Vectors of matroids over hyperfields

Laura Anderson

Binghamton University

Why we love covectors

1. Matroids over the sign hyperfield $\mathbb{S}$ are oriented matroids, and $\mathbb{S}$-covector sets satisfy the Topological Representation Theorem.
2. Matroids over a field F correspond to linear subspaces V of F^E, via:
   - Grassmann-Plücker function = Plücker coordinates
   - set $\mathcal{C}^*$ of F-cocircuits = set of elements of $V - \{\mathbf{0}\}$ of minimal support
   - set $\mathcal{C}$ of F-circuits = set of elements of $V^\perp - \{\mathbf{0}\}$ of minimal support

   So for matroids over general hyperfields, we hope for $\mathcal{V}$ and $\mathcal{V}^*$ with
   - $\mathcal{V}^* = V$
   - $\mathcal{V} = V^\perp$

We'll define H-vectors and H-covectors for matroids over a hyperfield H so that

1. when $H = \mathbb{S}$ we get the usual signed vectors and signed covectors of an oriented matroid,
2. when H is a field, and thus an H-matroid corresponds to a subspace V of H^E, we get $\mathcal{V}^* = V$ and $\mathcal{V} = V^\perp$.

This definition will capture the idea of an H-matroid being a "linear subspace of H^E", in two senses:

1. a linear subspace of a vector space is the span of a set of elements, and
2. a linear subspace of a vector space is the solution set to a system of linear equations $\mathbf{a} \cdot \mathbf{x} = 0$.

Linear algebra over hyperfields?

Naive linear algebra over hyperfields is a mess...
A linear combination of $X_1, \ldots, X_k \in H^E$ is a set $\boxplus_{i=1}^{k} a_i X_i \subseteq H^E$ with each $a_i \in H$. The span $\langle X_1, \ldots, X_k \rangle$ is the union of all linear combinations of the X_i.

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Nothing about this works well: for instance, we can have $Y \in \langle X_1, \ldots, X_k \rangle$ but $\langle Y, X_1, \ldots, X_k \rangle \neq \langle X_1, \ldots, X_k \rangle$.

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The inner product $X \cdot Y$ of two elements of H^E is the hypersum $\boxplus_{e \in E} X(e)Y(e)$. We say $X \perp Y$ if $0 \in X(e) \cdot Y(e)$.

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This is also problematic: for instance, we can have $X, Y \perp Z$ but $X \boxplus Y \not\perp Z$.
In general, sets $S^\perp$ and $\langle S \rangle$ don't seem to have any nice relationship to matroid theory.
However, H-matroids offer a less naive route to linear algebra over hyperfields.

Motivation: reduced row echelon forms

For a field F and rank r linear subspace V of F^n, $V = \mathrm{row}(A)$ for some $r \times n$ matrix $A = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ over F.
V has an underlying matroid M. The columns of A constitute a vector arrangement realizing M in the usual sense.
A is well-defined up to left multiplication by GL_r (i.e. up to change of coordinates for the vector arrangement).

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$B = \{b_1, \ldots, b_r\} \subseteq \{1, \ldots, n\}$ is a basis for M
$\Leftrightarrow$ there is a $G \in GL_r$ such that $(GA)_{b_1,\ldots,b_r} = I$.

We call GA the **reduced row-echelon form** (RREF) for V with respect to B.
The set of F-cocircuits of the F-matroid corresponding to V is exactly the set of scalar multiples of rows appearing in RREFs.

Now consider a hyperfield H and an H-matroid M.
For each basis B and each $a \in B$ there is a unique **fundamental H-cocircuit** $C_{a,B} \in \mathcal{C}^*$, i.e. a unique H-cocircuit satisfying $C_{a,B}(a) = 1$ and $C_{a,B}(c) = 0$ for all $c \in B - \{a\}$.

Definition

For each basis B define the RREF of M with respect to B to be $\{C_{a,B} : a \in B\}$.

Equivalently, this is the unique set of elements of $\mathcal{C}^*$ arising as the rows of a matrix over H in RREF with respect to B.

Theorem

For an H-matroid M with H-circuit set $\mathcal{C}$,

$$\mathcal{C}^{\perp} = \bigcap_{B} \langle C_{a,B} : a \in B \rangle$$

where the intersection is over all bases of M.

We define

$$\begin{aligned} \mathcal{V}^{*} &= \mathcal{C}^{\perp} \\ &= \text{the set of elements of } H^{E} \text{ which lie in the span} \\ &\quad \text{of every RREF for } M \end{aligned}$$

and $\mathcal{V} = (\mathcal{C}^{*})^{\perp}$.

H-vector axioms

Given $\emptyset \neq \mathcal{W} \subseteq H^E$, define a **basis** to be a minimal $B \subseteq E$ such that $B \cap \mathrm{s}upp(X) \neq \emptyset$ for each $X \in \mathcal{W} - \{\mathbf{0}\}$.
Then for every basis B there is a **near-RREF** in $\mathcal{W}$ corresponding to B, i.e. a set $\{D_{a,B} : a \in B\} \subseteq \mathcal{W}$ such that for every $a, c \in B$, $D_{a,B}(c) = 0$ iff $a \neq c$.

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The Vector Axiom

$\mathcal{W}$ is the H-vector set of an H-matroid if and only if $\mathcal{W}$is exactly the intersection of the spans of all near-RREF's in $\mathcal{W}$.

Equivalently, $\mathcal{W}$ has a RREF with respect to every basis B, and $\mathcal{W}$ is exactly the set of elements of H^E that can be written as a linear combination of each RREF in $\mathcal{W}$.

Theorem

This axiom is cryptomorphic to the other axiomatizations of strong H-matroids, via

- $\mathcal{V}^* = \mathcal{C}^\perp$
- $\mathcal{V} = (\mathcal{C}^*)^\perp$
- $\mathcal{C}$ *is the set of elements of* $\mathcal{V} - \{0\}$ *of minimal support, and*
- $\mathcal{C}^*$ *is the set of elements of* $\mathcal{V}^* - \{0\})$ *of minimal support.*

Further a morphism of hyperfields $H \to H'$ induces a map of H-vector sets and a map of H-covector sets from each H-matroid to the pushforward H'-matroid.

Compare to the usual oriented matroid axioms

$\mathbf{0} \in \mathcal{V}$ If $X \in \mathcal{V}$ then $-X \in \mathcal{V}$ }	These say that $\mathcal{V}$ is closed under scalar multiplication by elements of $\mathbb{S}$.(This clearly holds for all H-matroids.)
Composition Axiom Elimination Axiom }	These say that if $X, Y \in \mathcal{V}$ then certain elements of $X \boxplus Y$ are in $\mathcal{V}$.(Not so clear for H-matroids.)

The only reason we know that our new $\mathbb{S}$-vector axioms coincide with the usual signed vector axioms for oriented matroids is because both are known to give the cryptomorphism $\mathcal{V}^* = \mathcal{C}^\perp$. It would be good to have axioms (or even just results) similar to the Composition and Elimination Axioms for general H-matroids.

Composition Axioms

Definition

A composition operation *on a hyperfield H is a hyperoperation $\circ : H^E \times H^E \to 2^{H^E} - \{\emptyset\}$ defined for all finite E such that for every X_1 and X_2 in H^E*

- *if $Y \in X_1 \circ X_2$ then $\mathrm{supp}(Y) = \mathrm{supp}(X_1) \cup \mathrm{supp}(X_2)$, and*
- *if $X_1 \perp Z$ and $X_2 \perp Z$ then $X_1 \circ X_2 \perp Z$*

One might also want to include the condition

- $X_1 \circ X_2 \subseteq X_1 \boxplus X_2$

If a hyperfield H admits a composition operation, then every H-vector set is closed under that composition.

Example

If $H = \mathbb{R}$ then we can define $\circ$ by $X \circ Y = \{X + \epsilon Y : \epsilon < min(|X(e)/Y(e)| : e \in \mathrm{s}upp(X) \cap \mathrm{s}upp(Y))\}$. This is a composition on $\mathbb{R}$-matroids that pushes forward to the usual composition on oriented matroids.

Example

If H is the tropical hyperfield, then we can define a single-valued composition $\circ$ by

$$(X \circ Y) = \begin{cases} X(e) & \text{if } X(e) \geq Y(e) \\ Y(e) & \text{otherwise} \end{cases}$$

Example

Finite fields do not admit composition operations. Thus there is no axiomatization of H-covector sets over arbitrary H which includes a Composition Axiom.

Elimination Axioms?

Elimination Conjecture

Let $\mathcal{V}$ be an H-vector set on elements E, $e \in E$, and $X, Y \in \mathcal{V}$. If $0 \in X(e) \boxplus Y(e)$ then there is a $Z \in (X \boxplus Y) \cap \mathcal{V}$ such that $Z(e) = 0$.

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Additive Continuum Conjecture

Let $\mathcal{V}$ be an H-vector set on elements E, $e \in E$, and $X, Y \in \mathcal{V}$. If $\alpha \in X(e) \boxplus Y(e)$ then there is a $Z \in (X \boxplus Y) \cap \mathcal{V}$ such that $Z(e) = \alpha$.

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Mortifying Conjecture

Let $\mathcal{V}$ be an H-vector set and $X, Y \in \mathcal{V}$. Then $(X \boxplus Y) \cap \mathcal{V} \neq \emptyset$.

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We call a hyperfield H *perfect* if $\mathcal{V}^* = \mathcal{V}^\perp$ for all H-matroids.
Matt Baker and Nathan Bowler showed that all doubly distributive hyperfields are perfect.
Ting Su showed that for matroids over perfect hyperfields, $\mathcal{V}^*(M/A) = \{V \backslash A : X \in \mathcal{V}^*(M), X(A) = 0\}$ (and thus also $\mathcal{V}(M \backslash A) = \{V \backslash A : X \in \mathcal{V}(M), X(A) = 0\}$)

Conjecture

For matroids over perfect hyperfields,
$\mathcal{V}^*(M \backslash A) = \{V \backslash A : X \in \mathcal{V}^*(M)\}$

Topological Representation Theorems?

If a hyperfield H has a topology then this topology is inherited by every H-vector set $\mathcal{V} \subseteq H^E$.

Example

$\mathbb{S}$ has the topology with open sets $\emptyset$, $\{+\}$, $\{-\}$, $\{0, +, -\}$.
A theorem of McCord tells us that the induced topology on the set $\mathcal{V}^* - \{\mathbf{0}\}$ of nonzero covectors of a rank r oriented matroid has the weak homotopy type of the order complex of $\mathcal{V}^* - \{\mathbf{0}\}$. Hence a weak version of the Topological Representation is: the finite space $\mathcal{V}^* - \{\mathbf{0}\}$ has the weak homotopy type of S^{r-1}.

1. For every hyperfield H, the H-covector set of a rank r H-matroid on r elements is isomorphic to H^r.
2. If M and $M \backslash A$ have the same rank then we have a map $\mathcal{V}^*(M) - \{\mathbf{0}\} \to V^*(M \backslash A) - \{\mathbf{0}\}$.

Can we characterize the topological hyperfields H for which this map is a homotopy equivalence (and hence each $\mathcal{V}^*(M) - \{\mathbf{0}\}$ has the homotopy type of $H^r - \{0\}$)?

Can we characterize the hyperfield morphisms which (like $\mathbb{R} \to \mathbb{S}$) induce weak homotopy equivalences of the nonzero covector sets of matroids over these hyperfields?