On the existence of algebraic approximations of compact Kähler manifolds

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Plan

The Kodaira embedding theorem

Theorem (Kodaira '54)

Let X be a compact complex manifold. Equivalent statements :

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Definition

A Kähler form on X is a real 2-form ω of type $(1,1)$ which is closed and positive.

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A Kähler form on X is a real 2-form ω of type $(1, 1)$ which is closed and positive.
A Kähler manifold is a complex manifold with a Kähler form.

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Definition (Algebraic approximations)

Let X be a compact Kähler manifold. An algebraic approximation of X is a deformation

$$\Pi : \mathcal{X} \to \Delta$$

of X such that the subset parameterizing projective fibers is dense in Δ.

The Kodaira problem

Question (The Kodaira problem)

Does a compact Kähler manifold always have an algebraic approximation ?

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- $\dim \geq 4$: No (Voisin '04) ; (Oguiso '08)
  $\exists$ effective homotopical obstruction for being projective.

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The Kodaira problem and main statements

Other positive results :

- Some conic bundles over a surface. (Schrack '11)

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Conjectural positive result :

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Theorem (- '18)

Every compact Kähler threefold has an algebraic approximation.

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X : compact complex manifold.

Definition (Algebraic dimension)

The algebraic dimension $a(X)$ of X is defined by

$$a(X) = \mathrm{trdeg}(\mathcal{M}(X)).$$

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For non-projective compact Kähler manifolds, $a(X) \leq \dim X - 1$.

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Theorem (- '18)

Every compact Kähler manifold X *with* $a(X) = \dim X - 1$ *has an algebraic approximation.*

L. : Algebraic approximations of compact Kähler manifolds of algebraic codimension 1. arXiv :1809.03344, 2018.
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Theorem (Variant)

Every compact Kähler manifold bimeromorphic to the total space of an elliptic fibration $f : X' \to B$ *over a projective variety has an algebraic approximation.*

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Claudon, Höring, L. ('18) : Bimeromorphic version.

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Corollaries

Corollary

Let X be a compact complex variety with at worst rational singularities and bimeromorphic to a compact Kähler manifold. If $\dim X = 3$ or $a(X) = \dim X - 1$, then X has an algebraic approximation.

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$$f_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X \text{ and } R^1 f_* \mathcal{O}_{\tilde{X}} = 0,$$

so deformations of $\tilde{X} \rightsquigarrow$ deformations of f. (Ran)

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Corollary (Fundamental groups : Claudon, Höring, L. ('18))

Invariants of X that are preserved under small deformations (e.g. the fundamental group and the Hodge diamond) can be realized by some projective manifold.

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A general approach

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1. a bimeromorphic model $X \dashleftarrow\dashrightarrow X'$,
2. an alg. app. of X' which induces a deformation of $X \dashleftarrow\dashrightarrow X'$.

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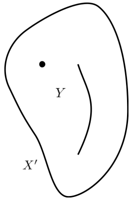

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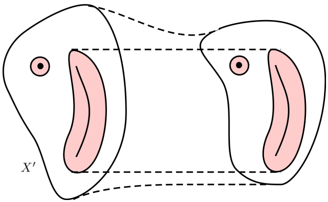
A deformation of X' in which a neighborhood of Y deforms trivially



induces a deformation of $\tilde{X}$, therefore (Ran) a deformation of X.

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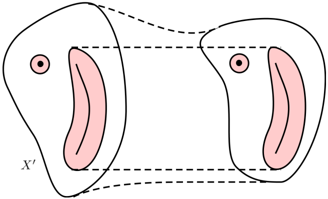
Definition (Locally trivial deformations)

Let $Y \subset X$ be a subvariety. A deformation $\Pi : \mathcal{X} \to \Delta$ of X is Y-locally trivial if a neighborhood of Y deforms trivially in Π along Δ.



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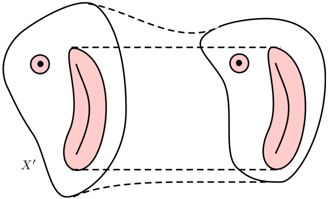

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Lemma

If X' is normal and has a Y-locally trivial algebraic approximation for every subvariety $Y \subset X'$ with $\dim Y \leq \dim X - 2$, then X has an algebraic approximation.

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$Y \subset X'$: a subvariety such that $\nu^{-1}_{|X' \setminus Y}$ is an isomorphism onto its image.

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Proposition

If X' has an algebraic approximation $\Pi : \mathcal{X}' \to \Delta$ such that the formal completion $\hat{Y}$ of X' along Y deforms trivially in Π, then X has an algebraic approximation.

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Proof based on Ancona-Tomassini-Bingener's work on formal modifications.

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Plan

Kähler MMP for threefolds

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The MMP is known for threefolds.

(Projective : Kollár, Mori et al. 8/90s ; Kähler : Campana, Höring, Peternell '15.)

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- $\kappa(X) \geq 0$: $X \dashrightarrow X_{\min}$ minimal model of X.

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- $\Phi_m : X_{\min} \xrightarrow{|mK_{X_{\min}}|} X_{\mathrm{can}} \subset \mathbf{P}^N$ is well-defined when $m \gg 0$.

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- $\Phi_m : X_{\min} \xrightarrow{|mK_{X_{\min}}|} X_{\mathrm{can}} \subset \mathbf{P}^N$ is well-defined when $m \gg 0$.
- General fiber F : connected (for $m \gg 0$) and $c_1(K_F)$ is torsion.

J. Kollár and S. Mori : Birational geometry of algebraic varieties, volume 134 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1998.
A. Höring and T. Peternell : Bimeromorphic geometry of Kähler threefolds. arxiv :1701.01653, 2017. To appear in "Proceedings of 2015 Summer Institute on Algebraic Geometry".

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Lemma

In case 1, $X \dashleftarrow\dashrightarrow \tilde{X}/G$ where G is a finite group and $\tilde{f} : \tilde{X} \to \tilde{B}$ is a G-equivariant fibration whose general fiber is an abelian surface.

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Case where F is non-algebraic :

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Theorem (Campana '06)

Let $f : X \to B$ be a fibration whose total space X is a compact Kähler manifold and whose general fiber F is a non-algebraic K-trivial surface. Then f is isotrivial.

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Proposition

In Case 2, $X \dashleftarrow\dashrightarrow \tilde{X}/G$ where G is a finite group and $\tilde{f} : \tilde{X} \to \tilde{B}$ is a G-equivariant smooth isotrivial fibration in K3 surfaces or 2-tori.

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Corollary

A standard conic bundle $f : X \to S$ over a surface with $a(S) = 0$ is a $\mathbf{P}^1$-bundle.

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X : compact complex manifold

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X is called algebraically connected if $\forall$ general $x, y \in X$, $\exists$ a connected compact algebraic curve $C \subset X$ such that $x, y \in C$.

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Smooth torus fibrations (after Claudon)

$f : X \to B$: smooth torus fibration of relative dimension g, B compact.

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$J \to B$: the associated Jacobian fibration.

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Exact sequence :

$$0 \longrightarrow \mathbf{H} \longrightarrow \mathcal{E} \xrightarrow{\exp} \mathcal{J} \longrightarrow 0$$

$$\mathbf{H} := R^{2g-1} f_* \mathbf{Z}, \quad \mathcal{E} := R^g f_* \Omega^{g-1}_{X/B}.$$

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There exists a family of J-torsors

$$\Pi : \mathcal{X} \xrightarrow{q} B \times V \to V$$

such that $t \in V$ parameterizes the J-torsor which corresponds to

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The family Π is called the tautological family associated to f.

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Theorem (Claudon 16')

Let $f : X \to B$ be a smooth fibration in abelian varieties.
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- ϕ is the projection of a HS of weight 2 to its (0,2)-part (Deligne). $\Rightarrow H^1(B, \mathbf{H}) \otimes \mathbf{Q} \to H^1(B, \mathcal{E})$ has dense image.

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B. Claudon : Smooth family of tori and linear Kähler groups. Annales de la Faculté des Sciences de Toulouse, 2016.

Smooth torus fibrations (after Claudon)

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Definition (Locally trivial deformations over the base)

A deformation

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of a fibration $f : X \to B$ (preserving the base B) is called locally trivial over B if there exists an open cover $\{U_i\}$ of B such that

$$q^{-1}(U_i \times V) \simeq f^{-1}(U_i) \times V$$

over $U_i \times V$.

Π is locally trivial over B

$\Rightarrow \Pi$ is C-locally trivial for every C contained in a finite union of fibers of f.

Fibrations in abelian varieties over a curve

$f : X \to B$: G-equivariant fibration in ab. varieties (not necessarily smooth).

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Hodge-theoretic ingredients :

- Zucker's theory on the VHS over a curve. (Zucker)
- Theory of generalized intermediate Jacobians (Deligne, El Zein, Zucker).

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Fibrations in abelian varieties over a curve

Corollary

The quotient

$$(\Pi/G) : \mathcal{X}/G \to (B/G) \times V \to V$$

of Π is an algebraic approximation of $X/G \to B/G$ which is locally trivial over B/G. In particular, it is C-locally trivial for every C contained in a finite union of fibers of $X/G \to B/G$.

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Corollary

The quotient $(\Pi/G) : \mathcal{X}/G \to V$ is an algebraic approximation of X/G which is C-locally trivial for every $C \subset X/G$ with $\dim C \leq 1$.

Corollary $\Rightarrow$ Case 2.

Elliptic fibrations

$f : X \to B$: G-equivariant elliptic fibration.

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Assumptions :

- f has local meromorphic sections at every point of B.
- The local monodromies of $\mathbf{H} = (R^1 f_* \mathbf{Z})_{|B \setminus \Delta}$ around Δ are unipotent.

Elliptic fibrations

Theorem (First part : Claudon-Höring-L.)

Up to replacing $f : X \to B$ by a bimeromorphic model of it, there exists a G-equivariant tautological family

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associated to f, which is an algebraic approximation of f. Furthermore, if $Y \to Z$ is a fibration contained in f which has a multi-section, then the tautological family Π contains a subfamily which is an algebraic approximation of f preserving $\hat{Y} \to \hat{Z}$.

N. Nakayama : On Weierstrass models. In Algebraic geometry and commutative algebra, Vol.II, pages 405-431. Kinokuniya, Tokyo, 1988.
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Main ingredient : N. Nakayama's work on elliptic fibrations.

Theorem + $\varepsilon \Rightarrow$ Case 1.

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Plan

Theorem (Ueno et al., '75)

Let X be a compact Kähler manifold. If $a(X) = \dim X - 1$, then

1. *X is bimeromorphic to an elliptic fibration over a projective variety.*

X : non-algebraic compact Kähler threefold.

Proposition ($\kappa = 0$ or 1)

If $\kappa(X) = 0$ or 1, then X is bimeromorphic to $\tilde{X}/G$ where G is a finite group and $\tilde{X}$ a smooth compact Kähler threefold which is one of the following :

2. *A G-equivariant fibration $\tilde{f} : \tilde{X} \to \tilde{B}$ whose general fiber is an abelian surface.*
3. *A G-equivariant smooth isotrivial fibration $\tilde{f} : \tilde{X} \to \tilde{B}$ in K3 surfaces or tori.*

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$f : X \to B$ has an algebraic approximation, which is G-equivariant and C-locally trivial for every $C \subset X$ with $\dim C \leq 1$.

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Algebraic approximation of f realized by the family of tautological families associated to $\{f_t : X_t \to B\}_{t \in \Delta}$ where

- $\mathcal{T} \to \Delta$ is an algebraic approximation of T
- f_t is a smooth isotrivial fibration in $\mathcal{T}_t$.

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Lemma follows from :

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Let S be a $K3$ surface and $G \curvearrowright S$ a finite group. Then S has a G-equivariant algebraic approximation which is C-locally trivial for every subvariety $C \subsetneq S$.

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(S, E) has an algebraic approximation. (Schrack)

Check local triviality.

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If E does not have any sub-sheaf of rank 1,

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Theorem

Assume that tr *is injective. Let* $\mathcal{S} \to \Delta$ *be a deformation of* S*. If* $\mathrm{tr}(\mathrm{At}(E)) \in H^1(S, \Omega^1_S)$ *remains of type* $(1,1)$ *along* Δ*, then* $\mathcal{S} \to \Delta$ *lifts to a deformation of* (S, E)*.*

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Theorem ($\Rightarrow$ Case 5)

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Plan

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Known examples X answering negatively the Kodaira problem all satisfy

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END

– END –