Maximum of branching random walk in random environment

Jiří Černý
University of Basel

with A. Drewitz (Köln)

August 27, 2018

Definition of the model

Random environment: $(\xi_x)_{x\in\mathbb{Z}}$ i.i.d. under $\mathbb{P}$, which is non-degenerate and elliptic,

$$0 \le c \le \xi_x \le C$$

Branching random walk in RE: Given (ξ_x),

- start with one particle at 0
- every particle performs a continuous time SRW on $\mathbb{Z}$
- when at x, every particle branches (binary) at rate ξ_x
- all particles move independently

Notation: $\mathtt{P}_0^\xi$ - quenched distribution of the BRWRE

Questions.

Questions.

- Behaviour of the fastest (right-most) particle.
- Relation to other models: PAM and randomized F-KPP equation.

Questions.

Questions.

- Behaviour of the fastest (right-most) particle.
- Relation to other models: PAM and randomized F-KPP equation.

Notation:

- N_t = the set of particles at time t
- $(Y_s)_{s \le t}$ trajectory of a particle $Y \in N_t$
- position of the maximal particle

$$M_t = \max\{Y_t : Y \in N_t\}$$

- m_t = median of M_t under $\mathtt{P}_0^\xi$ (random variable under $\mathbb{P}$)

Homogeneous BRW/BBM

In the homogeneous situation a lot is known

- LLN: $\frac{M_t}{t} \xrightarrow{t\to\infty} v_0$ a.s.
- precise asymptotics

$$m_t = v_0 t - \frac{3}{2} c \log t + O(1)$$

- tightness: $(M_t - m_t)_{t\geq 0}$ is tight
- point-process convergence: $\sum_{Y\in N_t} \delta_{Y_t - m_t}$ converges

'Failure' of the first moment prediction:

Set $N^{\geq}(t,x) = \#\{Y \in N_t : Y_t \geq x\}$ and define

$$\overline{m}_t = \sup\{x \in \mathbb{Z} : \mathtt{E}_0 N^{\geq}(t,x) \geq \tfrac{1}{2}\}.$$

Then

$$\overline{m}_t = v_0 t - \frac{1}{2} c \log t + O(1)$$

Homogeneous branching random walk II

Relation to the discrete F-KPP equation:

$$\partial_t w(t,x) = \Delta w(t,x) + w(t,x)(1-w(t,x))$$
$$w(0,x) = w_0(x)$$

Is solved by

$$w(t,x) = 1 - \mathtt{E}_0\Big[\prod_{Y\in N_t}(1-w_0(x-Y_t))\Big]$$

In particular, for $w_0 = \mathbf{1}_{-\mathbb{N}_0}$,

$$w(t,x) = \mathtt{P}_0(M_t \geq x).$$

Some other properties of the F-KPP equation.

- F-KPP equation has *travelling wave* solutions $w(t,x) = w_v(x-vt)$ for every $v \geq v_0$
- If $w_0 = \mathbf{1}_{-\mathbb{N}_0}$, then $w(t,x+m_t) \to w_{v_0}(x)$

Previous results

Related models

- BRW in temporarily varying environment (Bovier-Kurkova, Bovier-Hartung, Fang-Zeitouni)
- BRW in temporarily random environment (Malein-Miłoś 2015)

Maximal particle of BRW

- Comets-Popov 2007: Shape theorem for BRWRE in $\mathbb{Z}^d$
  $\implies$ LLN for the maximal particle

Previous results: Parabolic Anderson model

PAM: linear PDE with random coefficients

$$\begin{aligned} \partial_t u(t,x) &= \Delta u(t,x) + \xi(x) u(t,x) \\ u(0,x) &= \mathbf{1}_0(x) \end{aligned}$$

PAM gives the first moment of the BRWRE:

$$u(t,x) = \mathtt{E}^{\xi}[N(t,x)]$$

Lyapunov exponent: $\mathbb{P}$-a.s.

$$\lambda(v) = \lim_{t\to\infty} \frac{1}{t} \log u(t, \lfloor tv \rfloor), \qquad v \in \mathbb{R}.$$

Two important velocities:

- v_0: solution to $\lambda(v_0) = 0$, $v_0 > 0$.
- $v_c \geq 0$: minimal v s.t. λ is strictly convex on (v_c, ∞).

Breakpoint = the front of the PAM

$$\overline{m}_t = \sup\left\{ x \in \mathbb{Z} : \mathtt{E}_0^{\xi} \sum_{y \geq x} u(t,y) \geq \tfrac{1}{2} \right\}$$

Results for the BRWRE

Theorem (ČD'17)

Assume ξ is non-degenerate, elliptic and $v_0 > v_c$. Then

- *LLN (CP'07):* $\frac{M_t}{t} \to v_0$, $\mathbb{P} \times \mathtt{P}_0^{\xi}$*-a.s.*
- *FCLT for the breakpoint:*

$$\frac{\overline{m}_{nt} - v_0 nt}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} \mathsf{BM}_t$$

- *approximation for the median:* $\overline{m}_t \geq m_t$ *and*

$$\limsup_{t\to\infty} \frac{\overline{m}_t - m_t}{\log t} \leq C, \qquad \mathbb{P}\text{-a.s.}$$

  $\Longrightarrow$ *FCLT for the median*

- *approximation for the maximum:* $|M_t - m_t| \lesssim C \log t$
  $\Longrightarrow$ *FCLT for the maximum*
- *tightness: no proof yet*

Implications for the PAM

Theorem

- *CLT for the breakpoint*
- *For every* $v > v_c$

$$\frac{\log u(t, vt) - t\lambda(v)}{\sigma_v \sqrt{t}} \xrightarrow[t\to\infty]{\mathbb{P}} \mathcal{N}(0, 1).$$

Implication for the randomized F-KPP equation

Randomized F-KPP equation:

$$\partial_t w(t,x) = \Delta w(t,x) + \xi(x) w(t,x)(1-w(t,x))$$
$$w(0,x) = w_0(x)$$

Front of the solution:

$$\hat{m}_t = \sup\{x \in \mathbb{Z} : w(t,x) \geq 1/2\}$$

Previous results: Nolen 2012 gives CLT for $\hat{m}_t$ for initial conditions such that the speed of the front is $> v_0$.

Implication for the randomized F-KPP equation

Randomized F-KPP equation:

$$\partial_t w(t,x) = \Delta w(t,x) + \xi(x) w(t,x)(1 - w(t,x))$$
$$w(0,x) = w_0(x)$$

Front of the solution:

$$\hat{m}_t = \sup\{x \in \mathbb{Z} : w(t,x) \geq 1/2\}$$

Previous results: Nolen 2012 gives CLT for $\hat{m}_t$ for initial conditions such that the speed of the front is $> v_0$.

Relation of BRWRE and rF-KPP:

$$w(t,x) = \mathtt{P}_x^{\xi}[M_t \geq 0]$$

solves rF-KPP with $w_0 = \mathbf{1}_{-\mathbb{N}}$.

Theorem (CLT for the front)

$$\frac{\hat{m}_t - v_0 t}{\sigma \sqrt{t}} \xrightarrow[t \to \infty]{\mathbb{P}} \mathcal{N}(0,1).$$

Tools and ideas I

First and second moment of $N(t,x)$ can be computed with help of Feynman-Kac representation, resp. many-to-one formula (Harris-Roberts '17, O. Gün-König-Sekulović '13),

$$\mathtt{E}_0^\xi\Big[\Big|\Big\{Y \in N_t : \varphi_1(r) \le Y_r \le \varphi_2(r)\ \forall r \in [0,t]\Big\}\Big|\Big]$$
$$= E_0\Big[\exp\Big\{\int_0^t \xi(X_r)\,\mathrm{d}r\Big\}; \varphi_1(r) \le X_r \le \varphi_2(r)\ \forall r \in [0,t]\Big]$$

Tools and ideas I

First and second moment of $N(t,x)$ can be computed with help of Feynman-Kac representation, resp. many-to-one formula (Harris-Roberts '17, O. Gün-König-Sekulović '13),

$$\mathtt{E}_0^\xi\Big[\Big|\Big\{Y \in N_t : \varphi_1(r) \le Y_r \le \varphi_2(r)\ \forall r \in [0,t]\Big\}\Big|\Big] \\ = E_0\Big[\exp\Big\{\int_0^t \xi(X_r)\,\mathrm{d}r\Big\}; \varphi_1(r) \le X_r \le \varphi_2(r)\ \forall r \in [0,t]\Big]$$

$$\begin{aligned}
&\mathtt{E}_0^\xi\Big[\Big|\Big\{Y \in N_t : \varphi_1(r) \le Y_r \le \varphi_2(r)\ \forall r \in [0,t]\Big\}\Big|^2\Big] \\
&= E_0\Big[\exp\Big\{\int_0^t \xi(X_r)\,\mathrm{d}r\Big\}; \varphi_1(r) \le X_r \le \varphi_2(r)\ \forall r \in [0,t]\Big] \\
&+ 2\int_0^t E_0\Big[\exp\Big\{\int_0^s \xi(X_r)\,\mathrm{d}r\Big\}\xi(X_s)\mathbf{1}_{\varphi_1(r)\le X_r\le\varphi_2(r)\ \forall r\in[0,s]} \\
&\times\Big(E_{X_s}\Big[\exp\Big\{\int_0^{t-s}\xi(X_r)\,\mathrm{d}r\Big\}; \varphi_1(r+s) \le X_r \le \varphi_2(r+s)\ \forall r \le t-s\Big]\Big)^2\Big]\mathrm{d}s.
\end{aligned}$$

Why the CLT?

With H_i denoting the hitting time of i, one has

$$\mathtt{E}_0^{\xi} N(t, vt) = E_0\left[\exp\left\{\int_0^t \xi(X_r)\,\mathrm{d}r\right\}; X_t = vt\right]$$

$$= e^{-\eta t} E_0\left[\exp\left\{\int_0^t (\xi(X_r)+\eta)\,\mathrm{d}r\right\}; X_t = vt\right]$$

$$= e^{-\eta t} E_0\left[\exp\left\{\sum_{i=1}^{vt}\int_{H_{i-1}}^{H_i}(\xi(X_r)+\eta)\,\mathrm{d}r + \int_{H_{vt}}^{t}(\xi(X_r)+\eta)\,\mathrm{d}r\right\}; X_t = vt\right]$$

Pick $\eta = \eta(t, x)$ so that $X_t = x$ is a likely event to obtain

$$= e^{-\eta t}\prod_{i=1}^{vt} E_{i-1}\left[\exp\left\{\int_0^{H_i}(\xi(X_r)+\eta)\,\mathrm{d}r\right\}\right] \times \text{error}$$

Why the CLT?

With H_i denoting the hitting time of i, one has

$$\mathtt{E}_0^\xi N(t,vt) = E_0\Big[\exp\Big\{\int_0^t \xi(X_r)\,\mathrm{d}r\Big\}; X_t = vt\Big]$$

$$= e^{-\eta t} E_0\Big[\exp\Big\{\int_0^t (\xi(X_r)+\eta)\,\mathrm{d}r\Big\}; X_t = vt\Big]$$

$$= e^{-\eta t} E_0\Big[\exp\Big\{\sum_{i=1}^{vt}\int_{H_{i-1}}^{H_i}(\xi(X_r)+\eta)\,\mathrm{d}r + \int_{H_{vt}}^{t}(\xi(X_r)+\eta)\,\mathrm{d}r\Big\}; X_t = vt\Big]$$

Pick $\eta = \eta(t,x)$ so that $X_t = x$ is a likely event to obtain

$$= e^{-\eta t}\prod_{i=1}^{vt} E_{i-1}\Big[\exp\Big\{\int_0^{H_i}(\xi(X_r)+\eta)\,\mathrm{d}r\Big\}\Big] \times \text{error}$$

Problem: $\eta(t,x)$ is actually $\eta(t,x,\xi)$

Tools and ideas: homogeneous case

To understand the behaviour of the maximum one needs

- a precise large deviation estimate

$$P_0\left[X_t \sim vt\right] = \frac{c}{\sqrt{t}} e^{-I(v)t}(1 + o(1))$$

- Ballot theorem

$$P_0\left[X_t \sim vt, X_s \le vs \, \forall s \le t\right] \asymp \frac{1}{t} P_0\left[X_t \sim vt\right]$$

Tools and ideas: random case

To understand the behaviour of the maximum one needs

- a precise large deviation estimate

$$E_0\left[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \sim vt\right] \asymp \frac{c}{\sqrt{t}} e^{-I^{\xi}(v,t)}$$

- Ballot theorem

$$E_0\left[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \sim \overline{m}_t, X_s \leq \overline{m}_s \,\forall s \leq t\right]$$
$$\asymp \frac{1}{t^{\gamma}} E_0\left[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \in [vt-1, vt]\right]$$

The Ballot theorem in random environment

B_t and W_t be two independent Brownian motions, variances σ_B^2, σ_W^2.
Question. Understand the behaviour of

$$F(t) = P\Big[B_s + 1 \geq W_s \, \forall s \leq t \Big| W\Big]$$

The Ballot theorem in random environment

B_t and W_t be two independent Brownian motions, variances σ_B^2, σ_W^2.
Question. Understand the behaviour of

$$F(t) = P\left[B_s + 1 \geq W_s \,\forall s \leq t \middle| W\right]$$

Theorem (Malein-Miłoś, 2015)

$$\lim_{t\to\infty} \frac{\log F(t)}{\log t} = -\gamma(\sigma_B, \sigma_W), \qquad W - \textit{a.s.}$$

for some $\gamma(\sigma_B, \sigma_W) > \frac{1}{2}$.

Thank you