backslash$.

   $\longmapsto$ antipode(M) = $\sum_{P_N \leq P_M} (-1)^{\dim(P_N)} N = \pm\, \mathrm{Int}(P_M)$ (Aguiar-A. 17)

Applications.





$ij : e_i - e_j$

1. (Lafforgue 03) If a matroid polytope cannot be cut into smaller ones, its matroid has finitely many linear $\mathbb{F}$-representations for any fixed $\mathbb{F}$.

   $\longmapsto$ theory of matroid subdivisions (Derksen-Fink 10)

2. Deg(torus orbit in $Gr_{r,n}$) = Vol(matroid polytope).

   $\longmapsto$ combinatorial formula (A.-Benedetti-Doker 10)

3. (Joni-Rota 78) Hopf algebra of matroids via $\oplus$, $/$, $\backslash$.

   $\longmapsto$ antipode(M) = $\sum_{P_N \leq P_M} (-1)^{\dim(P_N)} N = \pm \operatorname{Int}(P_M)$ (Aguiar-A. 17)

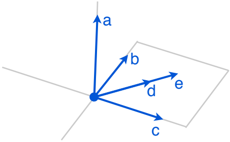
4. $\{e_i - e_j\}$ is the root system for the Lie algebra $\mathfrak{sl}_n$. Other types?

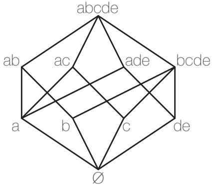
   $\longmapsto$ theory of Coxeter matroids (Gel'fand-Serganova 87)

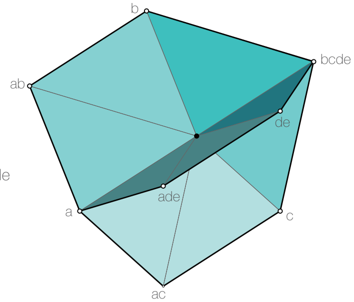
Model 2: Bergman fan

Def/Theorem. (A.-Klivans 06)
The *Bergman fan* Σ_M of M is the polyhedral complex with

- rays: $e_F := e_{f_1} + \cdots + e_{f_k}$ for each flat $F = \{f_1, \ldots, f_k\}$
- faces: $\text{cone}\{e_F : F \in \mathcal{F}\}$ for each flag $\mathcal{F} = \{\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_l \subsetneq E\}$.







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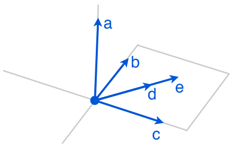

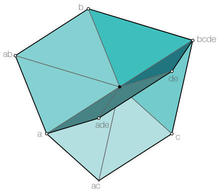

Bergman fans in “nature": Tropical geometry.

algebraic variety $V \mapsto$ Trop(V) polyhedral complex

Trop(V) still knows information about V, and can be studied combinatorially.

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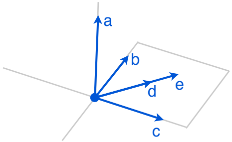

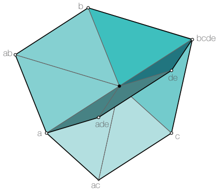

Bergman fans in "nature": Tropical geometry.

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Trop(V) still knows information about V, and can be studied combinatorially.

Question. (Sturmfels 02) Describe Trop(linear space).

Theorem. (A.-Klivans 06)
The tropicalization of a linear space $V \subseteq \mathbb{R}^n$ is the Bergman fan $\Sigma_{M(V)}$.

A tropical characterization of matroids

A **tropical variety** is a polyhedral complex “with zero-tension".
It has a **tropical degree**, and AlgDeg(V) = TropDeg(Trop V).

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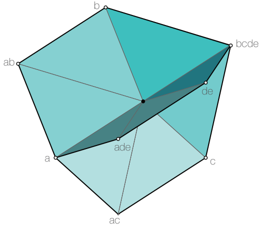
Theorem. (Fink 13) A tropical variety has degree 1 if and only if it is the Bergman fan of a matroid.

Definition. A **matroid** is a tropical variety of degree 1.

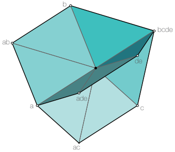


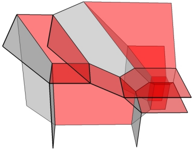
From this geometric viewpoint, all matroids are equally natural.
Again, matroids provide the correct level of generality!

Applications.

1. A **tropical manifold** is a tropical variety that looks locally like a (Bergman fan of a) matroid.

$\longmapsto$ theory of tropical manifolds (Mikhalkin, Rau, Shaw, ...)





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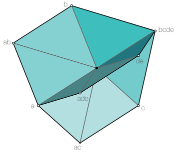

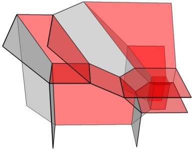

2. (Adiprasito-Huh-Katz 18) A combinatorial **Chow ring** of Σ_M behaves like the cohomology ring of a smooth projective variety. (**!!!**) This gives that the coefficients of the characteristic polynomial

$$\chi_G(q) = w_{v-1}q^{v-1} - w_{v-2}q^{v-2} + \cdots \pm w_1$$

are unimodal and log-concave:

$$w_1 \leq \cdots w_{k-1} \leq w_k \geq w_{k+1} \geq \cdots \geq w_{v-1}$$

$$w_{i-1}w_{i+1} \leq w_i^2 \qquad \text{for } i = 1, \ldots, v-2.$$

This was conjectured by Read (68) and Hoggar (74).

Model 3: conormal fan

Definition. (A.-Denham-Huh 17)
A **biflag** of M consists of a flag $\mathcal{F} = \{F_1 \subseteq \cdots \subseteq F_l\}$ of flats and a flag $\mathcal{G} = \{G_1 \supseteq \cdots \supseteq G_l\}$ of coflats (flats of $M^\perp$) such that

$$\bigcap_{i=1}^{l} (F_i \cup G_i) = E, \qquad \bigcup_{i=1}^{l} (F_i \cap G_i) \neq E.$$

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Question: Have you run into this before?!

All maximal biflags have length $n-2$ and we define:

Definition. (A.-Denham-Huh 17)
The *conormal fan* $\Sigma_{M,M^\perp}$ is the polyhedral complex in $\mathbb{R}^{E \sqcup E}$ with

- rays $e_F + f_G$ for each flat F and coflat G with $F \cup G = E$
- $cone(\mathcal{F}, \mathcal{G}) := cone\{e_{F_i} + f_{G_i} : 1 \leq i \leq l\}$ for each biflag $(\mathcal{F}, \mathcal{G})$.

Applications.

1. The conormal fan seems to be a Lagrangian analog of the Bergman fan.

Expectations:

- Conormal fans are the tropical Lagrangian linear spaces.
- They're the building blocks for tropical Lagrangian submanifolds Mikhalkin'18
- Again, (Lagrangian?) matroids provide the correct level of generality.

Applications.

1. The conormal fan seems to be a Lagrangian analog of the Bergman fan.

Expectations:
- Conormal fans are the tropical Lagrangian linear spaces.
- They're the building blocks for tropical Lagrangian submanifolds Mikhalkin'18
- Again, (Lagrangian?) matroids provide the correct level of generality.

2. (A.-Denham-Huh 18) The combinatorial **Chow ring** of $\Sigma_{M,M^\perp}$ **also** behaves like the cohomology ring of a smooth projective variety. (**!!!**) This gives that the coefficients of the shifted characteristic polynomial

$$\chi_G(q+1) = h_{v-1}q^{v-1} - h_{v-2}q^{v-2} + \cdots \pm h_1$$

are unimodal, log-concave, and flawless:

$$h_1 \leq \cdots h_{k-1} \leq h_k \geq h_{k+1} \geq \cdots \geq h_{v-1}$$

$$h_{i-1}h_{i+1} \leq h_i^2 \qquad \text{for } i = 1, \ldots, v-2.$$

$$h_i \leq h_{s-i} \qquad \text{for the nonzero entries.}$$

This was conjectured by Brylawski (82), Dawson (83) and Swartz (03).
It strengthens Adiprasito-Huh-Katz 18, and requires additional machinery.

merci beaucoup