Manifestations of localization in the random XXZ quantum spin chain

Abel Klein
University of California, Irvine

with Alexander Elgart and Günter Stolz

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$$H_\omega = \sum_{i\in\mathbb{Z}} \left\{ \tfrac{1}{4}\left(I - \sigma_i^z \sigma_{i+1}^z\right) - \tfrac{1}{4\Delta}\left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y\right) \right\} + \lambda \sum_{i\in\mathbb{Z}} \omega_i \mathcal{N}_i,$$

acting on $\bigotimes_{i\in\mathbb{Z}} \mathbb{C}_i^2$, $\mathbb{C}_i^2 = \mathbb{C}^2$ for all i, where

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We have $\sigma(H_\omega) = \{0\} \cup \left[1 - \frac{1}{\Delta}, \infty\right)$ almost surely.

XXZ chain Hamiltonian in finite intervals

Consider the finite interval $[-L, L] = [-L, L] \cap \mathbb{Z}$, $L \in \mathbb{N}$, and set

$$H_\omega^{(L)} = \sum_{i=-L}^{L-1} \left\{ \tfrac{1}{4} \left(I - \sigma_i^z \sigma_{i+1}^z \right) - \tfrac{1}{4\Delta} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y \right) \right\} + \lambda \sum_{i=-L}^{L} \omega_i \mathcal{N}_i$$

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- Unique ground state $\psi_0 = \psi_0^{(L)}$ determined by $\mathcal{N}_i \psi_0 = 0$ for all i.
- The spectrum of $H^{(L)} = H_\omega^{(L)}$ is almost surely simple, so that its normalized eigenvectors can be labeled as ψ_E, $E \in \sigma(H^{(L)})$.

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$$G_I = \{g : \mathbb{R} \to \mathbb{C} \ \text{ Borel measurable}, \quad |g| \le \chi_I\}.$$

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We will say that we have droplet localization in an interval I if the conclusions of the theorem hold in the interval I.

Best possible interval for droplet localization

We proved droplet localization on intervals

$$I_{1,\delta} = \left[1 - \tfrac{1}{\Delta}, (2-\delta)\left(1 - \tfrac{1}{\Delta}\right)\right] \subset \left[1 - \tfrac{1}{\Delta}, 2\left(1 - \tfrac{1}{\Delta}\right)\right).$$

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that is, we must have

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- Results on the random XXZ quantum spin chain by Beaud and Warzel (2017).

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♦ $H = H_\omega$ will be a random XXZ spin chain satisfying droplet localization in the interval $I = I_{1,\delta} = \left[1 - \frac{1}{\Delta}, (2-\delta)\left(1 - \frac{1}{\Delta}\right)\right]$.

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Thus, given an energy interval J, we consider the sub-Hilbert space $\operatorname{Ran} P_J^{(L)}$, spanned by the the eigenstates of $H^{(L)}$ with energies in J, and localize an observable X in the energy interval J by considering its restriction to $\operatorname{Ran} P_J^{(L)}$,

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Clearly
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$X_I = (X_{I_0})_I \implies$ the theorem holds with I substituted for I_0.

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Moreover, the estimate (1) *is not true without the counterterms.*

Correlators

We define the truncated time evolution of an observable X in the energy window I by ($H = H_\omega^{(L)}$),

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We are interested in quantities of the form ($K \subset I$)

$$R_K(\tau_t^I(X), Y) = \left(\tau_t^I(X)\, Y\right)_K - \left(\tau_t^I(X)\right)_K Y_K = \left(\tau_t^I(X)\, Y\right)_K - \tau_t(X_K)\, Y_K.$$

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While it is obvious where the first counterterm comes from, the same is not true of the second, where the time evolution seems to sit in the *wrong* place: it is $\tau_t^K(Y)$ and not $\tau_t^K(X)$. It turns out this term encodes information about the states above the energy window K, and the appearance of $\tau_t^K(Y)$ is related to the reduction of this data to P_0.

Particle number conservation

An important property of the XXZ chain is particle number conservation:

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It follows that

$$H^{(L)} = \bigoplus_{N=0}^{2L+1} H_N^{(L)} \quad \text{with respect to} \quad \mathcal{H}^{(L)} = \bigoplus_{N=0}^{2L+1} \mathcal{H}_N^{(L)}.$$

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- $V_\omega(x) = \sum_{j=1}^N \omega_{x_j}$ for $x \in \mathcal{X}_N$, a random potential.

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Then (in the sense of unitary equivalence)

$$\mathcal{H}_N^{(L)} \cong \ell^2\left(\mathcal{X}_N^{(L)}\right) \quad (x_1 < x_2 < \ldots < x_N \quad \text{are the sites with down spins})$$

$$H_N^{(L)} \cong -\tfrac{1}{2\Delta}\mathcal{L}_N^{(L)} + \left(1 - \tfrac{1}{\Delta}\right)\widetilde{W} + \lambda V_\omega + \left(\beta - \tfrac{1}{2}(1 - \tfrac{1}{\Delta})\right)\chi^{(L)}.$$

- $\left(\mathcal{L}_N^{(L)}\psi\right)(x) = \sum_{y \in \mathcal{X}_N^{(L)},\, |x-y|_1 = 1} (\psi(y) - \psi(x))$, the graph Laplacian.
- $\widetilde{W}(x) = 1 + \#\{j : \; x_{j+1} \neq x_j + 1\} \in \{1, 2, \ldots, N\}$ for $x \in \mathcal{X}_N$, the number of clusters in x.
- $V_\omega(x) = \sum_{j=1}^N \omega_{x_j}$ for $x \in \mathcal{X}_N$, a random potential.
- $\chi^{(L)} = \chi_{-L} + \chi_L$, the left and right boundary terms.

Local number operators

Recall $\mathcal{N}_i = \begin{cases} 1 & \text{if the spin at site } i \text{ is down} \\ 0 & \text{otherwise} \end{cases}$, and $\mathcal{N}_i = \bigoplus_{N=0}^{2L+1} \mathcal{N}_i^{(N)}$.

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Thus $\mathcal{N}_i^{(N)} \cong Q_i^{(N)}$, where $Q_i^{(N)}$ is the characteristic function of the set

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$$Q_N^{(L)}(i, j; I) = \sum_{E \in \sigma\left(H_N^{(L)}\right) \cap I} \left\| Q_i^{(N)} \psi_E \right\| \left\| Q_j^{(N)} \psi_E \right\|.$$

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It follows that

$$\sum_{E \in \sigma(H^{(L)}) \cap I} \|\mathcal{N}_i \psi_E\| \, \|\mathcal{N}_j \psi_E\| = \sum_{N=1}^{\infty} Q_N^{(L)}(i,j;I) \quad \text{almost surely.}$$

Reformulation of droplet localization

Theorem

Fix $0 < \delta < 1$, *and let* $I_{1,\delta} = \left[1 - \frac{1}{\Delta}, (2-\delta)(1 - \frac{1}{\Delta})\right]$.

There exists a constant $K > 0$ *with the following property: If*

$$\lambda\sqrt{\Delta - 1}\min\{1, (\Delta - 1)\} \geq K,$$

there exist constants $C < \infty$ *and* $m > 0$ *such that*

$$\sum_{N=1}^{\infty} \mathbb{E}(Q_N^{(L)}(i, j; I_{1,\delta})) \leq Ce^{-m|i-j|} \quad \text{for all} \quad -L \leq i, j \leq L,$$

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This reformulation reduces the proof of droplet localization in the droplet spectrum to establishing decay properties of the Green's functions associated with the random Schrödinger operators $H_N^{(L)}$.

Strategy for the proof of the theorem

The analysis is first done separately along the edge

$$\mathcal{X}_{N,1} = \left\{ x \in \mathcal{X}_N : \ \widetilde{W}(x) = 1 \right\} = \{ x = (x_1, x_1+1, \ldots, x_1+N-1) : \ x_1 \in \mathbb{Z} \}$$

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- In the bulk we use (purely deterministic) Combes-Thomas-type estimates.
- Along the edge we establish a fractional moment estimate.
- These estimates are combined to derive localization on a pair of "boxes", as in an energy interval multiscale analysis, from which we derive droplet localization.

Combes-Thomas-type estimates in the bulk

Theorem

Let $z \notin \sigma(H_N^{(L)})$ *and let*

$$\left\| \widetilde{W}^{\frac{1}{2}} (H_N^{(L)} - z)^{-1} \widetilde{W}^{\frac{1}{2}} \right\| \le \frac{1}{\eta_z}.$$

Then for all $\Phi, \Psi \subset \mathcal{X}_N^{(L)}$ *we have*

$$\left\| \chi_\Phi \widetilde{W}^{\frac{1}{2}} (H_N^{(L)} - z)^{-1} \widetilde{W}^{\frac{1}{2}} \chi_\Psi \right\| \le \frac{2}{\eta_z} \mathrm{e}^{-\log\left(1 + \frac{\eta_z \Delta}{2}\right) \mathrm{dist}_1(\Phi, \Psi)}.$$

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for all $N \in \mathbb{N}$, $E \in I_{1,\delta}$, $\epsilon \in \mathbb{R}$, and $u, v \in \mathcal{X}_{N,1}^{(L)}$.

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for all $N \in \mathbb{N}$, $E \in I_{1,\delta}$, $\epsilon \in \mathbb{R}$, *and* $u, v \in \mathcal{X}_{N,1}^{(L)}$.

Note that $|u - v|_\infty = |u_1 - v_1|$ for $u, v \in \mathcal{X}_{N,1}^{(L)}$.

Decomposition of local observables

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Given a local observable X, we define projections $P_{\pm}^{(X)}$ by

$$P_{+}^{(X)} = \bigotimes_{j \in \mathcal{S}_X} (1 - \mathcal{N}_j) \quad \text{and} \quad P_{-}^{(X)} = 1 - P_{+}^{(S)}.$$

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Moreover, since $P_{+}^{(X)}$ is a rank one projection on $\mathcal{H}_{\mathcal{S}_X}$, we must have

$$X^{+,+} = \zeta_X P_{+}^{(X)}, \quad \text{where} \quad \zeta_X \in \mathbb{C}, \ |\zeta_X| \leq \|X\|.$$

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In particular,

$$(X - \zeta_X)^{+,+} = 0 \quad \text{and} \quad \|X - \zeta_X\| \leq 2\|X\|,$$

so we can assume $X^{+,+} = 0$ in the proofs.

Droplet localization for general local observables

Droplet localization is defined in terms of the local number operators $\mathcal{N}_i$.
For proving the theorems we need to apply it to general local observables.

Lemma

Let X, Y be local observables, $\ell \geq 1$. Then

$$\mathbb{E}\left(\sup_{g\in G_{I_0}} \left\| P_-^{(X)} g(H) P_-^{(Y)} \right\|_1\right) \leq C\mathrm{e}^{-m\,\mathrm{dist}(X,Y)}$$

$$\mathbb{E}\left(\left\| P_-^{(Y)} P_-^{(X)} P_{I_0} \right\|_1\right) \leq C\mathrm{e}^{-\frac{1}{2}m\,\mathrm{dist}(X,Y)}$$

$$\mathbb{E}\left(\sup_{I\in G_I} \left\| P_-^{(X)} g(H) P_+^{(\mathcal{S}_{X,\ell})} \right\|_1\right) \leq C\mathrm{e}^{-m\ell}$$

$$\mathbb{E}\left(\sup_{g\in G_I} \left\| P_+^{(\mathcal{S}^c_{Y,\ell})} g(H) P_+^{(\mathcal{S}^c_{X,\ell})} \right\|_1\right) \leq C\mathrm{e}^{-m(\mathrm{dist}(X,Y)-2\ell)}$$

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The commutator is estimated by the Lieb-Robinson bound for small t.

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For $E, E' \in K$ we have

$$P_E \left(\int_{\mathbb{R}} \mathrm{e}^{-irH} Y \tau_r(X)\, \hat{f}(r)\,\mathrm{d}r\right) P_{E'} = P_E Y f(E + E' - H) X P_{E'}$$

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Note that $P_0 = h(H)$.

Interval for droplet localization- Sketch of proof

To prove: Droplet localization in $I = \left[1 - \frac{1}{\Delta}, \Theta_1\right] \implies \Theta_1 \leq 2(1 - \frac{1}{\Delta})$.

Sketch of proof: Let $\Theta_0 = 1 - \frac{1}{\Delta}$ and suppose $\Theta_1 > 2\Theta_0$. Let $K = [\Theta_0, \Theta_2]$, where $\Theta_0 < \Theta_2 < \Theta_1$, and $\varepsilon = \min\{\Theta_1 - 2\Theta_2, \Theta_0\} > 0$. Fix a Gevrey class function h such that

$$0 \leq h \leq 1, \ \operatorname{supp} h \subset (-\varepsilon, \varepsilon), \ h(0) = 1, \text{ and } \left|\hat{h}(t)\right| \leq C \mathrm{e}^{-c|t|^{\frac{1}{2}}}$$

Note that $P_0 = h(H)$.

Let X, Y be local observables with $X^{+,+} = Y^{+,+} = 0$. The Lemmas yield

$$\begin{aligned} \|(XP_0Y)_K\| &= \|(Xh(H)Y)_K\| \\ &\leq C \|X\| \|Y\| \mathrm{e}^{-m_1(\operatorname{dist}(X,Y))^{\frac{1}{2}}} + C' \sup_{r \in \mathbb{R}} \left\|(YP_{K_h}\tau_r(X))_K\right\|, \end{aligned}$$

where $K_h \subset [2\Theta_0 - \varepsilon, 2\Theta_2 + \varepsilon] \subset [\Theta_0, \Theta_1] = I$.

We can prove

$$\mathbb{E}\left(\sup_{r\in\mathbb{R}}\left\|\left(YP_{K_h}\tau_r\left(X\right)\right)_K\right\|\right) \leq C\left\|X\right\|\left\|Y\right\|\mathrm{e}^{-\frac{1}{8}m\,\mathrm{dist}(X,Y)},$$

We can prove

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so we conclude that

$$\mathbb{E}\left(\left\|\left(XP_0Y\right)_K\right\|\right)\leq C\left\|X\right\|\left\|Y\right\|\mathrm{e}^{-m_2(\mathrm{dist}(X,Y))^{\frac{1}{2}}}.$$

We can prove

$$\mathbb{E}\left(\sup_{r\in\mathbb{R}} \left\|\left(Y P_{K_h} \tau_r(X)\right)_K\right\|\right) \le C\,\|X\|\,\|Y\|\,\mathrm{e}^{-\frac{1}{8} m\,\mathrm{dist}(X,Y)},$$

so we conclude that

$$\mathbb{E}\left(\left\|\left(X P_0 Y\right)_K\right\|\right) \le C\,\|X\|\,\|Y\|\,\mathrm{e}^{-m_2(\mathrm{dist}(X,Y))^{\frac{1}{2}}}.$$

In particular, it follows that we have, uniformly in L,

$$\mathbb{E}\left(\left\|\left(\sigma_i^x P_0^{(L)} \sigma_j^x\right)_K\right\|\right) \le C\mathrm{e}^{-m_2(|i-j|)^{\frac{1}{2}}} \quad \text{for all} \quad i,j \in [-L,L]. \qquad (2)$$

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But we can show that for all $i,j\in\mathbb{Z}$ with $|i-j|\ge R_K$, we have

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(2) and (3) give a contradiction $\implies$ $\Theta_1 \le 2\Theta_0$.

Non-spreading of information- Sketch of proof

To prove: Given a local observables X, $t \in \mathbb{R}$ and $\ell > 0$, there is a local observable $X_\ell(t) = (X_\ell(t))_\omega$ with support $\mathcal{S}_{X,\ell}$ satisfying

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \left\| \left(X_\ell(t) - \tau_t\left(X\right)\right)_{I_0} \right\|_1 \right) \le C\|X\|\mathrm{e}^{-\frac{1}{16}m\ell}.$$

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Sketch of proof: Let $\mathcal{S}_X = [s_X, r_X]$, recall $\mathcal{S}_{X,\ell} = [s_X - \ell, r_X + \ell]$, and set

$$\mathcal{O} = [-L, L] \setminus \mathcal{S}_{X,\frac{\ell}{2}} = [-L, s_X - \tfrac{\ell}{2}) \cup (r_X + \tfrac{\ell}{2}, L]$$

$$\mathcal{T} = \mathcal{S}_{X,\ell} \cap \mathcal{O} = [s_X - \ell, s_X - \tfrac{\ell}{2}) \cup (r_X + \tfrac{\ell}{2}, r_X + \ell]$$

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We first prove that

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \left\| \left(P_+^{(\mathcal{O})}\tau_t\left(X_{I_0}\right)P_+^{(\mathcal{O})} - \tau_t\left(X\right)\right)_{I_0} \right\|_1 \right) \leq C\|X\|\mathrm{e}^{-\frac{1}{16}m\ell}.$$

We now observe that for all observables Z we have

$$P_+^{(\mathcal{O})} Z P_+^{(\mathcal{O})} = \tilde{Z} P_+^{(\mathcal{O})} = P_+^{(\mathcal{O})} \tilde{Z},$$

where $\tilde{Z}$ is an observable with $\mathcal{S}_{\tilde{Z}} = \mathcal{S}_{X,\frac{\ell}{2}}$ and $\|\tilde{Z}\| \leq \|Z\|$.

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We conclude that

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \left\| \left(P_+^{(\mathcal{O})} \widetilde{\tau_t\left(X_{I_0}\right)} - \tau_t\left(X\right) \right)_{I_0} \right\|_1 \right) \leq C\|X\| \mathrm{e}^{-\frac{1}{16} m\ell}.$$

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We conclude that

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Since $P_+^{(\mathcal{O})} \widetilde{\tau_t(X_{I_0})}$ does not have support in $\mathcal{S}_{X,\ell}$, we now define

$$X_\ell(t) = P_+^{(\mathcal{T})} \widetilde{\tau_t(X_{I_0})} \quad \text{for} \quad t \in \mathbb{R},$$

an observable with support in $\mathcal{S}_{X,\frac{\ell}{2}} \cup \mathcal{T} = \mathcal{S}_{X,\ell}$,

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We conclude that

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an observable with support in $\mathcal{S}_{X,\frac{\ell}{2}} \cup \mathcal{T} = \mathcal{S}_{X,\ell}$, and prove

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \left\| \left(P_+^{(\mathcal{O})} \widetilde{\tau_t\left(X_{I_0}\right)} - X_\ell(t)\right)_{I_0} \right\|_1 \right) \leq C\, \|X\|\, \mathrm{e}^{-\frac{1}{4} m\ell}.$$

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