Gibbs-non Gibbs transitions in different geometries: The Widom-Rowlinson model under stochastic spin flip dynamics

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Outline

Gibbs on lattice, sequentially Gibbs, marked Gibbs point processes

Hard-core and soft-core Widom-Rowlinson model

Dynamical Gibbs-non Gibbs transitions

Infinite volume Gibbs measures on the lattice

$\{-1, 0, 1\}$ local state space, particles with spin, and holes
$\mathbb{Z}^d$ lattice site space
$\Omega = \{-1, 0, 1\}^{\mathbb{Z}^d}$ infinite volume configurations

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Specification: a candidate system for conditional probabilities of an infinite volume Gibbs measure μ (probability measure on Ω) to be defined by DLR equations

$$\mu(\gamma_\Lambda(f|\cdot)) = \mu(f)$$

Definition (Specification)

Family of proper probability kernels $\gamma = (\gamma_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$ with

Consistency:

$$\gamma_\Delta(\gamma_\Lambda(d\omega|\cdot)|\tilde{\omega}) = \gamma_\Delta(d\omega|\tilde{\omega})$$

for all finite volumes $\Lambda \subset \Delta \Subset \mathbb{Z}^d$

Measurability: $\gamma_\Lambda(f|\cdot) \in \mathcal{F}_{\Lambda^c}$

Properness: $\gamma_\Lambda(1_A|\cdot) = 1_A$ for $A \in \mathcal{F}_{\Lambda^c}$

Quasilocality (regularity): $\omega \mapsto \gamma_\Lambda(f|\omega)$ should be quasilocal for f quasilocal

Gibbsian specifications on the lattice

A **Gibbsian specification** on $\Omega = \{-1, 0, 1\}^{\mathbb{Z}^d}$ for interaction potential $\Phi = (\Phi_A)_{A \Subset \mathbb{Z}^d}$ and apriori measure $\alpha \in \mathcal{M}_1(\{-1, 0, 1\})$ has kernels

$$\gamma_{\Lambda,\Phi,\alpha}(\omega_\Lambda | \omega_{\Lambda^c}) := \frac{1}{Z_\Lambda(\omega_{\Lambda^c})} e^{-\sum_{A \cap \Lambda \neq \emptyset} \Phi_A(\omega)} \prod_{i \in \Lambda} \alpha(\omega_i)$$

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First statistical mechanics task:
Given the specification $\gamma = (\gamma_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$, find the **Gibbs measures**

$$\mathcal{G}(\gamma) = \{\mu \in \mathcal{M}_1(\Omega), \mu\gamma_\Lambda = \mu, \text{ for all } \Lambda \Subset \mathbb{Z}^d\}$$

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If $|\mathcal{G}(\gamma)| > 1$ we say that the specification γ has a **phase transition**

Hardcore and softcore Widom-Rowlinson model

Hardcore Widom Rowlinson model on $\mathbb{Z}^d$ (Higuchi-Takei 2004)

$$\gamma^{hc}_{\Lambda,\alpha}(\omega_\Lambda|\omega_{\Lambda^c}) := \frac{1}{Z^{hc}_\Lambda(\omega_{\Lambda^c})} I^{hc}_\Lambda(\omega_\Lambda\omega_{\Lambda^c})\prod_{i\in\Lambda}\alpha(\omega_i),$$

hardcore-indicator $I^{hc}_\Lambda(\omega) = \prod_{i\in\Lambda} I_{(\omega_i\omega_j\neq -1,\, \forall j\sim i)}$ forbids $+-$ neighbors

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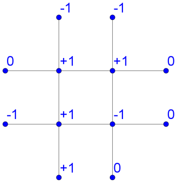

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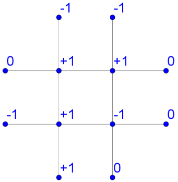

Soft-Core Widom Rowlinson model with repulsion parameter $\beta > 0$

$$\gamma^{sc}_{\Lambda,\beta,\alpha}(\omega_\Lambda|\omega_{\Lambda^c}) := \frac{1}{Z^{sc}_\Lambda(\omega_{\Lambda^c})} e^{-\beta \sum_{\{i,j\}\in\mathcal{E}^b_\Lambda} I_{(\omega_i\omega_j=-1)}} \prod_{i\in\Lambda} \alpha(\omega_i).$$

punishes $+-$ neighbors

Mean-Field Soft-Core Widom-Rowlinson model

Finite-volume Gibbs measure of size $N \in \mathbb{N}$ for the **Mean-Field** Soft-Core Widom-Rowlinson model with repulsion parameter $\beta > 0$

$$\mu_{N,\beta,\alpha}(\omega_{[1,N]}) := \frac{1}{Z_{N,\beta,\alpha}} e^{-\frac{\beta}{2N}\sum_{1\le i,j\le N} I_{(\omega_i\omega_j=-1)}} \prod_{j=1}^{N} \alpha(\omega_j)$$

Sequential Gibbsianness for mean-field (and Kac-models on torus)

Take $(\mu_N)_{N\in\mathbb{N}}$ a sequence of exchangeable probability measures on $\{-1,0,1\}^N$.

The model is called **sequentially Gibbs** iff

$$\lim_{N\uparrow\infty} \mu_N(d\omega_1|\omega_{[2,N]}) = \gamma(d\omega_1|\nu)$$

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for all **limiting empirical distributions of conditionings** $\nu \in \mathcal{M}_1(\{-1,0,1\})$.

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Marked Gibbs Point Processes in Euclidean space: Some analogy to lattice theory

$\{-1, 1\}$ mark space (has no zeros)
Ω locally finite subsets of $\mathbb{R}^d$ (for spatial degrees of freedom)
marked particle configuration $\underline{\omega} = (\omega^-, \omega^+)$
where each $\omega^-, \omega^+ \in \Omega$
$\underline{\Omega}$ marked point configurations, configuration space

$\underline{\mathcal{F}}, \underline{\mathcal{F}}_\Lambda$: σ-algebras for marked particles generated by counting variables

Specification

Candidate system for conditional probabilities of Gibbs measure μ to be defined by DLR equations

$$\mu\gamma_\Lambda = \mu$$

A family of proper probability kernels $\gamma = (\gamma_\Lambda)_{\Lambda \Subset \mathbb{R}^d}$ with consistency

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Quasilocality: Compatibility of γ_Λ with local topology

(Convergence means that sequences of marked particle clouds stabilize locally)

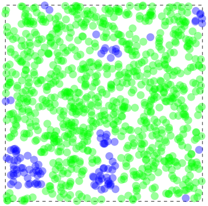
Widom-Rowlinson model in Euclidean space (1970)

Spatial dimension $d \geq 2$, two color local spin space $\{-,+\}$
base measure $\underline{P}$: two-color homogenous **Poisson Point Process**,
intensities λ_+ for plus colors and λ_- for minus colors The (hardcore)

Widom-Rowlinson specification is the Poisson-modification

$$\gamma_\Lambda(d\underline{\omega}_\Lambda|\underline{\omega}_{\Lambda^c}) := \frac{1}{Z_\Lambda(\underline{\omega}_{\Lambda^c})}\chi(\underline{\omega}_\Lambda\underline{\omega}_{\Lambda^c})\underline{P}_\Lambda(d\underline{\omega}_\Lambda)$$

where indicator χ is one iff interspecies distance is bigger or equal than $2a$



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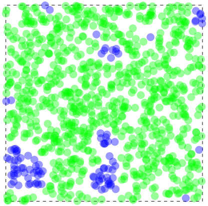

Chayes-Chayes-Kotecky 95, Ruelle 71, Bricmont-Kuroda-Lebowitz 84:
$d \geq 2$, $\lambda_+ = \lambda_-$ large $\Rightarrow$ the continuum WiRo has a **phase transition**

Hyperedge potentials

General concepts to define **Gibbsian specifications for point particles**: physical multibody interactions OR **hyperedge potentials**

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Dereudre, Drouilhet, Georgii PTRF 2012: existence theory

Jahnel-K preprint 2017: representation theorems from μ to Φ, in spirit of Kozlov-Sullivan

Independent spinflip dynamics

Define continuous time stochastic dynamics:

Particle locations stay fixed, holes stay fixed.

$+ \leftrightarrow -$ flips at rate one independently over the sites

Every site has the transition kernel

$$p_t(+,-) = \frac{1}{2}(1 - e^{-2t})$$

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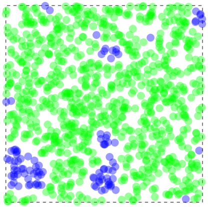
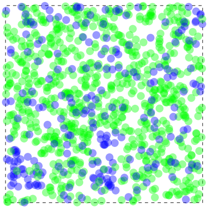
Symmetric Euclidean WiRo under spin-flip time-evolution





Dynamical Gibbs-non Gibbs transitions

We say the model shows a **dynamical Gibbs-non Gibbs transition**
if the initial measure μ is Gibbs for a quasilocal specification,
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Dynamical Gibbs-non Gibbs transitions for Ising:
Enter, Fernandez, den Hollander, Redig 02: lattice
K-LeNy 07: sequential Gibbs, broken symmetry of bad configurations
Ermolaev-K 10: mean field low temperature dynamics, Lagrangian view
Enter, Fernandez, Hollander, Redig 10: Feng-Kurtz Hamiltonian view
Enter, Ermolaev, Iacobelli, K 12: tree
Fernandez, den Hollander, Martinez 14: Kac-model
Kraaij, Redig, van Zuijlen preprint 17: mean-field Hamilton-Jacobi point of view

Relation to disordered systems

nG indicated by very long-range dependencies in **conditional probabilities:**
$\eta \mapsto \mu_t(\eta_i | \eta_{\mathbb{Z}^d \setminus i})$ behaves discontinuously w.r.t. local topology

Useful strategy (for independent dynamics): Consider **two-layer measure**

$$\bar{\mu}_t(d\omega, d\eta) = \mu(d\omega) P_t(\omega, d\eta)$$

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Analyze **hidden phase transitions**
in **first layer measure constrained on the future** η:

$$\bar{\mu}_t(d\omega|\eta)$$

Relation to disordered systems:
$\eta = (\eta_i)_{i\in\mathbb{Z}^d}$ plays the role of quenched disorder configuration

Definitions for Euclidean model

A marked infinite-volume configuration $\underline{\omega} \in \underline{\Omega}$ is called **good** for specification γ iff for any Euclidean ball B we have

$$\left|\gamma_B(f|\underline{\omega}'_{B^c}) - \gamma_B(f|\underline{\omega}_{B^c})\right| \to 0$$

as $\underline{\omega}' \Rightarrow \underline{\omega}$ in the sense of local convergence.

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μ is called **ql** (quasilocally Gibbs) iff there exists γ such that $\underline{\Omega}(\gamma) = \underline{\Omega}$

μ is called **aql** (almost surely quasilocally Gibbs) iff there exists γ with $\mu(\underline{\Omega}(\gamma)) = 1$

Gibbsian transitions in time and intensity for μ^+

Reentrance time into Gibbs $t_G := \frac{1}{2}\log\frac{\lambda_+ + \lambda_-}{\lambda_+ - \lambda_-}$ for $\lambda_+ > \lambda_-$

high-intensity (percolating) regime: $\mu^+(B \leftrightarrow \infty) > 0$ for some ball B

	$\lambda_+ > \lambda_-$			$\lambda_+ = \lambda_-$
time	$0 < t < t_G$	$t = t_G$	$t_G < t \leq \infty$	$0 < t \leq \infty$
high	non-asq	asq, non-q	q	non-asq
low	asq, non-q	asq, non-q	q	asq, non-q

Jahnel, K in AAP 2017:

uses cluster representations for time-evolved conditional probabilities

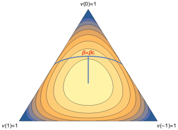
Main features: Immediate loss, full measure discontinuities

Mean-field softcore WiRo model

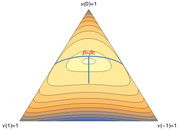
Pressure

$$p(\beta,\alpha) = \lim_{N\to\infty} \frac{1}{N} \log\left(\int e^{-N\beta L_N^1(\omega)L_N^{-1}(\omega)} \prod_{j=1}^{N} \alpha(d\omega_j)\right)$$
$$= \sup_{\nu\in\mathcal{M}_1(\{-1,0,1\})} \left(-\beta\nu(1)\nu(-1) - I(\nu|\alpha)\right)$$

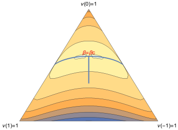
Theorem (with Sascha Kissel). The symmetric model at any $\alpha(1)=\alpha(-1)>0$ has a second order phase transition driven by repulsion strength $\beta>0$ at $\beta_c = 2 + e\frac{\alpha(0)}{\alpha(1)}$.



$\beta = 0$



$\beta = 4$



$\beta = 5 (> \beta_c = 2 + e)$

Rate-function contours, equidistribution $\alpha(0)=\alpha(1)=\alpha(-1)$

Blue line: Possible maximizers for any $\beta \geq 0$

Solution for mean-field WiRo model at time $t = 0$

Parametrize empirical spin distribution ν via coordinates (x, m)

x = occupation density, m = magnetization on occupied sites

$$\begin{pmatrix} \nu(-1) \\ \nu(0) \\ \nu(1) \end{pmatrix} = \begin{pmatrix} \frac{x}{2}(1-m) \\ 1-x \\ \frac{x}{2}(1+m) \end{pmatrix}$$

Parametrize a priori measure α via coordinates (h, l)

"magnetic field" $h := \frac{1}{2}\log\left(\frac{\alpha(1)}{\alpha(-1)}\right)$

Bias on occupations $l := \log\frac{1-\alpha(0)}{\alpha(0)}$

Pressure

$$p(\beta,\alpha) = \log \alpha(0) + \sup_{0\le x,|m|\le 1} \Big(\underbrace{-\frac{\beta x^2}{4} + x(I - \log(2\cosh(h)) - J(x)}_{\text{part for occupation density}}$$

$$+ x(\underbrace{\frac{\beta x m^2}{4} + hm - I(m) - \log 2}_{\text{Ising part at occupation-dependent temperature}}) \Big)$$

with entropies for spins and occupations

$$I(m) = \frac{1-m}{2}\log(\frac{1-m}{2}) + \frac{1+m}{2}\log(\frac{1+m}{2})$$

$$J(x) = (1-x)\log(3(1-x)) + x\log(\frac{3x}{2})$$

The antiferromagnet $\beta < 0$: Holes matter

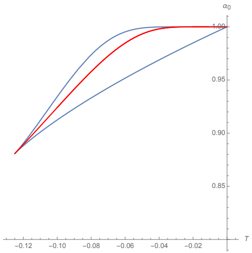
Theorem. The symmetric antiferromagnetic model has a first order transition when crossing the red Maxwell line in $\beta, \alpha(0)$-space.

Jumps occur in occupation density x, at fixed zero magnetization $m = 0$.

Pressure becomes

$$p(\beta, \alpha(0)) = \log \alpha(0) + \sup_{0 \le x \le 1} \underbrace{(-\frac{\beta}{4}x^2 - J(x) + x(I - \log(2)))}_{V(x;\beta,\alpha(0))}$$

Bifurcation set $B = \{(\beta, I), \exists x \in (0,1) : V'(x) = V''(x) = 0\}$



$\alpha(0)$ vs. β^{-1}: BLUE line bifurcation set, RED line Maxwell-line $V(x_1) = V(x_2)$

Solution by parametrization, possibly asymmetric model

Theorem. Repulsion parameter β

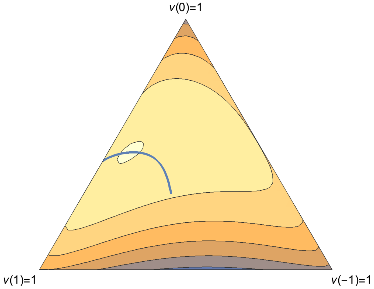
a priori measure $\alpha = \alpha(h, l)$

and typical values (m, x) of the empirical distribution

are related via the parametrization

$$\beta = \beta(m; \alpha) = \frac{2}{m}(l'(m) - h)(1 + e^{-l + \log(2\cosh(h)) + \frac{1}{m}(l'(m) - h) - m l'(m) + l(m)})$$

$$x = x(m; \alpha) = (1 + e^{-l + \log(2\cosh(h)) + \frac{1}{m}(l'(m) - h) - m l'(m) + l(m)})^{-1}$$



Critical exponents

Corollary: The model has **mean-field critical exponents:**

Fix any $\alpha(0) \in (0, 1)$.

Let β_c be the corresponding critical value for the symmetric model.

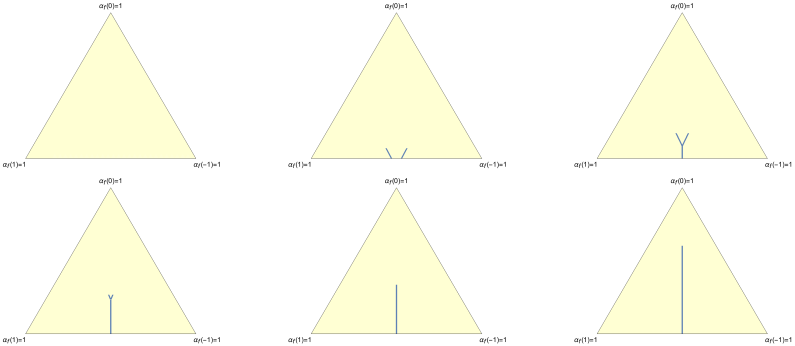
Then

$$\lim_{\beta \downarrow \beta_c} \frac{m(\beta, h = 0)}{(\beta - \beta_c)^{\frac{1}{2}}} = c$$

$$\lim_{h \downarrow 0} \frac{m(\beta_c, h)}{h^{\frac{1}{3}}} = c'$$

Dynamical GnG: Bad empirical measures of time-evolved symmetric WiRo

Bad empirical measures, time-evolution of WiRo for $\beta = 5 > 3$



Kissel-K 18:

The bad measures α_f in the time-evolved mean-field WiRo model after time t satisfy for any symmetric a priori measure α

$$B^{WiRo}(\beta, t) = \Big\{\alpha_f \in \mathcal{M}_1(\{-1,0,1\}),\; \frac{\alpha_f(1)-\alpha_f(-1)}{\alpha_f(\{1,-1\})} \in B^{Ising}\big(\frac{\beta\alpha_f(\{1,-1\})}{2}, t\big)\Big\}.$$

where $B^{Ising}(\beta_I, t)$ are bad magnetizations for the time-evolved Curie-Weiss Ising model (K-LeNy-CMP 07) with initial inverse temperature β_I.

Kissel-K 18:

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Consequences: Short-time Gibbs for all β, α

Small β-Gibbs for all times t, for $\beta \leq 2$

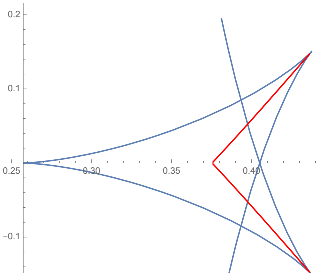
B^{WiRo} have dimension one on the simplex

β, t-regimes of disconnected curves, Y-shapes, growing antenna

Bifurcation set of first layer rate time-evolved symmetric Ising model

$B^{Ising}(\beta_I, t)$ symmetric pair OR magnetization zero OR empty

Obtained via conditional first layer rate function $M \mapsto \Psi(M; \beta_I^{-1}, t, m_{final})$
BLUE: Fixed-t slices of bifurcation set in $(\beta_I^{-1}, m_{final})$-plane
RED: Maxwell-line giving set of bad configurations $B^{Ising}(\beta_I, t)$

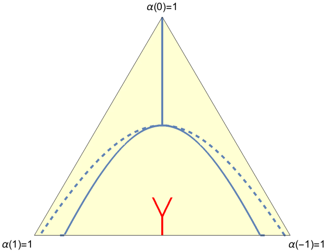


Almost surely sequentially Gibbs in time-evolved WiRo?

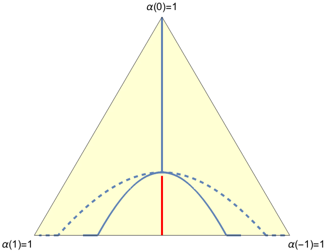
Analytical principle for Ising-systems in mean-field:
Atypicality of bad configurations follows from preservation of semiconcavity for time-evolved rate-function via integrals over Lagrange densities (Kraaij, Redig, van Zuijlen preprint 2017)

Fixed-β-typical configurations, any symmetric α, are above BLUE DOTTED line

BLUE: after time-evolution   RED: set of bad configurations at same time



very low $\beta = 4 > 3$



low $\beta = 2.8$

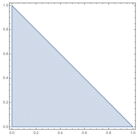
Lattice Soft-Core Widom-Rowlinson model, Dobrushin uniqueness

Let $\gamma := (\gamma_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$ be a quasilocal specification on the lattice
Dobrushin interdependence matrix

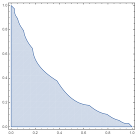
$$C_{ij}(\gamma) = \sup_{\omega_{\mathbb{Z}^d \setminus \{j\}} = \eta_{\mathbb{Z}^d \setminus \{j\}}} \|\gamma_{\{i\}}(\cdot|\omega) - \gamma_{\{i\}}(\cdot|\eta)\|_{TV,i}.$$

Theorem (Dobrushin). If $c(\gamma) := \sup_{i \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} C_{ij}(\gamma) < 1$ then $|\mathcal{G}(\gamma)| = 1$

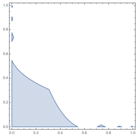
Dobrushin region for homogeneous model on $\mathbb{Z}^2$, in space of $\alpha \in \mathcal{M}(\{-1, 0, 1\})$



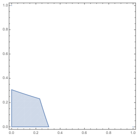
$\beta < \frac{1}{2}$



$\beta = 0.75$



$\beta = 1.05$



$\beta = 2$

Lattice Soft-Core, short-time Gibbs via Dobrushin uniqueness

Theorem (homogeneous model) (1) Let $0 \le \beta d < 1$. Then for all $\alpha \in \mathcal{M}(\{-1, 0, 1\})$ the Soft-Core Widom Rowlinson model satisfies the Dobrushin condition.

(2) For every $\beta > 0$ there exists an $\epsilon := \epsilon(\beta) > 0$ such that the Soft-Core model satisfies the Dobrushin condition if $d_{TV}(\alpha, \delta_1) < \epsilon$ or $d_{TV}(\alpha, \delta_{-1}) < \epsilon$.

Theorem (short-time Gibbs) Let $\alpha \in \mathcal{M}(\{-1, 0, 1\})$, $\beta \ge 0$, and $\mu \in \mathcal{G}(\gamma_{\beta,\alpha})$. Then there exists a time $t_c > 0$ such that for all $t < t_c$ the time evolved measure μ_t is a Gibbs measure for some quasilocal specification γ_t.

(Kissel-K, extension of Opoku-K to degenerate time-evolutions,
uses (2) to control all first-layer models for possible end-condionings)

Symmetric soft-core model: μ_t^+ is nG for t large
(since fully occupied checkerboard configuration is bad)

THANK YOU!