Metastability for Glauber Dynamics on Random Graphs

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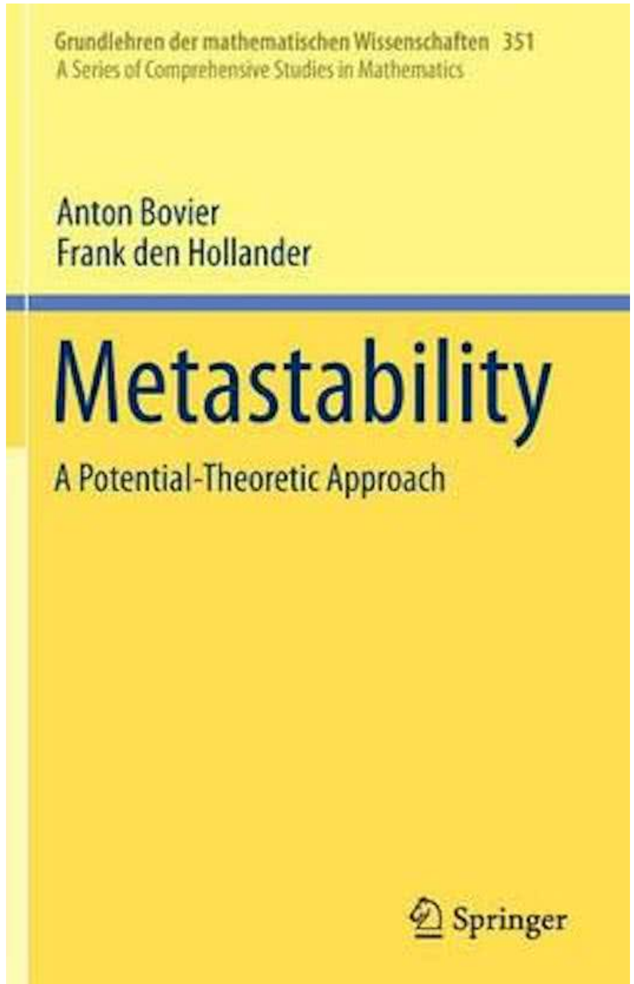
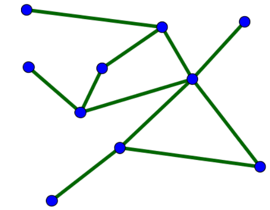


A magnum opus in 2015

OUTLINE:

- Glauber dynamics on graphs. setting
- Metastability. basic facts
- Complete graph. old stuff
- Erdős-Rényi random graph. new stuff

§ GLAUBER DYNAMICS ON GRAPHS



Let $G = (V, E)$ be a connected graph. Ising spins are attached to the vertices V and interact with each other along the edges E.

1. The energy associated with the configuration $\sigma = (\sigma_i)_{i \in V} \in \mathcal{X} = \{-1, +1\}^V$ is given by the Hamiltonian

$$H(\sigma) = -J \sum_{(i,j) \in E} \sigma_i \sigma_j - h \sum_{i \in V} \sigma_i$$

where $J > 0$ is the ferromagnetic interaction strength and $h > 0$ is the external magnetic field.

2. Spins flip according to Glauber dynamics

$$\forall \sigma \in \mathcal{X}, \forall j \in V: \quad \sigma \to \sigma^j \text{ at rate } e^{-\beta[H(\sigma^j)-H(\sigma)]_+}$$

where σ^j is the configuration obtained from σ by flipping the spin at vertex j, and $\beta > 0$ is the inverse temperature.

3. The Gibbs ensemble

$$\mathbb{P}(\sigma) = \frac{1}{Z} e^{-\beta H(\sigma)}, \qquad \sigma \in \mathcal{X},$$

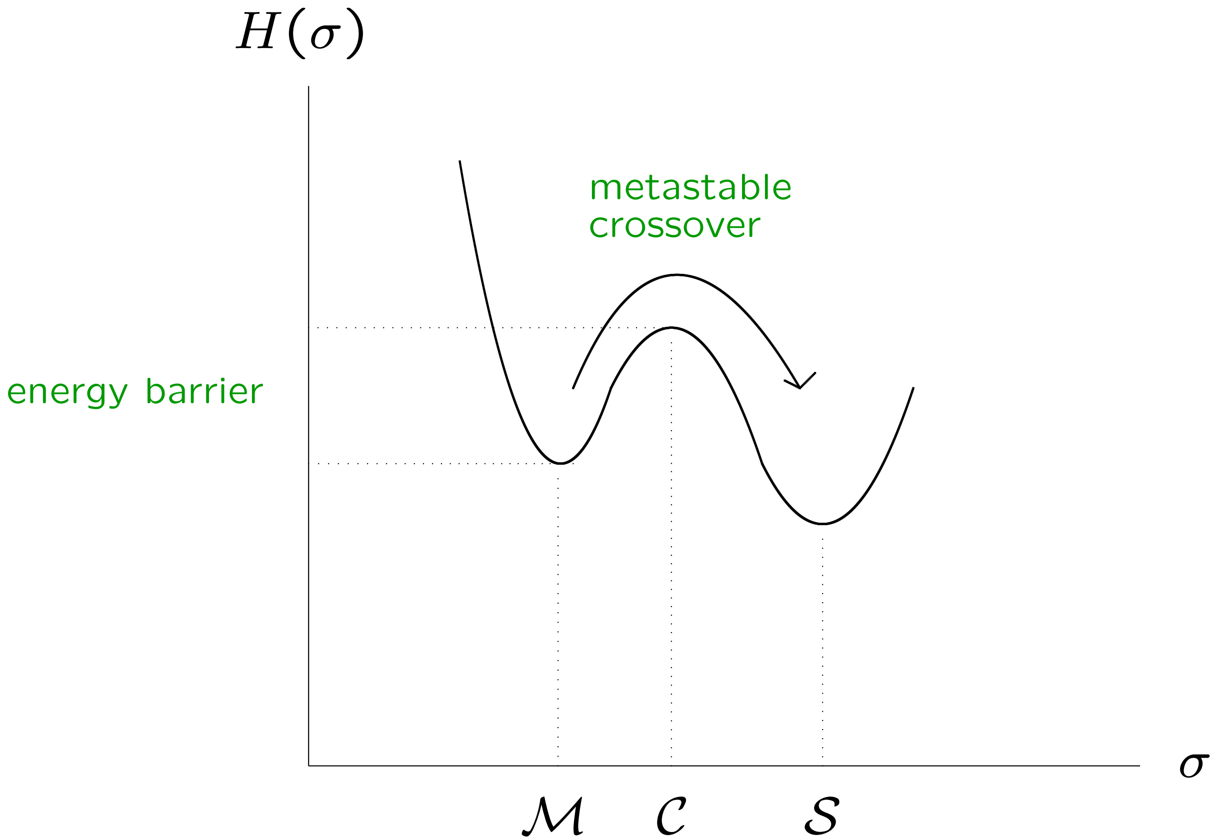
is the reversible equilibrium of this dynamics.

4. Three sets of configurations play a central role:

$\mathcal{M}$ = metastable state

$\mathcal{C}$ = crossover sate

$\mathcal{S}$ = stable state.



Caricature of the energy landscape

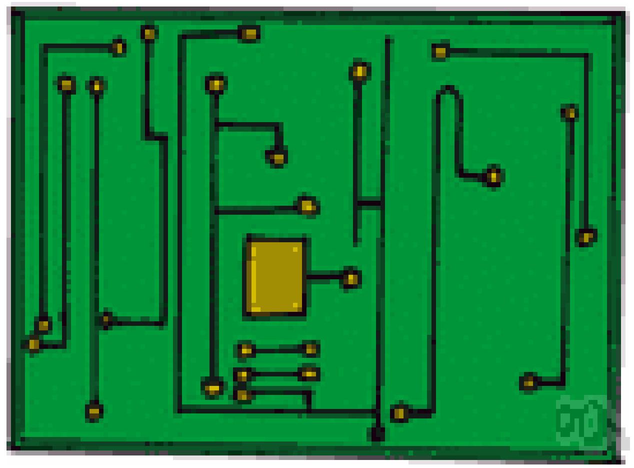
§ METASTABILITY

While the system resides in the valley around $\mathcal{M}$, it makes many unsuccessful attempts to cross over to the valley around $\mathcal{S}$. It manages to do so only after it reaches the set $\mathcal{C}$ of saddle-point configurations, called critical droplets.

Key question: How long does Glauber dynamics need to achieve the crossover from $\mathcal{M}$ to $\mathcal{S}$?

Potential theory provides the answer!

Bovier, Eckhoff, Gayrard, Klein 2001–2004



Random Dynamics $\Longleftrightarrow$ Electric Network

spin configurations	$\longrightarrow$	nodes
spin flips	$\longrightarrow$	links
transition rates	$\longrightarrow$	conductances

Metastable Crossover Time $\Longleftrightarrow$ Effective Resistance

THEOREM: Bovier, Eckhoff, Gayrard, Klein 2001

Let $\mathbb{P}_{\mathcal{M}}$ *denote the probability distribution on path space of the Glauber dynamics starting at* $\mathcal{M}$*. Let* $\tau_{\mathcal{S}}$ *denote the first hitting time of* $\mathcal{S}$*. Then*

$$\boxed{\mathbb{E}_{\mathcal{M}}(\tau_{\mathcal{S}}) = \frac{1 + o(1)}{Z\text{cap}(\mathcal{M}, \mathcal{S})}}$$

with $\text{cap}(\mathcal{M}, \mathcal{S})$ *the capacity of the pair* $(\mathcal{M}, \mathcal{S})$*.*

Here, $o(1)$ refers to a parameter regime where the system is metastable, e.g. for low temperature or large volume.

The capacity can be expressed as the variational principle

$$\mathrm{cap}(\mathcal{M},\mathcal{S}) = \inf_{\phi\in\Phi_{\mathcal{M},\mathcal{S}}} \frac{1}{Z} \sum_{\sigma,\sigma'\in\mathcal{X}} e^{-\beta[H(\sigma)\vee H(\sigma')]} [\phi(\sigma') - \phi(\sigma)]^2,$$

where

$$\Phi_{\mathcal{M},\mathcal{S}} = \{\phi\colon\ \mathcal{X}\to[0,1]\colon\ \phi(\mathcal{M})=0,\ \phi(\mathcal{S})=1\}.$$

The sum under the infimum is the Dirichlet form associated with the dynamics.

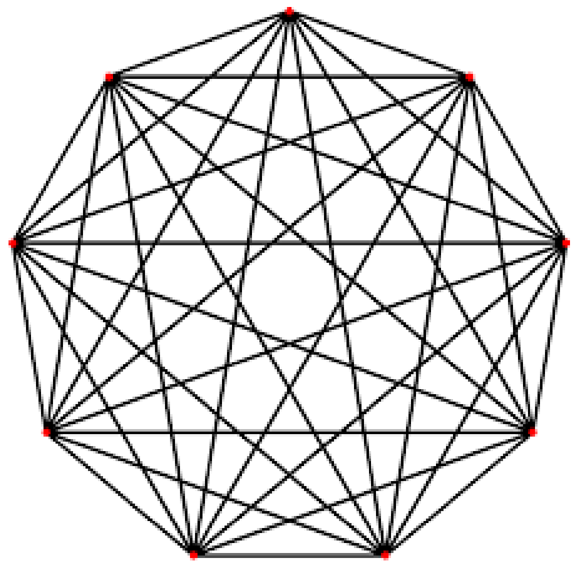
Dirichlet

What makes this variational principle so powerful is that upper and lower bounds on $\mathrm{cap}(\mathcal{M}, \mathcal{S})$ can be obtained by combining physical insight with a clever choice of

- test function ϕ
- truncation of $\mathcal{X}$

The key idea is that in metastable regimes the high-dimensional Dirichlet form is largely controlled by the low-dimensional set of critical droplets.

§ COMPLETE GRAPH



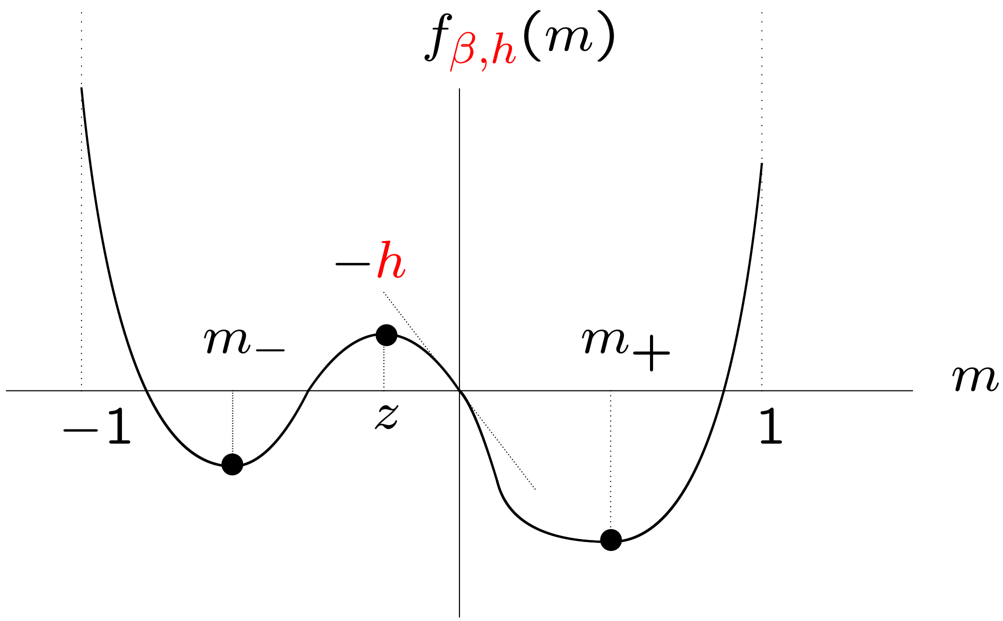
Complete graph: Curie-Weiss

In the limit as $N \to \infty$, the free energy per vertex when the magnetization is m equals

$$f_{\beta,h}(m) = -\tfrac{1}{2}m^2 - hm + \beta^{-1} I(m)$$

with

$$I(m) = \tfrac{1}{2}(1+m)\log(1+m) + \tfrac{1}{2}(1-m)\log(1-m).$$



THEOREM: Bovier, Eckhoff, Gayrard, Klein 2001

On the complete graph with N vertices, for $J = 1/N$, $h \in (0, \Theta(\beta))$ and $\beta > 1$,

$$\mathbb{E}_{\mathcal{M}_N}(\tau_{\mathcal{S}_N}) = K\, e^{N\Gamma}[1 + o(1)], \qquad N \to \infty,$$

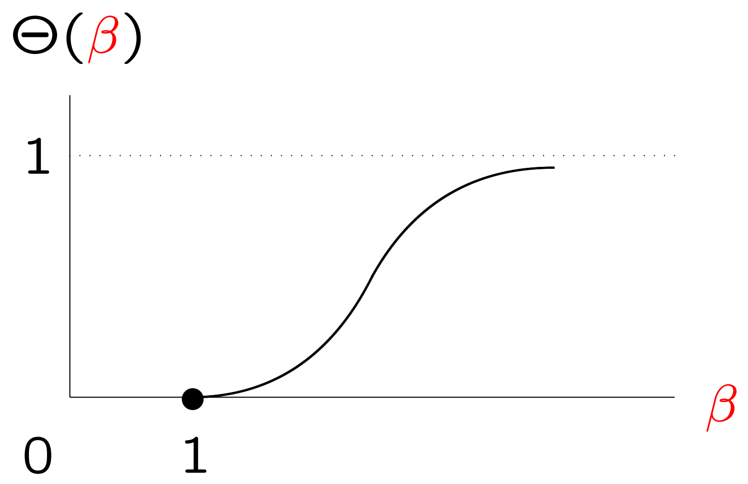
where $\mathcal{M}_N, \mathcal{S}_N$ are the sets of configurations for which the magnetization tends to m_-, m_+,

$$\Gamma = \beta\,[f_{\beta,h}(z) - f_{\beta,h}(m_-)]$$

$$K = \pi\beta^{-1}\sqrt{\frac{1-z}{1+z}\,\frac{1}{1-m_-^2}\,\frac{1}{[-f''_{\beta,h}(z)]f''_{\beta,h}(m_-)}}$$

and

$$\Theta(\beta) = \sqrt{1-\tfrac{1}{\beta}} - \tfrac{1}{2\beta}\log\left[\beta\left(1+\sqrt{1-\tfrac{1}{\beta}}\right)^2\right].$$



The conditions on J, h, β are needed to ensure that the free energy $m \mapsto f_{\beta,h}(m)$ has a double-well structure.

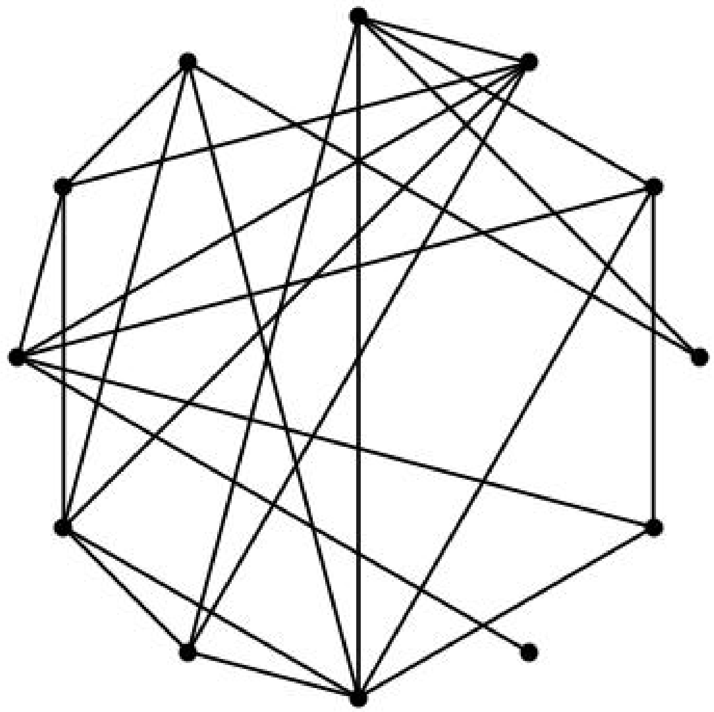
The proof uses a lumping technique typical for mean-field models: for finite N the magnetization performs a random walk on the $\frac{2}{N}$-grid in $[-1, 1]$ in a potential close to $f_{\beta,h}$.

In the limit as $N \to \infty$, the system behaves like

Brownian motion in a double-well potential

analysed by Kramers 1940.

§ ERDŐS-RÉNYI RANDOM GRAPH



Erdős-Rényi random graph: edge percolation

Take the complete graph with N vertices and retain edges with probability $p \in (0, 1)$.

THEOREM: dH, Jovanovski 2018

On the Erdős-Rényi random graph with N vertices, for $J = 1/N$, $h \in (0, p\Theta(\beta p))$ and $\beta > 1/p$,

$$\mathbb{E}_{\mathcal{M}_N}(\tau_{\mathcal{S}_N}) = O(N^5)\, e^{N\Gamma(p)}, \qquad N \to \infty,$$

where $\Gamma(p)$ is the same as Γ on the complete graph, but with $J = 1/N$ replaced by $J = p/N$.

The conditions on J, h, β are again needed to ensure that the free energy $m \mapsto f_{\beta,h,p}(m)$ has a double-well structure.

Apart from a polynomial error term, the crossover time is the same as on the complete graph with average interaction strength.

From the shape of Θ, we find that for $\beta \to \infty$ any $h \in (0, p)$ is metastable, while for $\beta \downarrow 1/p$ or $p \downarrow 0$ no h is metastable.

The latter observation also explains why we do not consider the non-dense Erdős-Rényi random graph with $p = p_N \downarrow 0$ as $N \to \infty$.

On the complete graph the prefactor is constant in N and is computable. On the Erdős-Rényi random graph it is more involved, and for now we only know that it is $O(N^5)$.

We do not expect the prefactor to be of order 1. It may very well be random.

The prefactor is related to the entropy of the set of critical droplets.

RANDOM MAGNETIC FIELD

An interesting model is where the randomness sits on the vertices rather than on the edges, namely,

$$H(\sigma) = -\frac{1}{N} \sum_{1 \leq i,j \leq N} \sigma_i \sigma_j - \sum_{1 \leq i \leq N} h_i \sigma_i,$$

where h_i, $1 \leq i \leq N$, are i.i.d. random variables drawn from a common probability distribution ν on $\mathbb{R}$.

Bovier, Eckhoff, Gayrard, Klein 2001 ν discrete

Bianchi, Bovier, Ioffe 2009 + 2012 ν continuous

The prefactor turns out to be constant in N and to be a somewhat involved function of ν.

Our model is harder because the interaction runs along the set of edges, which has an intricate spatial structure.

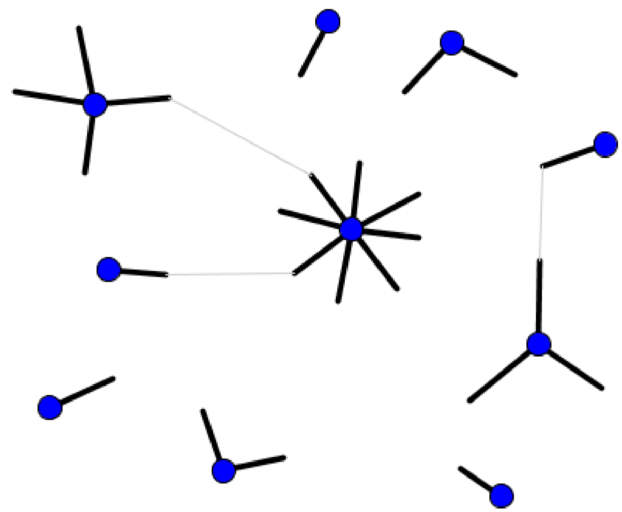
Lumping technniques cannot be used (in the above papers the magnetization is monitored on the level sets of the magnetic field).

Our proof relies on elaborate coupling techniques, together with concentration estimates for the Erdős-Rényi random graph.

FUTURE CHALLENGES

- Elucidate the nature of the prefactor, which should be inversely proportional to the cardinality of the set of critical droplets.
- What can be said in the sparse regime after a proper scaling of the interaction strength?

RELATED WORK



Rough estimates for the average metastable crossover time are known for the configuration model (a random graph with prescribed degrees) when N, J, h are fixed and $\beta \to \infty$.

Dommers 2017

Dommers, dH, Jovanovski, Nardi 2017

TAKE-HOME MESSAGE

Prefactors of average metastable crossover times are delicate objects for random graphs, because they depend in an intricate manner on the underlying geometry.

Prefactors are controlled by the cardinality of the set of critical droplets and may be random.

Very little is known so far

and much remains to be done!