The replica trick in the frame of replica interpolation

Francesco Guerra
Dipartimento di Fisica
Università di Roma "La Sapienza"
Istituto Nazionale di Fisica Nucleare
Sezione di Roma
Centro Studi e Ricerche Enrico Fermi,
Roma

Advances in Statistical Mechanics

CIRM, Luminy, August 27-31, 2018

dedicated to Anton Bovier
on occasion of his sixtieth birthday

As it is very well known, for disordered models of statistical mechanics, the celebrated "replica trick" is based on the idea that the annealed averages for replicated systems give some information on the original system.

We give a new interpretation of the replica trick in the general frame of interpolation on the number of replicas, extending on the traditional exploitation of the replica trick as connected with analytic continuation toward zero replicas, a procedure with controversial meaning.

As an example of this new general strategy we give some application concerning the nature of the so called Almeida-Thouless line in the Sherrington-Kirkpatrick model.

We consider simple disordered systems of statistical mechanics, as the Sherrington-Kirkpatrick spin-glass mean field model, and associated models, and the Derrida Random Energy Model.

We consider N spin Ising configurations

$$\sigma : (1, 2, \ldots, N) \ni i \to \sigma_i = \pm 1.$$

There are 2^N Ising configurations on N sites. We are interested in the limit $N \to \infty$.

For each configuration σ we introduce random variables $\sigma \to \mathcal{K}(\sigma)$. In the simplest cases, we can assume these 2^N variables as Gaussian, with zero averages, and covariances given for example by

$$\mathbb{E}(\mathcal{K}(\sigma)\mathcal{K}(\sigma')) = q^2_{\sigma\sigma'},$$

where $q_{..}$ are the configuration overlaps defined by

$$q_{\sigma\sigma'} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i \sigma'_i,$$

in the celebrated Sherrington-Kirkpatrick model.

Let us recall that this model was introduced in the far 1975, and has been the subject of intensive research. Thousands of papers are dedicated to it.

On the other hand, in the case of Derrida Random Energy Model, we define $q_{\sigma\sigma'} = 1$ if the two configurations are equal and $q_{\sigma\sigma'} = 0$ if they are different, *i.e.* $q_{\sigma\sigma'} = \delta_{\sigma\sigma'}$.

The random variables $\mathcal{K}(\sigma)$ are exploited to define the energy associated to each configuration σ in the form

$$\mathcal{H}(\sigma) = -\sqrt{\frac{N}{2}}\mathcal{K}(\sigma),$$

where the term $\sqrt{N}$ is introduced for serious thermodynamic reasons, and $\sqrt{2}$ for pure aesthetic reasons, as it will be shown in the following.

In the well known Boltzmann-Gibbs scheme, the partition function is

$$Z_N(\beta) = \sum_\sigma \exp(-\beta\mathcal{H}(\sigma)) = \sum_\sigma \exp(\beta\sqrt{\frac{N}{2}}\mathcal{K}(\sigma)),$$

where β is the inverse of the temperature.

We have performed the sum over all configurations. Therefore, the partition function does depend only on the random noise present in the $\mathcal{K}(\sigma)$'s.

The (random) free energy is

$$-\beta F_N(\beta) = \log Z_N(\beta).$$

The rescaling $\sqrt{N}$ in the definition of the energy is introduced in order to assure a good thermodynamic behavior for the free energy per site, in the limit $N \to \infty$.

In fact, it is not difficult to prove that the limit

$$\lim_{N \to \infty} \frac{1}{N} \log Z_N(\beta)$$

does exist almost surely in the probability space where all $\mathcal{K}(\sigma)$ are defined. We call $A(\beta)$ this limit, where any random character has been lost.

It turns out that the limit $A(\beta)$ can be calculated also through the quenched averages

$$A(\beta) = \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_N(\beta),$$

where $\mathbb{E}$ is the average with respect to the noise due to the $\mathcal{K}(\sigma)$'s.

The equality between the probabilistic limit and the quenched limit is due to a moderate statistical fluctuation of the free energy in the limit, which can be easily proved through elementary interpolation methods.

There is a deep physical motivation at the basis of the metallurgic terminology. In the partition function $Z_N(\beta)$ we perform only the sum over the σ's, according to Boltzmann prescriptions. Thefeore, the noise in the $\mathcal{K}(\sigma)$'s acts as external noise, which is not involved in the thermodynamic equilibrium, but affects thermodynamic equilibrium of the σ'. Then, we take the log, and at the end the average $\mathbb{E}$.

Obviously we can take also the (annealed) average, before taking the log, so that the external noise does participate to the the thermodynamic equilibrium

$$\bar{A}(\beta) = \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z_N(\beta).$$

This annealed expression is easily calculated

$$\mathbb{E} Z_N(\beta) = \mathbb{E} \sum_{\sigma} .. = \sum_{\sigma} \mathbb{E} .. = \exp(N(\log 2 + \frac{1}{4}\beta^2)),$$

since for each σ we have

$$\mathbb{E} \exp(\beta \sqrt{\frac{N}{2}} \mathcal{K}(\sigma)) = \exp(\frac{1}{2}\beta^2 \frac{N}{2} \mathbb{E}(\mathcal{K}^2(\sigma))) =$$

$$\exp(\frac{1}{4}\beta^2 N).$$

The term $\log 2$ comes from the final sum over the σ's.

The annealed expression is not correct in general. In any case it is a rigorous upper bound, uniform in N. In fact, from the concavity of the log

$$\mathbb{E} \log .. \leq \log \mathbb{E} ..$$

we have

$$\frac{1}{N} \mathbb{E} \log Z_N(\beta) \leq \frac{1}{N} \log \mathbb{E} Z_N(\beta) = \log 2 + \frac{1}{4} \beta^2,$$

preserved in the limit

$$A(\beta) = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\log Z_N(\beta) \leq \log 2 + \frac{1}{4}\beta^2.$$

We are interested in the explicit expression for $A(\beta)$, in the form of a variational principle. It will be a long journey.

Let us introduce the concept of **replicas**. For $s = 1, 2, ..$ (s positive integer) the s-replicated system has a configuration space which is the s time product space of the original system. Therefore, now the variables are

$$\sigma_i^a = \pm 1,\ i = 1, 2, ..N,\ a = 1, 2, .., s,$$

where the index i denoted the sites, and the index a denotes the replicas. Therefore, now overall there are sN sites.

The energy is now defined as the sum of the energies for each single replica, with the *same* randomness. The Boltzmann factor is therefore factorized, and for the new partition function we have a simple product

$$\bar{Z}_{s,N}(\beta) = \sum_{\sigma^1} \exp(\beta\sqrt{\frac{N}{2}}\mathcal{K}(\sigma^1)) \sum_{\sigma^2} \exp(\beta\sqrt{\frac{N}{2}}\mathcal{K}(\sigma^2))...$$

$$...\sum_{\sigma^s} \exp(\beta\sqrt{\frac{N}{2}}\mathcal{K}(\sigma^s)) = Z_N^s(\beta),$$

since every sum on the variables σ's gives the same contribution. Therefore the partition function of the new system is simply the product of identical terms corresponding to the partition function of the original nonreplicated system.

The free energy per site, and its quenched average, of the replicated system is the same as for the original system. In fact, the logarithm of a product is the sum of the logarithms, each with the same contribution

$$\log \bar{Z}_{s,N} = s \log Z_N.$$

Therefore, trivially

$$\frac{1}{sN} \log \bar{Z}_{s,N} = \frac{1}{N} \log Z_N.$$

However, if we take the annealed expressions, we have a nontrivial dependence on the number of replicas s. We are induced to introduce an auxiliary function

$$\phi_N(s,\beta) = \frac{1}{sN}\log \mathbb{E}\bar{Z}_{s,N} = \frac{1}{sN}\log \mathbb{E}Z_N^s(\beta),$$

$$s = 1, 2, ...,$$

with a deep motivation made explicit in the following. Obviously, if $s = 1$ we have simply the annealed case, considered above.

It is immediately possible to establish a small industry in order to study the thermodynamic limit $N \to \infty$ di $\phi_N(s, \beta)$, with very interesting results.

The limit, $\phi(s, \beta)$, does exist for any integer $s = 1, 2, ...$, and can be explicitely expressed through a variational principle. We only give the general structure. We have to specify the order parameters, and the trial function.

Let us consider firstly the Sherrington-Kirkpatrick model. For a given integer s, for each couple of replicas we introduce the system of order parameters $q_{ab} \geq 0$, $a < b$. There are $s(s-1)/2$ order parameters. The case $s=1$ does not require order parameters, and the auxiliary function is given directly. For $s=2$ there is only one order parameter q_{12}. For $s=3$ three order parameters appear q_{12}, q_{13}, q_{23}, and so on.

The trial function $\tilde{\phi}(s,\beta;q_{..})$, which we omit to write explicitely here, is a function replica symmetric in the q_{ab}'. If the replicas are permuted its value does not change. The variational principle states

$$\lim_{N\to\infty} \phi_N(s,\beta) = \phi(s,\beta) = \sup_{q_{..}} \tilde{\phi}(s,\beta;q_{..}).$$

The variational principle enjoys a remarkable property, as it is very well known. In fact, the sup is realized for values of the order parameters, where all $q_{..}$ have the same value $\bar{q} \geq 0$. There is full replica symmetry for the optimal values.

Replica symmetry allows to state the variational principle in a very simple form. Introduce the replica symmetric trial function $\phi_{RS}(s, \beta; \bar{q})$, as a function of the trial order parameter $\bar{q} \geq 0$,

$$\tilde{\phi}_{RS}(s, \beta; \bar{q}) =$$

$$\log 2 + \frac{1}{s} \log \int (\cosh(\beta\sqrt{\bar{q}}z))^s \; d\mu(z)) +$$

$$\frac{\beta^2}{4}(1 - 2\bar{q} - (s-1)\bar{q}^2),$$

where $d\mu(z)$ is the unit Gaussian measure on the real line. Then we have in the infinite volume limit

$$\lim_{N\to\infty} \phi_N(s,\beta) = \phi(s,\beta) =$$

$$\sup_{\bar{q}} \tilde{\phi}_{RS}(s,\beta;\bar{q}) = \phi_{RS}(s,\beta),$$

for integer values $s = 1, 2, 3...$

There exists a deep and complex successive development, conventionally called "replica trick", introduced by the pioneers of the study of these systems. The first step is to extend the definition of the auxiliary function $\phi_N(s,\beta)$, from the integers $s = 1, 2, ...$ to any real value of s. For the sake of simplicity we consider only the case $s > 0$.

This extension is easily obtained by noticing that the very definition

$$\phi_N(s,\beta) = \frac{1}{sN} \log \mathbb{E} Z_N^s(\beta),$$

originally introduced only for integer values of s, in the replica frame, has a perfect rigorous meaning also for any $s > 0$.

For the β derivative, we easily find

$$\frac{\partial}{\partial \beta} \phi_N(s, \beta) = \frac{\beta}{2}(1 + (s-1) < q^2_{\sigma\sigma'} >),$$

for an appropriately defined average $<>$, involving two replicas

$$< q^2_{\sigma\sigma'} > = \mathbb{E}(Z^{s-2} \Omega(q^2_{\sigma\sigma'}))/\mathbb{E}(Z^s).$$

Here Ω is the Boltzmann-Gibbs average for two replicas.

Notice the presence of the term $(s-1)$. Its sign changes at $s=1$. It is responsible of many notable inversions. Superadditivity vs subadditivity, inf vs sup, ...

The hope is that the experience accumulated in the study of $\phi_N(s,\beta)$, and its limit $\phi(s,\beta)$, for integer s, can produce some information in the case of generic values $s>0$.

What is the interest in generic values of s. Here we have a first deep aspect of the "replica trick". As a matter of fact, we have in the limit $s \to 0$

$$\lim_{s\to 0} \phi_N(s,\beta) = \frac{1}{N}\mathbb{E}\log Z_N(\beta),$$

a very important relation which holds also in the thermodynamic limit $N \to \infty$

$$\lim_{s\to 0} \phi(s,\beta) = A(\beta).$$

Therefore, the auxiliary function ϕ, for very small values of s, reduces to the expression of the quenched free energy we are interested in.

An intuitive proof can be given in few lines. For small values of s we have

$$Z_N^s(\beta) = \exp(s \log Z_N) \simeq 1 + s \log Z_N,$$

and therefore

$$\log \mathbb{E} Z_N^s \simeq \log(1 + s\mathbb{E} \log Z_N) \simeq s\mathbb{E} \log Z_N.$$

No finger crossing is necessary. In the traditional treatment one considers the expression

$$\frac{Z_N^s - 1}{Ns},$$

which well reproduces $\log Z_N/N$ if the limit $s \to 0$ is taken firstly, but does lead surely to disasters if it is the limit $N \to \infty$ which is taken firstly. Finger crossing is needed in this case, in the hope that by proceeding formally at the end something significant will be achieved.

The extension of $\phi_N(s, \beta)$ to all values $s > 0$ is well motivated.

By a systematic exploitation of the interpolation methods it is easy to establish the following important properties of $\phi_N(s, \beta)$.

It turns out that $N\phi_N(s,\beta)$ is subadditive in N for $s \geq 1$, and superadditive for $s \leq 1$. The thermodynamic limit $N \to \infty$ follows in the form

$$\phi(s,\beta) = \lim_{N\to\infty} \phi_N(s,\beta) =$$

$$= \inf_N \phi_N(s,\beta),$$

for $s \geq 1$, and

$$= \sup_N \phi_N(s,\beta),$$

for $s \leq 1$.

The functions $\phi_N(s,\beta)$ and $\phi(s,\beta)$ are monotone nondecreasing in the parameter s

$$\phi(s,\beta) \leq \phi(s',\beta) \quad for \quad s \leq s'.$$

The functions $\phi_N(s,\beta)$ and $\phi(s,\beta)$ are convex in β, for any fixed value of s. Here the proof is elementary (a simple calculation) and does not involve subtle properties of the Ghirlanda-Guerra identities.

The functions $\phi_N(s,\beta)$ and $\phi(s,\beta)$ are convex in $1/s$, for any value of β, as a consequence of Holder inequality.

In order to explore the potentialities of the "replica trick", now we shift to the laboratory of the Random Energy Model. Here we will show that a new interpretation of the "trick", not based on analytic continuation for $s \to 0$, gives the right order parameter, the right trial function and the right variational principle, for any value of s, starting only from the elementary variational principle at integer values $s = 1, 2, 3, ...$.

We show that in this case replica symmetry is minimally broken. The deep reason for spontaneous replica symmetry breaking arises quite naturally.

Let us recall the expressions of the partition function and the auxiliary function in the Random Energy Model

$$Z_N(\beta) = \sum_{\sigma} \exp(\beta\sqrt{\frac{N}{2}}J(\sigma)),$$

$$\mathbb{E}(J(\sigma)J(\sigma')) = \delta_{\sigma\sigma'},$$

$$\phi_N(s,\beta) = \frac{1}{Ns} \log \mathbb{E}(Z_N(\beta)^s).$$

Recall that at $s = 1$ we have the annealed value

$$\phi(1,\beta) = \log 2 + \frac{1}{4}\beta^2.$$

Let us establish the variational principle for integer values of s.

We have

$$\phi_N(s,\beta) = \frac{1}{Ns}\log \mathbb{E}(Z_N(\beta)^s) =$$

$$\frac{1}{Ns}\log \mathbb{E}\sum_{\sigma_i^a} \exp(\beta\sqrt{\frac{N}{2}}(J(\sigma^1) + .. + J(\sigma^s))).$$

Now we can exchange freely the $\mathbb{E}$ and the $\sum$. Therefore we are led to the calculation of

$$\mathbb{E}\exp(\beta\sqrt{\frac{N}{2}}(J(\sigma^1)+..+J(\sigma^s))) =$$

$$= \exp(\frac{1}{2}\beta^2\frac{N}{2}\mathbb{E}(J(\sigma^1)+..+J(\sigma^s))^2).$$

It turns out that

$$\mathbb{E}(J(\sigma^1)+..+J(\sigma^s))^2 = s + 2\sum_{a<b}\delta_{ab},$$

where the first term s comes from the diagonal terms in the square, while $\delta_{ab} = 1$ if the configurations σ^a and σ^b are equal, and zero otherwise, $1 \leq a < b \leq s$.

By collecting all terms we have

$$\phi_N(s,\beta) = \frac{1}{4}\beta^2 + \frac{1}{Ns}\log\sum_{\sigma_i^a}\exp(\frac{1}{2}N\beta^2\sum_{a<b}\delta_{ab}).$$

Now we split all possible configurations for the σ variables into the sum of K bubbles, $K = 1, 2, ..., s$, each made of s_r replicas, with $r = 1, ..., K,\ s_r \geq 1,\ \sum_r s_r = s$, in such a way that the σ's are all equal in each bubble, and all different for different bubbles.

The order parameters are therefore $K, s_1, ..., s_K$. For each of these specifications the $\sum_{a<b}$ in the exponent reduces to

$$\sum_{a<b} \delta_{ab} = \sum_r \frac{1}{2} s_r (s_r - 1),$$

while the residual $\sum_\sigma$ gives 2^{NK}.

Clearly in the infinite volume limit $\phi_N(s, \beta)$ will be bigger that each contributing term, and equal to the highest one.

By collecting all information, we have the order parameters $K, s_1, ..., s_K$, and the trial functional

$$\tilde{\phi}(s, \beta; K, s_1, ..., s_K) =$$

$$= \frac{1}{4}\beta^2 + \frac{\beta^2}{2s}\sum_r \frac{1}{2} s_r(s_r - 1) + \frac{K}{s}\log 2 =$$

$$= \frac{\beta^2}{4s}\sum_r s_r^2 + \frac{K}{s}\log 2,$$

so that

$$\phi(s, \beta) = \sup_{K, s_1, ..., s_K} (\frac{\beta^2}{4s}\sum_r s_r^2 + \frac{K}{s}\log 2).$$

In the trial functional the first term has the meaning of an energy, the second is the entropy. The variational principle is an entropy principle. The entropy is maximum with the constraint of a given energy.

The sup is easily found. We considere the two extremal cases, $K = 1, s_1 = s$, $K = s, s_r = 1, r = 1, ..., s$, and immediately check that the $\sup_{K,s_1,...,s_K}$ can be found by considering only the sup between the two extremal cases.

For $K = 1$, when all $\delta_{..} = 1$, the value of the trial function is $\frac{\beta^2}{4}s + \frac{1}{s}\log 2$. For $K = s$, when all $\delta_{..} = 0$ the value is $\frac{\beta^2}{4} + \log 2$.

We see that there are transition points $\beta_c^2(s) = \frac{4\log 2}{s}$, such that

$$\phi(s, \beta) = \log 2 + \frac{\beta^2}{4},$$

for $\beta \leq \beta_c(s)$, and

$$\phi(s, \beta) = \frac{1}{s}\log 2 + \frac{\beta^2}{4}s,$$

for $\beta \geq \beta_c(s)$.

The replica symmetry is never broken. The overlaps are all zero in the first region and all one in the second region. Remember that we are considering for the moment only integer values $s = 1, 2, ...$

Now we come to the main point. We show that the variational principle at integer s, gives a strong hint at what sould be the variational principle for all $s > 0$. This is our interpretation of the "trick".

Consider the values taken by the trial function

$$\tilde{\phi}(s, \beta; K, s_1, ..., s_K) = \frac{\beta^2}{4s} \sum_r s_r^2 + \frac{K}{s} \log 2,$$

for various specifications of the order parameters $K, s_1, ..., s_K$.

Let us start from the simple inequality

$$\frac{1}{K} \sum_r s_r^2 \geq (\frac{1}{K} \sum_r s_r)^2 = (\frac{s}{K})^2,$$

so that

$$\frac{1}{s} \sum_r s_r^2 \geq \frac{s}{K}.$$

Therefore, by defining $s/K = m$, so that $1 \leq m \leq s$, we have the estimate

$$\tilde{\phi}(s, \beta; K, s_1, ..., s_K) \geq \frac{1}{m} \log 2 + \frac{\beta^2}{4} m.$$

This expression is really remarkable. It suggests to consider the convex trial function for the order parameter m

$$0 < m \to \tilde{\phi}(m, \beta) = \frac{1}{m} \log 2 + \frac{\beta^2}{4} m,$$

independent of s, and such that, at least for integer values of s

$$\phi(s,\beta) = \sup_{1\leq m\leq s} \left(\frac{1}{m}\log 2 + \frac{\beta^2}{4}m\right).$$

The trial function is independent of s, only the range of the variational parameter m is taken to depend on s. It is impressive to see that the variational values of $\tilde{\phi}(s,\beta;K,s_1,...,s_K)$ for different integer values of s, but at the same β, do still have as a common lower bound the same reduced trial function $\tilde{\phi}(m,\beta)$, only the range for m may change, according to s.

Therefore, the order parameter m and the reduced trial function $\tilde{\phi}(m, \beta)$ are suggested, in the present interpretation of the "trick", by the variational values of the complete trial function $\tilde{\phi}(s, \beta; K, s_1, ..., s_K)$. The variational parameters $(K, s_1, ..., s_K)$, which strongly depend on the value of s, are collapsed to a unique essential variational parameter m.

The suggestion of the "trick" is proficuous. In fact, through some work, one can easily prove the following Theorem.

For the order parameter $m > 0$, introduce the convex trial function

$$\tilde{\phi}(m,\beta) = \frac{1}{m}\log 2 + \frac{\beta^2}{4}m.$$

Then, we have in the infinite volume limit for the auxiliary function $\phi(s,\beta)$ for any $s \geq 0$ the following variational principle:

$$\phi(s,\beta) = \sup_{1\leq m\leq s} \tilde{\phi}(m,\beta),$$

for $s \geq 1$, and

$$\phi(s,\beta) = \inf_{s\leq m\leq 1} \tilde{\phi}(m,\beta),$$

for $0 \leq s \leq 1$.

The inversion from a sup to an inf, by crossing the $s = 1$ line is completely analogous to the mentioned inversion from subadditivity to superadditivity.

Since $\tilde{\phi}(m, \beta)$ is convex, the sup for $s \geq 1$ can be reached only at the boundaries $m = 1$ or $m = s$, and the replica symmetry can not be broken. On the other hand, when $s < 1$, it can happen that the minimum for $\tilde{\phi}(m, \beta)$ is in the interval $s \leq m \leq 1$, and replica symmetry is broken.

Globally the space (s, β), for $s \geq 0$ is split into three regions. For $s \geq 1$ we are always in the replica symmetric case. For $\beta < \beta_c(s)$, with $\beta_c^2(s) = \frac{4 \log 2}{s}$, we have

$$\phi(s, \beta) = \log 2 + \frac{\beta^2}{4}.$$

From

$$\frac{\partial}{\partial \beta} \phi_N(s, \beta) = \frac{\beta}{2}(1 + (s-1) < \delta_{\sigma\sigma'} >),$$

we see that here $< \delta_{\sigma\sigma'} >= 0$ in the limit. For $\beta > \beta_c(s)$ we have

$$\phi(s, \beta) = \frac{1}{s} \log 2 + \frac{\beta^2}{4} s,$$

and here $< \delta_{\sigma\sigma'} >= 1$ in the limit.

These results extend the given expression from the case s integer, to any $s \geq 0$. Notice that the line at $\beta_c(s)$ is a first order transition line. The function $\phi(s, \beta)$ is continuous, as it should be, because of the convexity in β. But its derivative in β has a sudden jump.

For $s < 1$ the situation is more complicated. There are two second order transition lines, the first at $\beta_c = 2\sqrt{\log 2}$, the second at $\beta'_c(s) =$

$2\sqrt{\log 2}/s$. The two merge at $s = 1$, but in general $\beta_c < \beta_c'(s)$.

For $\beta \leq \beta_c$, replica symmetry holds,

$$\phi(s, \beta) = \log 2 + \frac{\beta^2}{4},$$

and $< \delta_{\sigma\sigma'} >= 0$.

For $\beta \geq \beta_c'(s)$, replica symmetry is restored in the form

$$\phi(s, \beta) = \frac{1}{s} \log 2 + \frac{\beta^2}{4} s,$$

but now $< \delta_{\sigma\sigma'} >= 1$.

In the region $\beta_c \leq \beta \leq \beta'_c(s)$ replica symmetry is broken, and we have

$$\phi(s, \beta) = \beta\sqrt{\log 2},$$

independently of s.

Now the formula

$$\frac{\partial}{\partial\beta}\phi_N(s, \beta) = \frac{\beta}{2}(1 + (s-1) < \delta_{\sigma\sigma'} >),$$

gives

$$< \delta_{\sigma\sigma'} >= \frac{1}{1-s}(1 - \frac{2\sqrt{\log 2}}{\beta}),$$

with a smooth interpolation between the value $< \delta_{\sigma\sigma'} >= 0$ at $\beta = \beta_c$, and $< \delta_{\sigma\sigma'} >= 1$ at $\beta = \beta_c'(s)$.

We can see that replica symmetry breaking is not connected to a difficulty in the analytic continuation of the replica symmetric solution.

In fact let us take $\beta > \beta_c$ and a large value of s, where

$$\phi(s,\beta) = \frac{1}{s}\log 2 + \frac{\beta^2}{4}s.$$

At fixed β, there is no problem in the analytic continuation of this expression to all values of $s > 0$ well inside the region of symmetry breaking, for $s < s_c = 2\sqrt{\log 2}/\beta$. However, for $s < s_c$, the equation

$$\phi(s,\beta) = \frac{1}{s}\log 2 + \frac{\beta^2}{4}s$$

can no longer be true for a very simple reason.

In fact, at fixed β, the function

$$\frac{1}{s}\log 2 + \frac{\beta^2}{4}s$$

is decreasing, with decreasing s, up to the point $s = s_c$, where there is an inversion, and the function starts to increase with decreasing s.

Notice that the derivative

$$\frac{\partial}{\partial s}(\frac{1}{s}\log 2 + \frac{\beta^2}{4}s) = -\frac{1}{s^2}\log 2 + \frac{\beta^2}{4}$$

is positive in s for $s \geq s_c$, becomes zero at $s = s_c$, and becomes negative in s for $s \leq s_c$.

Since $\phi(s, \beta)$ must be increasing in s, we surely have

$$\phi(s,\beta) \leq (\frac{1}{s_c}\log 2 + \frac{\beta^2}{4}s_c).$$

As a matter of fact equality holds here, since for $s = s_c = 2\sqrt{\log 2}/\beta$ we have exactly

$$\frac{1}{s_c}\log 2 + \frac{\beta^2}{4}s_c = \beta\sqrt{\log}2.$$

We say that in this case that replica symmetry is minimally broken. Replica symmetry holds everywhere, with the exception of the region where this can not be true. Then, necessarily

$$\phi(s, \beta) = \phi(s_c(\beta), \beta).$$

This ends our discussion of the "replica trick" for the Random Energy Model. We have seen how to reach the right order parameter, and the right trial function, through a direct inspection of the variational behavior of the trial function for the system at integer values of s.

A straightforward extension of this interpretation of the “trick” for the Sherrington-Kirkpatrick model, seems to point out to a minimal replica symmetry breaking also in this case. But the situation is open, and will be considered in some other occasion. In any case, if the replica symmetric solution has an inversion point for some value of $s = s_c$, then a new upper bound is easily established for the auxiliary function for all values of $s \leq s_c$, and in particular at $s = 0$.

We give a simple but unexpected example about the so called Almeida-Thouless line for the Sherrington-Kirkpatrck model in external field. Let us sketch the argument.

Consider the simplest case, where the partition function is a function of two parameters $x \geq 0, t \geq 0$, connected with the strength of a one body and two body interaction respectively

$$Z_N(x,t) = \sum_{\sigma} \exp(\sqrt{t}\sqrt{\frac{N}{2}}\mathcal{K}(\sigma) + \sqrt{x}\sum_{i=1}^{N} J_i\sigma_i),$$

where the J_i are independent unit Gaussian random variables.

It is very well know that there is a large replica symmetric region in the x, t plane.

Consider $x_0 \geq 0$, and the order parameter

$$\bar{q} = \int \tanh^2(\sqrt{x_0} z)\ d\mu(z).$$

Consider the flow lines

$$x(t) = x_0 - \bar{q}t,$$

for $0 \leq t \leq t_m = x_0/\bar{q}$. These lines do not intersect for different values of x_0. Therefore, for any value of x, t, there is a unique value of x_0, such that x, t are on its flow line.

Introduce the replica symmetric function

$$A_{RS}(x,t) = \log 2 + \int \log\cosh(\sqrt{x_0}z)\, d\mu(z) + \frac{t}{4}(1-\bar{q})^2.$$

By standard interpolation techniques it is immediately shown that in the infinite volume limit for $A(x,t) = \lim_{N\to\infty} N^{-1}\mathbb{E}\log Z_N(x,t)$ there is the inequality

$$A(x,t) \le A_{RS}(x,t),$$

in general, while in the replica symmetric region equality holds.

It is universally believed that replica symmetry holds up to the Almeida-Thouless line, in the region

$$0 \leq t \leq t_{AT},$$

where t_{AT} is defined so that

$$t_{AT} \int \frac{1}{cosh^4(\sqrt{x_0} z)}\ d\mu(z) = 1.$$

On the other hand, our interpretation of the replica trick, gives the suggestion that replica symmetry is broken for a smaller value $0 \leq t \leq t_c < t_{AT}$, in general.

In fact for this model the replica symmetric solution $\phi_{RS}(s; x, t)$ has an inversion in s, *i.e.*, its derivative in s becomes negative for some value of $s = s_c(x,t)$, provided $t > t_c$. The best way to be convinced about this fact is to calculate

$$\frac{\partial}{\partial s}\phi_{RS}(s;x,t)$$

at $s = 0$ and find that it is negative.

Through a straightforward forward calculation one can see the following.

Introduce the one step symmetry breaking trial function, corresponding to the order parameter $[0,1] \ni q \to x(q) \in [0,1]$ defined by

$x(q) = m$ for $0 \le q \le \bar{q}$, and $x(q) = 1$ for $0 < q \le 1$,

$$\tilde{\phi}(x, t; m, \bar{q}) =$$

$$\log 2 + \frac{1}{m} \log \int (\cosh(\sqrt{x_0} z))^m \, d\mu(z)) +$$

$$\frac{\beta^2}{4}(1 - 2\bar{q} - (m-1)\bar{q}^2).$$

Then it is immediately seen that

$$\frac{\partial}{\partial s} \phi_{RS}(s; x, t)$$

at $s = 0$ equals

$$\frac{\partial}{\partial m}\tilde{\phi}(x, t; m, \bar{q})$$

at $m = 0$.

Notice that $\tilde{\phi}(x, t; m, \bar{q})$ at $m = 0$ is equal to $A_{RS}(x, t)$. By the well known broken replica bounds the inequality

$$A(x, t) \leq \tilde{\phi}(x, t; m, \bar{q})$$

holds. Therefore, if

$$\frac{\partial}{\partial m}\tilde{\phi}(x, t; m, \bar{q})$$

is negative at $m = 0$, then surely

$$A(x,t) < A_{RS}(x,t),$$

and replica symmetry is broken.

We see that once the right trial $\tilde{\phi}(x,t;m,\bar{q})$ is attempted, then the broken replica bounds alone can give the replica symmetry breaking. But it is the replica trick which suggests the right trial.

We are left with the calculation of

$$\frac{\partial}{\partial m}\tilde{\phi}(x,t;m,\bar{q})$$

at $m = 0$.

To this purpose, let us start from the definition of $\tilde{\phi}(x, t; m, \overline{q})$ given above.

Notice that

$$(\cosh(\sqrt{x_0}z))^m = e^{m \log \cosh} =$$

$$1 + m \log \cosh + \frac{1}{2} m^2 (\log \cosh)^2 + r_3,$$

where r_3 is an error of order $O(m^3)$. Therefore

$$\int (\cosh(\sqrt{x_0}z))^m \; d\mu(z) =$$

$$1 + m\int \log\cosh + \frac{1}{2}m^2\int(\log\cosh)^2 + r_3.$$

By taking into account that

$$\log(1+x) = x - \frac{1}{2}x^2 + r_3,$$

we have

$$\log\int(\cosh(\sqrt{x_0}z))^m =$$

$$m\int\log\cosh + \frac{1}{2}m^2\left(\int(\log\cosh)^2 - \left(\int\log\cosh\right)^2\right) + r_3.$$

By substituting into the definition of $\tilde{\phi}(x,t;m,\bar{q})$, we find

$$\tilde{\phi}(x,t;m,\bar{q}) = A_{RS}(x,t) + m\left(\frac{1}{2}\Delta^2 - \frac{1}{4}t\bar{q}^2\right) + r_2,$$

where Δ^2 is the variance

$$\Delta^2 = \int (\log\cosh(\sqrt{x_0} z))^2 \; d\mu(z) -$$

$$(\int \log\cosh(\sqrt{x_0} z)) \; d\mu(z))^2.$$

We have calculated $\tilde{\phi}$ up to the first order in m. We see that the m derivative at $m = 0$ becomes negative if $t > t_c$, where t_c is defined by

$$t_c = 2\Delta^2 / \bar{q}^2.$$

It is immediate to see that $t_c < t_{AT}$ in general. For example, at small x_0, we can easily calculate the first terms in the asymptotic developments

$$t_{AT} = 1 + 2x_0 - 3x_0^2 + r_3,$$

$$t_c = 1 + 2x_0 - \frac{14}{3}x_0^2 + r_3,$$

which give $t_c < t_{AT}$ for small positive values of x_0.

Further additional work will be necessary to fully understand the nature of the transition.