From 1 to 6 in Branching Brownian Motion

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Advances in Statistical Mechanics, CIRM, 27.08.2018

Outline

1. Definition variable speed BBM
2. Previous results
3. Closing in on the straight line
4. Main results
5. Ideas of proof

Definition BBM

1. Start a Brownian motion x in 0.

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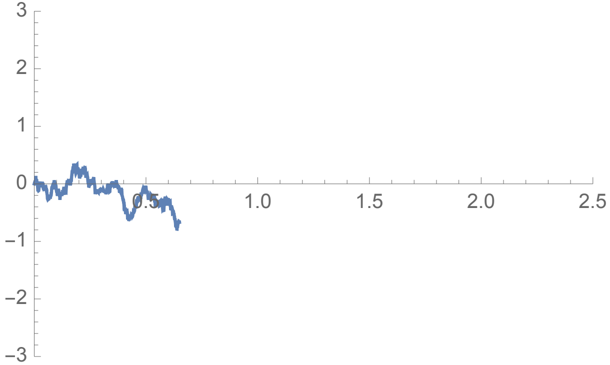
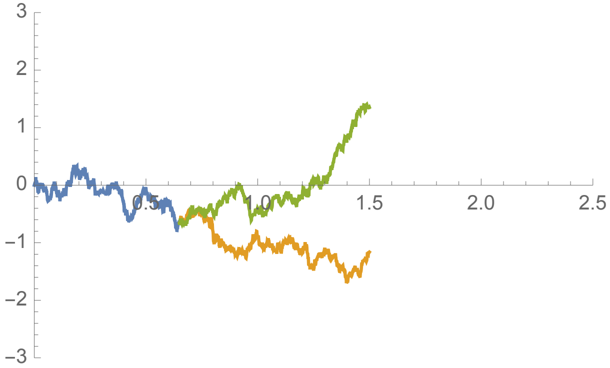


Definition BBM

2. After exponential holding time T particle splits into 2 offsprings.
3. Each performs independent Brownian motion starting at x(T).



Definition BBM

4. The new particles are subject of the same splitting rule.

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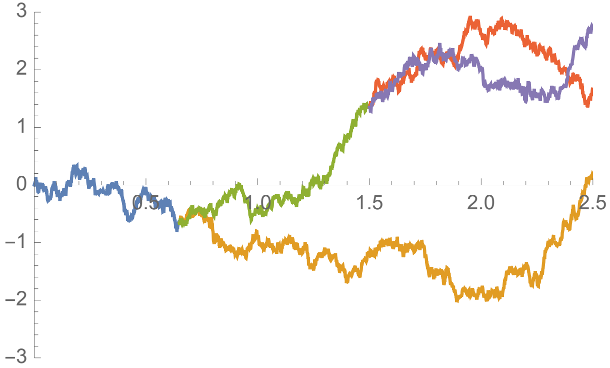
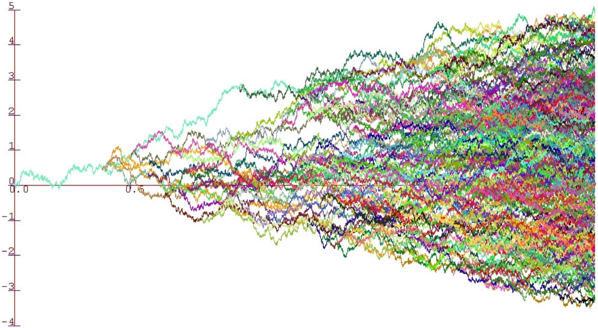


Definition BBM

And after some time...



Picture by Matt Roberts, Bath

Why are we interested in this process?

- Prototype continuous spatial branching process
- Connection to F-KPP equation
- Extreme value theory for correlated Gaussian processes

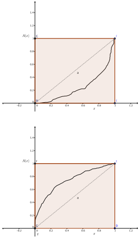
Variable speed BBM

Let $A : [0,1] \to [0,1]$ be increasing. Define

$$\Sigma^2(s) = tA(s/t).$$

Brownian motion with speed function Σ^2

$$B_s^{\Sigma} = B_{\Sigma^2(s)}.$$



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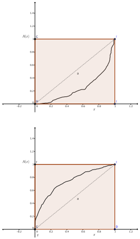

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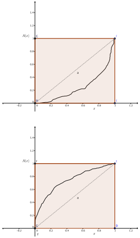

Variable speed BBM:
same splitting rules, but if a particle splits at time $s < t$:
law of movement independent copies of $\{B^\Sigma_r - B^\Sigma_s\}_{t \geq r \geq s}$

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Example for Gaussian process labelled by tree

- A time-homogeneous tree. Label individuals at time t as $\mathbf{i}_1(t), \ldots, \mathbf{i}_{n(t)}(t)$.

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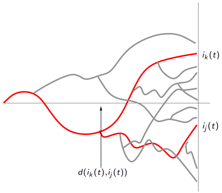

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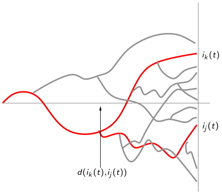

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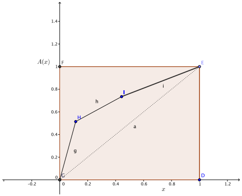
- Canonical tree-distance: $d(\mathbf{i}_\ell(t), \mathbf{i}_k(t)) \equiv$ time of most recent common ancestor of $\mathbf{i}_\ell(t)$ and $\mathbf{i}_k(t)$
- For fixed time horizon t, define Gaussian process, $(x_k^t(s), k \leq n(t), s \leq t)$, with covariance

$$\mathbb{E} x_k^t(r) x_\ell^t(s) = tA(t^{-1} d(\mathbf{i}_k(r), \mathbf{i}_\ell(s)))$$

for $A : [0,1] \to [0,1]$, increasing.



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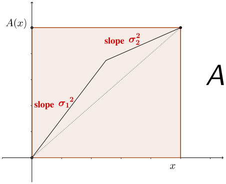
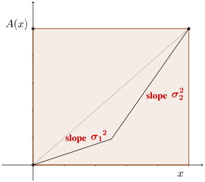
Previous results

In this talk we focus for clarity on the case of two-speed BBM, where



$$A(x) = \sigma_1^2 x + \mathbb{1}_{x \geq 1/2}(x - 1/2)(\sigma_2^2 - \sigma_1^2)$$

$$\sigma_1^2 + \sigma_2^2 = 2$$



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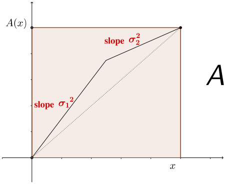

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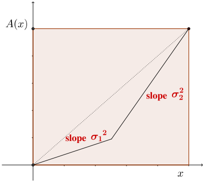

Convergence of the maximum:

$$\lim_{t \uparrow \infty} \mathbb{P}\left(\max_{k \leq n(t)} x_k(t) \leq m(t) + y \right) = \mathbb{E}\left[\mathrm{e}^{-CV\mathrm{e}^{-\sqrt{2}y}} \right]$$

with

- $m(t) = m_{\sigma_1}(t)$ depending on σ_1
- $C = C(\sigma_1)$ a constant depending on σ_1
- $V = V(\sigma_1)$ a random variable depending on σ_1

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Typical results

Moreover:

- $m_{\sigma_1}(t) = \sqrt{2}M(\sigma_1)t - \frac{c(\sigma_1)}{2\sqrt{2}} \ln t$, where
  1. $$M(\sigma_1) = \begin{cases} 1, & \text{if } \sigma_1^2 \leq 1, \\ \frac{\sigma_1+\sigma_2}{2}, & \text{if } \sigma_1^2 \geq 1, \end{cases}$$
     is continuous in σ_1
  2. $$c(\sigma_1) = \begin{cases} 1, & \text{if } \sigma_1^2 < 1, \\ 3, & \text{if } \sigma_1^2 = 1, \\ 3(\sigma_1 + \sigma_2), & \text{if } \sigma_1^2 > 1, \end{cases}$$
     is discontinuous at $\sigma_1^2 = 1$.

Typical results

And the random variable:

- V is the limit of the McKean martingale if $\sigma_1^2 < 1$
- V is the limit of the derivative martingale if $\sigma_1^2 \geq 1$

and the constant:

- $C \equiv \lim_{r\uparrow\infty} \sqrt{\frac{2}{\pi}} \mathrm{e}^{-a^2 r/2} \int_0^\infty u(r, y+\sqrt{2}r)\mathrm{e}^{(\sqrt{2}+a)y}(1-\mathrm{e}^{-2ay})dy$, with $a = \sqrt{2}(\sigma_2 - 1)$ if $\sigma_1^2 < 1$, u solution of $F-KPP$
- $C \equiv \lim_{r\uparrow\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty u(r, y+\sqrt{2}r)\mathrm{e}^{\sqrt{2}y} y dy$, if $\sigma_1^2 = 1$, u solution of $F-KPP$
- More complicated, if $\sigma_1^2 > 1$.

Closing in on the discontinuity

We have seen that various quantities behave discontinuously at the borderline case $A(x) = x$.
To analyse this singular behaviour more closely, we consider functions A that depend on t, specifically,

$$\sigma_1^2 = \sigma_1^2(t) \equiv 1 \pm t^{-\alpha}, \qquad \alpha > 0$$

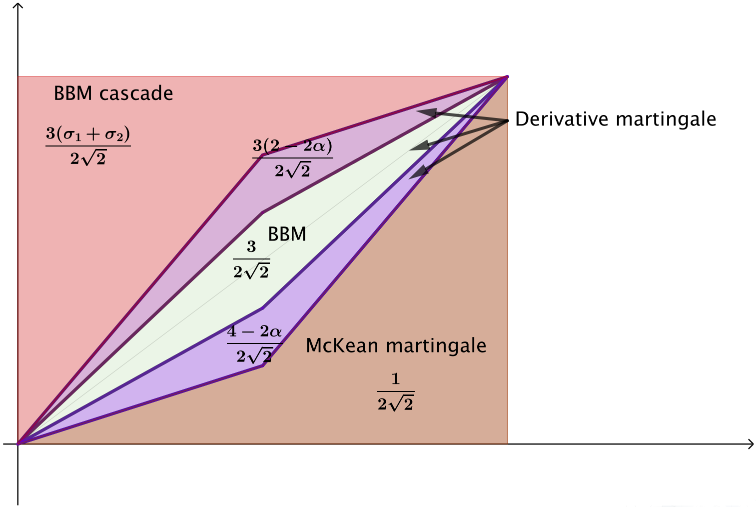
$$\sigma_2^2 = \sigma_2^2(t) \equiv 1 \mp t^{-\alpha}, \qquad \alpha > 0$$

Closing in on the discontinuity

Theorem [B, H, '18] With the notation above, the following facts hold:

1. If $\alpha \geq 1/2$, then everything is the same as in BBM ($\sigma_1^2 = 1$)
2. If $\alpha \in (0, 1/2)$, $\sigma_1^2 = 1 - t^{-\alpha}$,
   - $$m(t) = \sqrt{2}t - \frac{1+4\alpha}{2\sqrt{2}} \ln t$$
   - V is the derivative martingale
   - C is the same as in *BBM*
3. If $\alpha \in (0, 1/2)$, $\sigma_1^2 = 1 + t^{-\alpha}$,
   - $$m(t) = \sqrt{2}t\frac{\sigma_1 + \sigma_2}{2} - \frac{6(1-\alpha)}{2\sqrt{2}} \ln t$$
   - V is the derivative martingale
   - C is $2/\sqrt{\pi}$ times the constant in *BBM*

The phase diagram



Ingredients of the proof

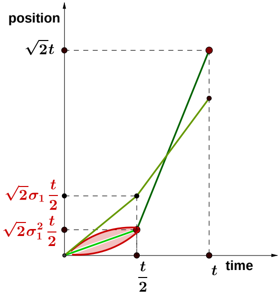
The proof is based on three basic ingredients:

1. Localisation of the ancestors of extremal particles at the time of the speed change
2. Tree recursions
3. Tail asymptotics

Localisation: the t-independent case

Two-speed BBM

$\sigma_1 < 1$



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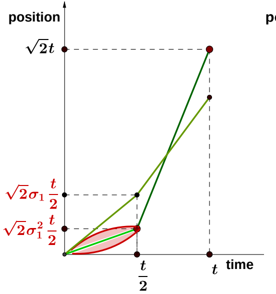
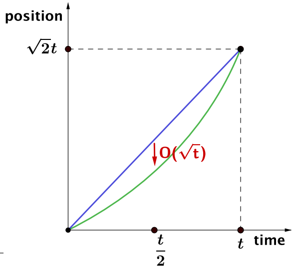


Standard BBM



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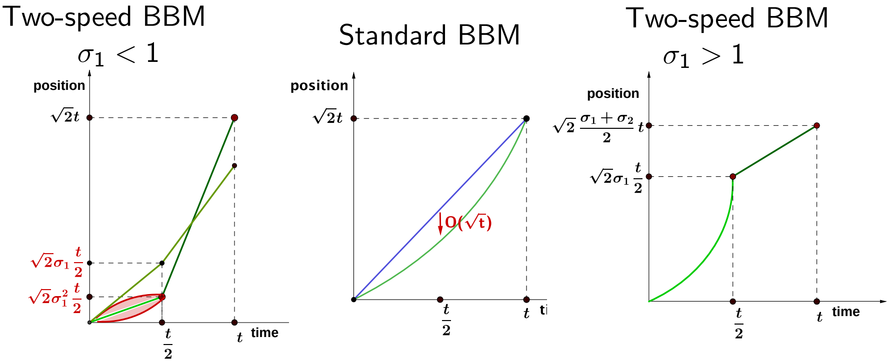

A barrier for (standard) BBM

A priori information on all paths (Lalley-Sellke)

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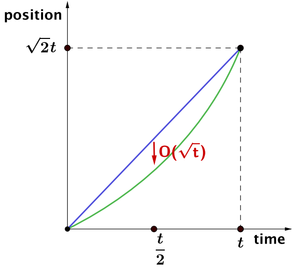
With high probability

$$\forall k \leq n(t)\ \forall r < s \leq t : x_k(s) < \sqrt{2}s$$



- All particles have to stay essentially below blue line
- Path of maximal particle $\approx$ Brownian bridge $0 \to \sqrt{2}t$ in time t staying below straight line
  $\Rightarrow$ Path is $\approx O(\sqrt{t})$ **below barrier**

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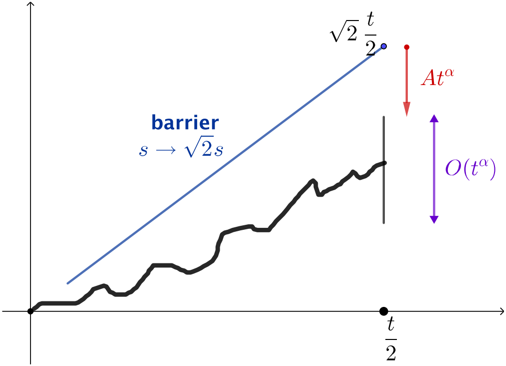
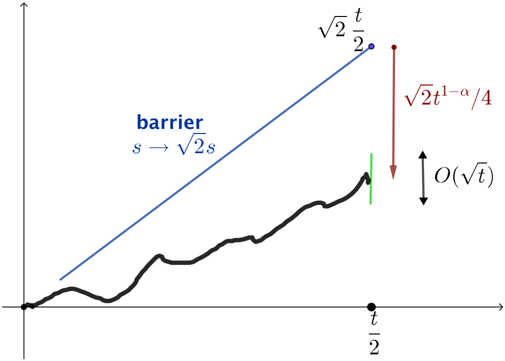
Localisation: $\sigma_1^2 = 1 + t^{-\alpha}$



To reach height $m(t)$ at time t

- Stay below barrier up to time $t/2$
- Be $O(t^{\alpha})$ below $\sqrt{2}\frac{t}{2}$ at time $\frac{t}{2}$

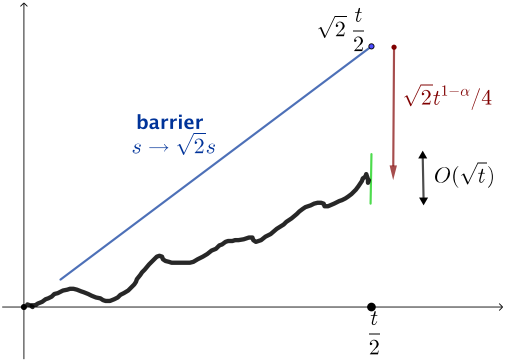
Localisation: $\sigma_1^2 = 1 - t^{-\alpha}$



To reach height $m(t)$ at time t

- Stay below barrier up to time $t/2$
- Be $\frac{t^{1-\alpha}}{2\sqrt{2}} \pm O(\sqrt{t})$ below $\sqrt{2}\frac{t}{2}$ at time $t/2$

Localisation: $\sigma_1^2 = 1 - t^{-\alpha}$



To reach height $m(t)$ at time t

- Stay below barrier up to time $t/2$
- Be $\frac{t^{1-\alpha}}{2\sqrt{2}} \pm O(\sqrt{t})$ below $\sqrt{2}\frac{t}{2}$ at time $t/2$

Localisation impose the size of the logarithmic corrections via probabilities of Brownian bridges to satisfy barrier conditions!

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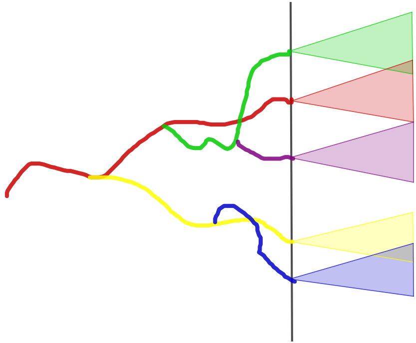
Branching property



$$\mathbb{P}\left[\max_{1\leq k\leq n(t)} \tilde{x}_k(t) - m(t) \geq y\right]$$

$$= \mathbb{P}\left[\max_{\substack{k\leq n(t/2)\\ \ell\leq n^k(t/2)}} \sigma_1 x_k(t/2) + \sigma_2 x_\ell^k(t/2) - m(t) \geq y\right],$$

Asymptotics

Lemma (Bramson '83)

For $x = x(t)$ *such that* $\lim_{t\uparrow\infty} x(t)/t = 0$

$$\mathbb{P}\left(\max_{k\le n(t)} x_k(t) > x + \sqrt{2}t\right) \sim Cx\mathrm{e}^{-\sqrt{2}x} e^{-x^2/2t} t^{-3/2}, \quad \text{as } t\uparrow\infty$$

where C *is a strictly positive constant.*

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Combining everything

$$\mathbb{P}\left[\max_{\substack{k\leq n(t/2)\\ \ell\leq n^k(t/2)}} \sigma_1 x_k(t/2) + \sigma_2 x_\ell^k(t/2) - m(t) \geq y\right],$$

- Use tail asymptotics.
- Use localization.
- Use the independence given through the branching property to put the pieces together.

Concluding remarks

- Analogous results for the extremal process: Same as in BBM

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Concluding remarks

- Analogous results for the extremal process: Same as in BBM
- Kistler and Schmidt have shown a transition form 1 to 3 with a sequence of step functions with step length $t^{-1/(1-\alpha)}$ converging to the straight line. Here extremal process remains Poisson for all $\alpha < 1$.
- One can consider a lot of different sequences of functions A_t that converge to $A(x) = x$ and obtain lots of different rescalings.
- Gaussian processes considered as functions of the function A have a very complex discontinuity at the identity function. BBM and other log-correlated processes are natural borderlines.

Thank you for your attention!