Remarks on the Riemann Hypothesis

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Introduction

For a function $f \geq 0$ with $\int_{-\infty}^{\infty} f(u)du < \infty$; let

$$L_{f,\lambda}(w) := \int_{-\infty}^{\infty} e^{wu} e^{\lambda u^2} f(u)du$$

for $w \in \mathbb{C}, \ \lambda \in \mathbb{R}$ where possible (e.g., $\lambda < 0$); and

$$L_{\rho,\lambda}(w) := \int_{\mathbb{R}} e^{wu+\lambda u^2} d\rho(u).$$

We take f (or ρ) to be even (and ρ is usually a probability measure).

Riemann Hypothesis (RH) — 1859

For a **specific** function Φ,

RH $\Longleftrightarrow$ zeros in $\mathbb{C}$ of $L_{\Phi,0}$ all pure imaginary;

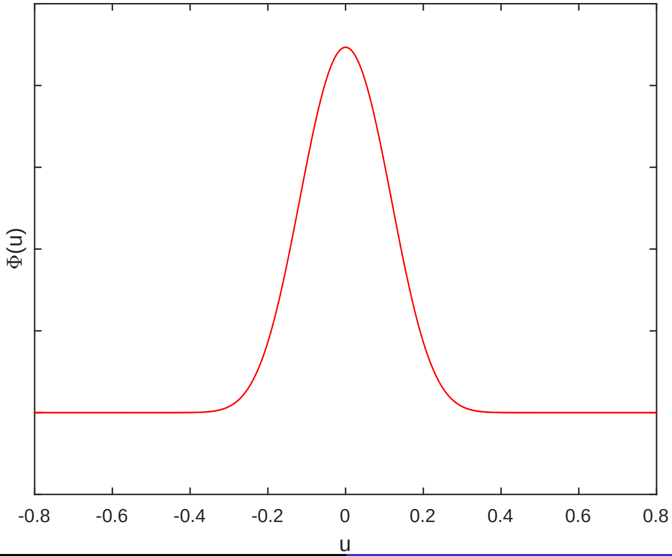
we'll say $L_{\Phi,0}$ is PIZ. Φ is defined so that

$$L_{\Phi,0} = Cs(s-1)\pi^{-s/2}\Gamma(s/2)\sum_{1}^{\infty} n^{-s}\Big|_{s=1/2+w/2}$$

and its explicit formula is

$$\Phi = \sum_{1}^{\infty}\left(n^4\pi^2 e^{9u} - \frac{3}{2}n^2\pi e^{5u}\right)e^{-n^2\pi e^{4u}}$$

Graph of Φ



Some History

- Polya '20s: hoped that $L_{\Phi,\lambda}$ is PIZ $\forall\lambda \in \mathbb{R}$; he proved that PIZ for $\lambda_1 \implies$ PIZ for $\lambda \geq \lambda_1$. (I.e., increasing/decreasing λ helps/hurts PIZ.)
- de Bruijn '50: $L_{\Phi,\lambda}$ is PIZ for $\lambda \geq 1/2$. (Based on zeros of $L_{\Phi,0}$ being in critical strip.)
- N. '76: $\exists\lambda$ s.t. $L_{\Phi,\lambda}$ is **not** PIZ and thus $\exists\Lambda \in (-\infty, 1/2]$ s.t. PIZ for $\lambda \geq \Lambda$ but not for $\lambda < \Lambda$. Λ is now called the **de B-N constant**.

$$\text{RH} \iff \Lambda \leq 0$$

Some History

- de B. '50: $\Lambda \leq 1/2$,
- N. '76: $\Lambda > -\infty$.

There is also

> N. '76 Conjecture: $\Lambda \geq 0$;
> i.e., the RH, if true, is only barely so.

$\exists$ series of bounds on Λ better than $\Lambda > -\infty$:

- $\Lambda > -50$ (Csordas-Norfolk-Varga '88), ... ,
- $\Lambda > -4.3 \times 10^{-6}$ (Csordas-Smith-Varga '94), ... ,
- $\Lambda > -1.1 \times 10^{-11}$ (Saouter-Gourdon-Demichel '11);

also $\Lambda < 1/2$ (Ki-Kim-Lee '09).

Update

- B. Rodgers - T. Tao (arXiv 18 Jan 2018):

> Proof of N. Conjecture: $\Lambda \geq 0$

Methods — extend Csordas-Smith-Varga work to study motion in t of zeros of $L_{\Phi,t}$.

- New Project (see terrytao.wordpress.com) to improve upper bound $\Lambda < 1/2$ of Ki-Kim-Lee: this is Polymath 15 project; as of 04/30/18: $\Lambda < 0.28$.

Mathematical Physics Background

Math. Phys. interest starts from the '52 Ising model Thm. of Lee and Yang that generates ρ's s.t. $L_{\rho,\lambda}$ is PIZ for $\lambda \geq 0$.

For Euclidean Field Theory, would like $f = e^{-V}$ s.t. $L_{f,\lambda}$ is PIZ also for all $\lambda < 0$; call such an f "perfect".

Example, Polya '20s, Simon-Griffiths '73

$$e^{-au^4 - bu^2} \text{ for } a > 0, b \in \mathbb{R} \text{ is perfect.}$$

Motivated by $e^{-a\cosh(u)}$, N '76 determined all perfect f's; they did **not** include $e^{-a\cosh(u)}$ or Φ of RH (which proved $\Lambda > -\infty$).

Some related results

Theorem A (N., Wei WU '17)

If $\int e^{\lambda u^2} d\rho = \infty \ \forall \ \lambda > 0$*; then for every* $\lambda < 0$, $L_{\rho,\lambda}$ *is not PIZ.*

Proof is based on a surprising weak convergence result (Thm. B below). ($\exists$ also a connection to Gaussian Multiplicative Chaos.)

Some related results

Definition

A random variable X is in $\mathscr{L}$ if:
(i) $X \overset{d}{=} -X$, and (ii) $E[e^{bX^2}] < \infty$ for some $b > 0$, and
(iii) $E(e^{zX})$ has only PIZ.

Theorem B (N., WU '17)

*If each $X_n \in \mathscr{L}$ (**with $\mathbf{b = b(X_n)}$**) and $X_n \overset{d}{\Longrightarrow} X$, then $X \in \mathscr{L}$.*

How Th. B$\Longrightarrow$ Th. A: If conclusion of Th. A not valid, then $\rho_{\lambda_0} \equiv C_{\lambda_0} e^{\lambda_0 u^2} d\rho \in \mathscr{L}$ for some $\lambda_0 < 0$; then by Polya would be in $\mathscr{L}$ $\forall \lambda \in (\lambda_0, 0)$, but $\rho_\lambda \to \rho$ as $\lambda \uparrow 0$. So by Th. B, $\rho \in \mathscr{L}$. But $\rho \notin \mathscr{L}$ since by assumptions of Th. A, it doesn't satisfy (ii).

Proof of Theorem B

Key to the proof of Th. B is a Hadamard factorization:

$$X \in \mathscr{L} \Rightarrow E(e^{zX}) = e^{Bz^2} \prod_k (1 + \frac{z^2}{y_k^2})$$

with $B \geq 0$, $y_k \in \mathbb{R}$, $\sum 1/y_k^2 < \infty$ and $E(X^2) = 2(B + \sum 1/y_k^2)$.

Remark about N. '76:

A perfect $f(u)$ must be of form

$$K u^{2m} e^{-au^4 - bu^2} \prod (1 + \frac{u^2}{y_k^2}) e^{-u^2/y_k^2}$$

with $\sum 1/y_k^4 < \infty$, $a > 0$, $b \in \mathbb{R}$ (or $a = 0$, $b + \sum 1/y_k^2 > 0$).

Thanks!