One-sided versus two-sided

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The difference between one-sided and two-sided points of view.

One-sided versus two-sided.
Stochastic Systems (Processes) .
Two flavours:
Time, discrete.
(Dynamical Systems, asymmetric description).
Past and future, one-sided (SRB).
versus
Space, discrete, here one-dimensional.
(Mathematical Physics, symmetric description).
Left and right, two-sided (DLR).
Question:
When are both descriptions equivalent?
When not equivalent?

Introduction:

Simple background.

Markov modeling (for the short-sighted..)

Time:

Probability and Statistics (Markov chains).

Future independent of past, given the *present*.

Ergodic Theory, Dynamical Systems.

Ahistoric, forget history.

(Henry Ford: All history is bunk...)

Space:
Statistical (Mathematical) Physics.
Markov: Inside independent of outside,
given the *border.*
(Take control of your borders..)
Forget about everywhere else.
2-state Markov chains
-timelike-
versus
1-dimensional, nearest neighbour, spin
(e.g. Ising) models
-spacelike-.

Probability measures on e.g.
two-symbol sequences,
configuration space $\Omega = \{-,+\}^Z$.

Theorem:
(well-known, see e.g.
Wikipedia lemma "Markov Property",
see further Georgii).
Stationary Markov chains, i.e.
invariant Markov measures on histories,
and n.n. Gibbs measures,
in dimension 1,
are the **same** objects.
(Brascamp,Spitzer,...)

Warning: This is about objects (measures)
on *infinite* time/space.

Question:
If we try to be a bit more far-sighted
and change independent to
weakly dependent(continuous, almost Markov),
does this sameness remain true?
(Fernández, Gallo, Maillard, Verbitskiy, Redig,
Pollicott, Walters...)

Answers:

With extra regularity conditions: Yes.
(SRB, Thermodynamic Formalism..).
Without those: NO!
Neither class includes the other.
One direction known (since 2011),
(Fernández, Gallo, Maillard)
other direction new (here).

Time version:

Class of Stochastic Processes,
rediscovered repeatedly,
under a variety of names:
(g-measures=
Chains of Infinite Order=
Chains with Complete Connections=
Uniform Martingales/Random Markov
Processes).
(Keane 70's, Harris 50's,
Onicescu-Mihoc and Doeblin-Fortet 30's,
Kalikow 90's).
Studied in Ergodic Theory, Probability.

Spatial version:
Gibbs (=DLR) measures
= Gibbs = " almost" Markov random fields.
Discovered independently,
in East (mathematics)
and West (physics),
(Dobrushin, Lanford-Ruelle 60's).
Mathematical Physics.
Here two-state -Bernoulli- variables,
(= **Ising** spins:)
$\omega_i = \pm$, for all $i \in Z$.
(Can be much more general.)
Warning:
DLR Gibbs $\neq$ SRB Gibbs.

Gibbs measures:

Let G be an infinite graph, here Z.

Configuration space:

Space of sequences: $\Omega = \{-,+\}^G$.

Probability measures on Ω,

labeleled by **interactions**.

An interaction is a collection of functions,

$\Phi_X(\omega)$, dependent on $\{-,+\}^X$,

where the X are subsets of G.

Let Λ be a finite subset of G.

We write $\Omega_\Lambda = \{-,+\}^\Lambda$.

Energy (**Hamiltonian**)
$H_{\Lambda}^{\Phi,\tau}(\omega) = \sum_{X \cap \Lambda \neq \emptyset} \Phi_X(\omega_\Lambda \tau_{\Lambda^c})$.
Sum of interaction energy terms.
A measure μ is *Gibbs* iff:
(A version of) the
conditional probabilities of
finite-volume configurations,
given the outside configuration, satisfies:
$\mu(\omega_\Lambda | \tau_{\Lambda^c}) = \frac{1}{Z_{\Lambda}^{\tau}} \exp - \sum_{X \cap \Lambda \neq \emptyset} \Phi_X(\omega_\Lambda \tau_{\Lambda^c})$.
for ALL configurations ω,
boundary conditions τ
and finite volumes Λ.

Gibbsian form.
Rigorous version of
"$\mu = \frac{1}{Z} \exp -H$",
Gibbs canonical ensemble.
Larger energy means
exponentially smaller probability.
Nearest-neighbour interaction means that
$\Phi(X) = 0$,
except when $X = \{i, i+1\}$ or $X = i$,
for some $i \in Z$.
A Gibbs measure for a nearest-neighbour
model satisfies a
spatial Markov property:
$\mu(\omega_{\{1,\ldots,n\}} | \tau_{\{1\ldots n\}^c}) = \mu(\omega_{\{1\ldots n\}} | \tau_0 \tau_{n+1})$.
Conditioned on the border spins,
at 0 and $n+1$,
inside and *outside* are independent.

A two-state Markov chain is again
a measure on the same sequence space Ω.
Now it has to satisfy the " ordinary"
(timelike) Markov property:
$\mu(\omega_{\{1...n\}}|\tau_{\{-\infty,....,0\}}) = \mu(\omega_{\{1...n\}}|\tau_0)$.
One can describe this via a product
of 2-by-2 stochastic matrices P
with non-zero entries:
$P(k,l) = P(\omega_i = k \rightarrow \omega_{i+1} = l)$.
Here $k, l = \pm$ and i is any site (=time) in Z.
There is a one-to-one connection between
stationary (time-invariant)
2-state Markov Chains
and (space-translation-invariant) nearest-neighbor
Ising Gibbs measures.

Continuity (=almost Markov = quasilocality).
Product topology:
Two sequences are close if they are
equal on a large enough finite interval.
Topology metrisable,
metric e.g. by:
$d(\omega, \omega') = 2^{-|n|}$,
where n is the site with
minimal distance from origin such that
$\omega_n \neq \omega'_n$.
A function is continuous,
if it depends weakly on sites far away
and mostly on what happens not too far,
(or not too long ago)
whatever it is.

Processes (time):
$\mu(\sigma_0 = \omega_0 | \omega_{Z-}) = g(\omega_0 \omega_{Z-})$,
with g-function continuous.
Probability of getting ω_0, given the past.
Continuous dependence on the **past**.
Continuity studied since the 30's
(Doeblin-Fortet).
Claim!?:
Continuity implies uniqueness (Harris(50's)).
Mistake in proof pointed out by Keane (70's).
Counterexamples due to Bramson-Kalikow (90's).
Sharper criterion Berger-Hoffman-Sidoravicius (2003-2017).

Gibbs measures:
Continuity of conditional probabilities
corresponds to summability of interactions.
$\sum_{0 \in X} ||\Phi_X|| < \infty$.
Continuous dependence on **outside**
beyond the border.
(Quasilocality).
No action at a distance.
(No observable influence from behind the moon)
Plus: "non-nullness".
Any finite change in the -infinite- system
costs a finite amount of energy.
Any configuration in finite domain
occurs with finite probability,
whatever is happening outside.
Gibbs measures satisfy (equivalently) a
finite-energy condition.
Equivalence holds (Kozlov-Sullivan):
Finite-energy + continuity = Gibbs.

Our Counterexample:
(Gibbs, non-g-measure).
Gibbs measures for Dyson models.
Low temperatures.
Long-range Ising models.
Ferromagnetic pair interactions.
$\Phi_{i,j}(\omega) = -J|i-j|^{-\alpha}\omega_i\omega_j$.
Interesting regime $1 < \alpha \leq 2$.
Phase transition for large J,
at low temperatures:
There exist then **two** different
Gibbs measures,
for the same interaction,
called μ^+ and μ^-, for such Φ.
Spatially continuous conditional probabilities.
Warning:
Impossible for Markov Chains or Fields,
always uniqueness.

Claim:
At low T and for $\alpha^* < \alpha < 2$
Dyson Gibbs measures are not g-measures.
Here technical condition $\alpha^* = 3 - \frac{\ln 3}{\ln 2}$.
Proof uses technically rather hard Input,
perturbative, cluster expansions, from others,
giving the α^* condition,
plus three simple Observations.

Input:
Interface result for Dyson models
(Cassandro,Merola,Picco,Rozikov).
Take interval $[-L,+L]$,
all spins to the left are minus,
all spins to the right are plus.
Then there is an interface point **IF**, such that:
1) To the left of the interface
we are in the minus phase (μ^-),
to the right of the interface
we are in the plus phase (μ^+).
2) With overwhelming probability the location
of the interface is at most $O(L^{\frac{\alpha}{2}})$ from the center.
$... - - - - - m....|\mathbf{IF}| + m.....| + + + + + ...$

Observation 1:
If I change all spins left of a length-N interval of minuses, the effect from the left on the central O(L) interval is bounded by $O(LN^{1-\alpha})$, thus small for N large.
Consequence:
A large interval of minuses (size N) will have a moderately large (size L) interval of minus phase on both sides. Interfaces are pushed away.

Observation 2:
If I decouple a comparatively small interval,
of size $L_1 = o(L)$,
in the beginning of my minus-phase interval,
this hardly changes the interface location.
(Cost of **IF** shift by εL is larger, namely $O(L^{2-\alpha})$.
Shown by Cassandro et al.)

Observation 3:
If I make in this L_1 interval
an alternating configuration
+-+-+-+-+-...
then the total energy (influence)
on its complement
is bounded by the double sum
$\sum_{i=1....L_1, j>L_1}(|j-i|^{-\alpha} - |j+1-i|^{-\alpha})=$
$\sum_{i=1....L_1, j>L_1}(O(|j-i|^{-(\alpha+1)})=$
$\sum_{i=1....L_1} O(|i|^{-\alpha})$
which is bounded, uniformly in L_1.
Therefore finite, small effect.

Remark:
Effect only at positive temperature.
Entropic Repulsion.
A large alternating interval,
preceded by a MUCH
larger interval of minuses,
cannot shield the influence
of this homogeneous minus interval.
But this means precisely that
the conditional probability of finding a plus (or a minus),
at a given site, conditioned on an alternating past,
is not continuous.
Thus two-sided continuity
occurring at the same time
as one-sided discontinuity.

Alternating configuration is discontinuity point,
due to cancellations of pluses and minuses.
Set of discontinuity points may have measure zero, but nonremovable.

$... - - - - - + - + - + - X$ (- N, alt_L intervals)

versus

$... + + + + - + - + - + - X$ (+N, alt_L intervals)

Expected value of X differs,
by more than cst,
uniformly in L and N(L).
Direct influence from Deep Past.

Analogies with higher-dimensional
Gibbs measures.
Analogy g-measures:
Global Markov property.
Conditioning on infinite-volume
(like half-line) events.
There are Markov fields
which are not Globally Markov.
Other analogy:
There are Markov fields which depend
discontinuously on lexicographic past.

Open Question (Bethuelsen-Conache),
trigger: Schonmann projection
(one-dimensional marginal,
of 2d Ising measures).
Entropic repulsion in one or two directions?
Is it a g-measure?
Partial results so far, suggesting different behaviour.
Non-Gibbs, possibly g-measure.

Conclusion:

Two-sided continuous dependence
-spacelike- does *not* imply
one-sided continuous dependence
-timelike.

Summary:

Controlling borders is NOT the same as
control of history,
except for the shortsighted.

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Further questions:
1) Get rid of the technical restriction on α, and large n.n.term, with Bissacot, Endo, Kimura, Ruszel.
(Kimura, Littin-Picco)
2) Understand $\alpha = 2$ case.

Anton has, for me as for many, been a reliable and inspiring combination of friend, colleague, guide and mentor into probability and disorder.

For this many thanks.

Congratulations!

And many more years.