Percolation methods for nodal lines

V. Beffara and D. Gayet — Université Grenoble Alpes

CIRM, 30.8.2018

Outline

Spherical harmonics / Laplacian eigenfunctions

Random polynomials / Kostlan ensemble

Percolation

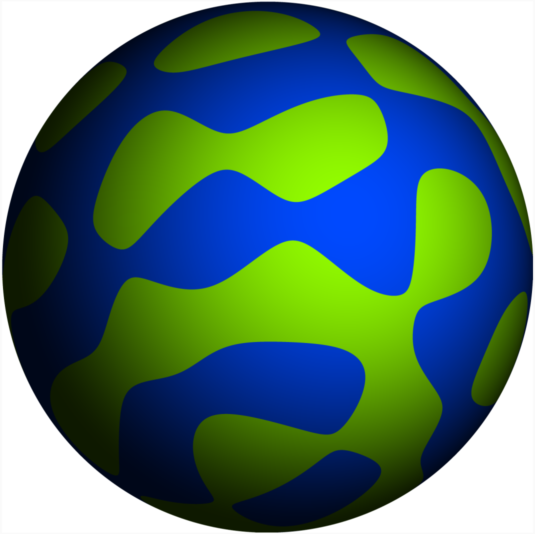
Russo-Seymour-Welsh

Negatively correlated fields

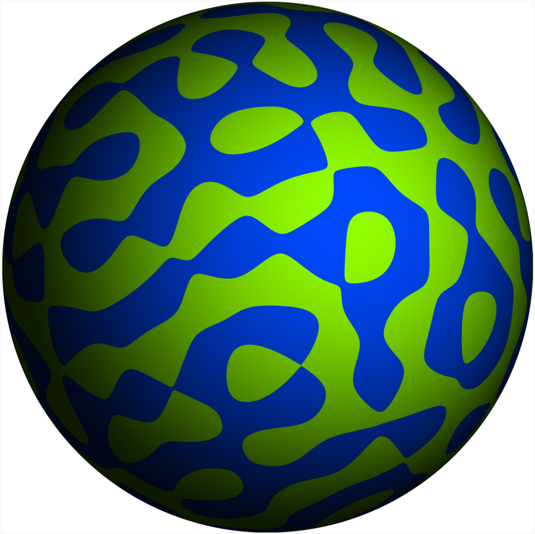
Spherical harmonics / Laplacian eigenfunctions

Random eigenfunction of the Laplacian on the sphere

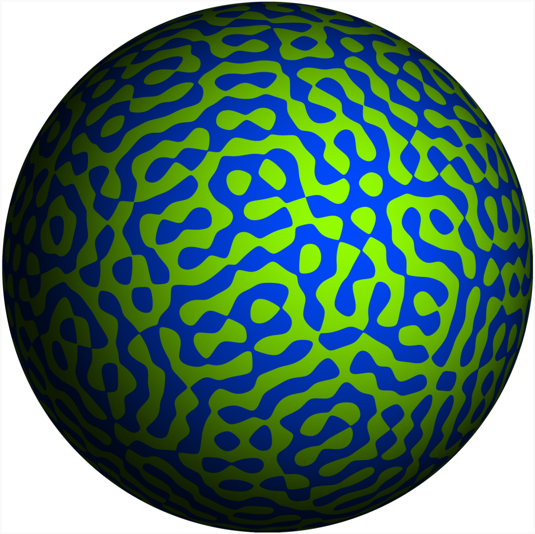
Random eigenfunction of the Laplacian on the sphere



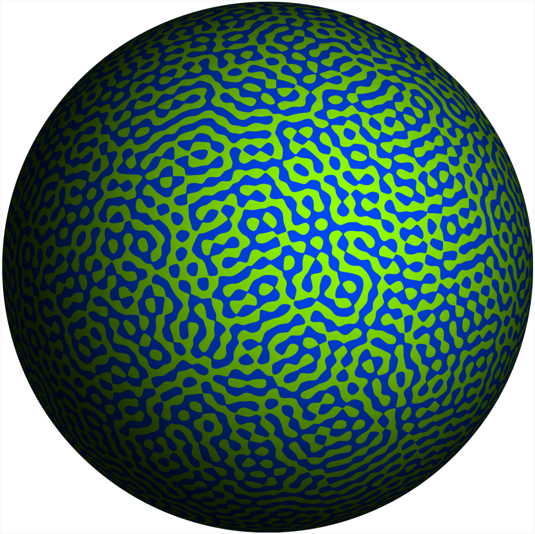
Random eigenfunction of the Laplacian on the sphere



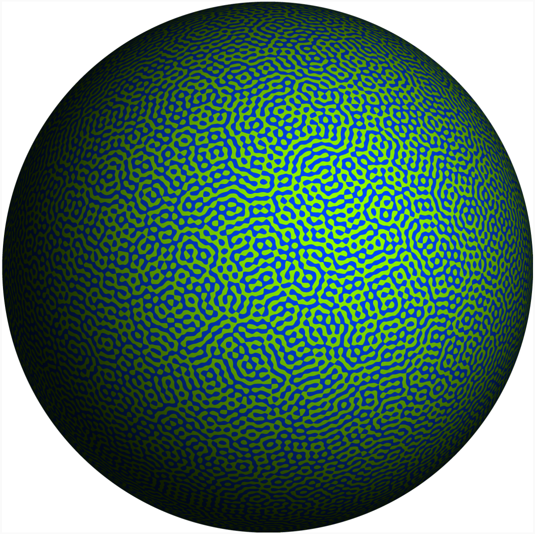
Random eigenfunction of the Laplacian on the sphere



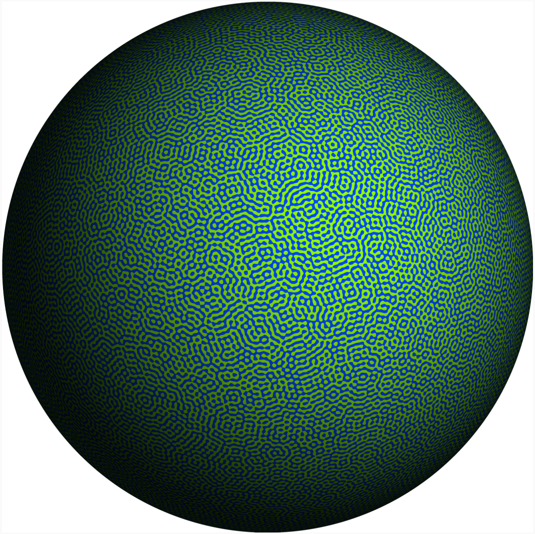
Random eigenfunction of the Laplacian on the sphere



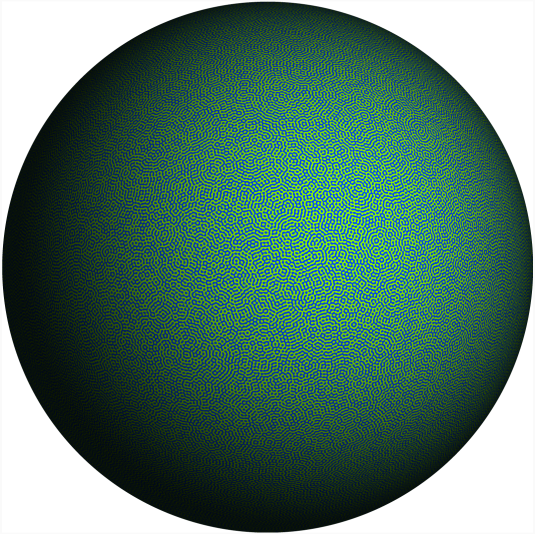
Random eigenfunction of the Laplacian on the sphere



Random eigenfunction of the Laplacian on the sphere



Random eigenfunction of the Laplacian on the sphere



Plane waves on $\mathbb{R}^2$

Consider solutions of the equation

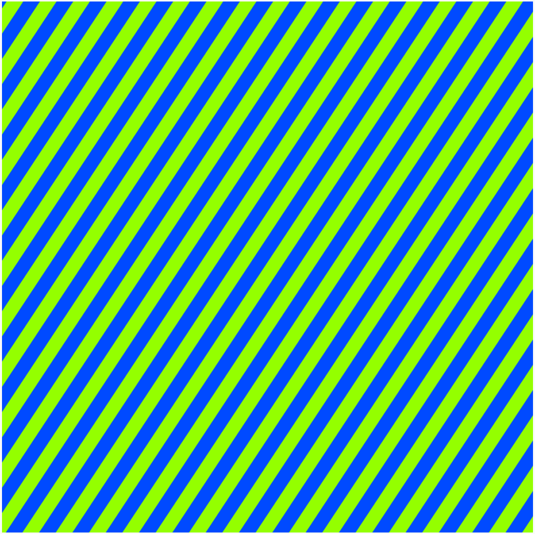
$$\Delta f + \lambda f = 0$$

on the plane. Particular solutions are given by

$$f_{\alpha,\beta}(x, y) = \cos(\alpha x + \beta y + \varphi)$$

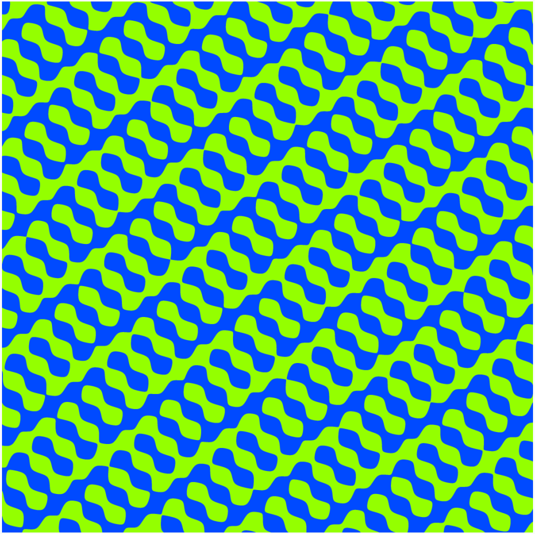
with $\alpha^2 + \beta^2 = \lambda$. By linearity, one can consider linear combinations of the $f_{\alpha,\beta}$.

Plane waves : one component

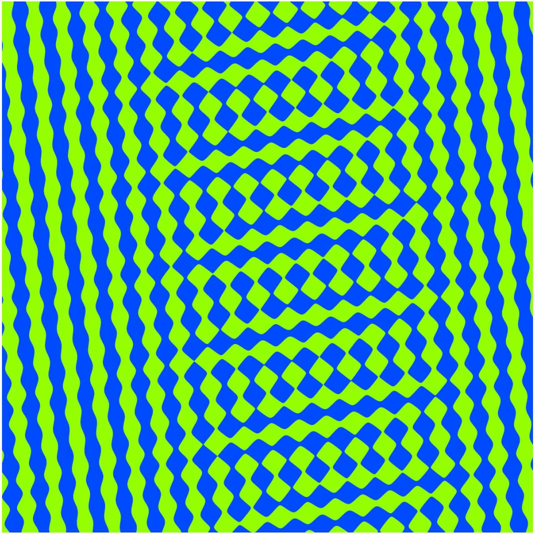


Plane waves : two components

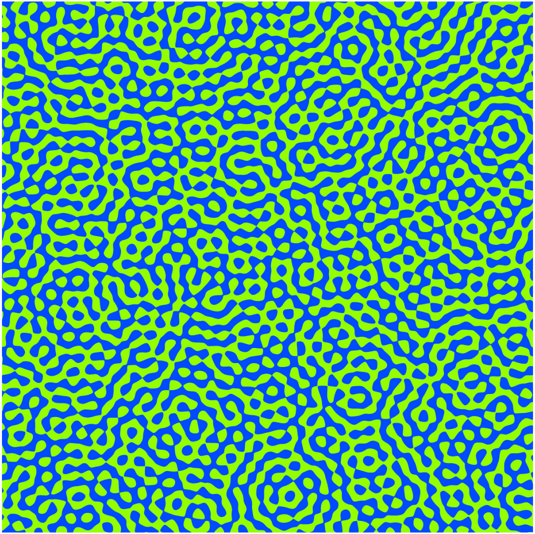
Plane waves : three components



Plane waves : four components



Infinitely many components / local limit on the sphere



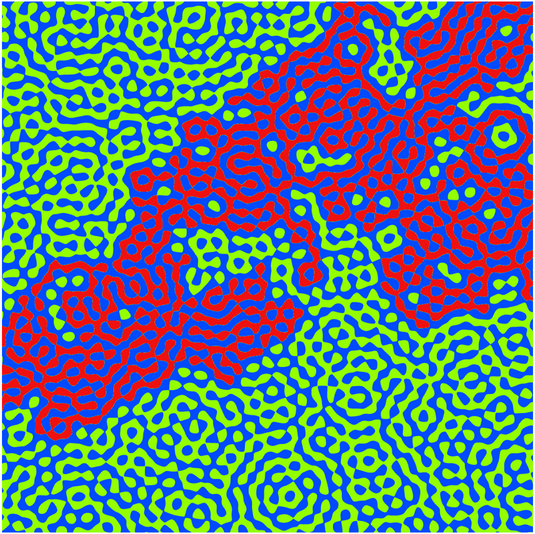
The limit as a Gaussian field

The local limit of random eigenfunctions of Δ as $\lambda \to \infty$ is given by a Gaussian field ϕ of covariance

$$Cov[\phi(x), \phi(y)] = J_0(\|y - x\|)$$

The covariance oscillates, and decays as $1/\sqrt{\|y - x\|}$.

One large connected component



Random polynomials / Kostlan ensemble

Random polynomial

Define a random homogeneous polynomial on $\mathbb{R}^3$ by

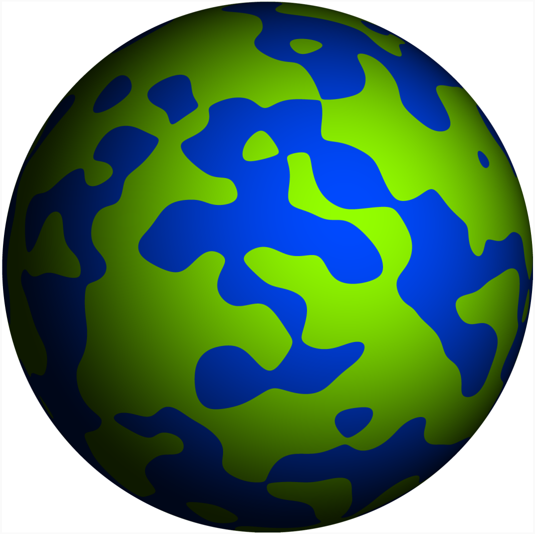
$$P_d(X) = \sum_{|I|=d} a_I \sqrt{\frac{(d+2)!}{I!}} X^I$$

where the a_I are i.i.d. Gaussians.

Restrict it to the unit sphere.

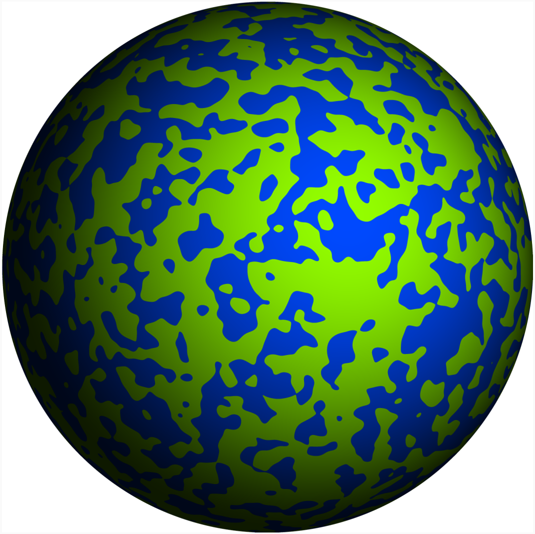
Restriction to the sphere (d=30)

Restriction to the sphere (d=100)

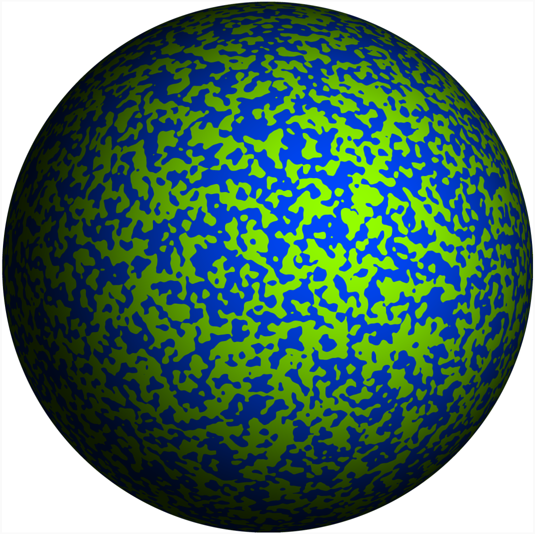


Restriction to the sphere (d=200)

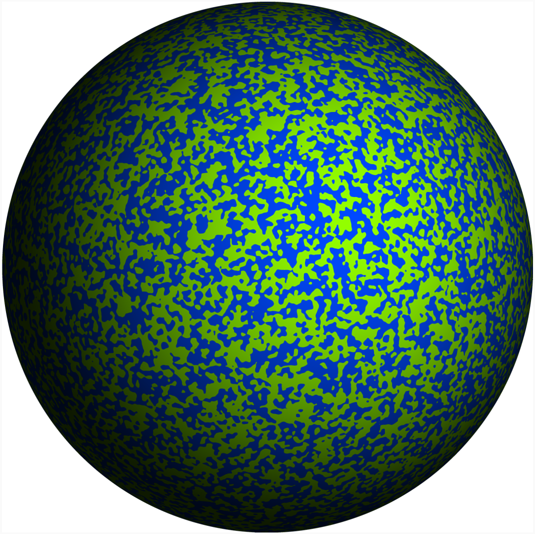
Restriction to the sphere (d=1000)



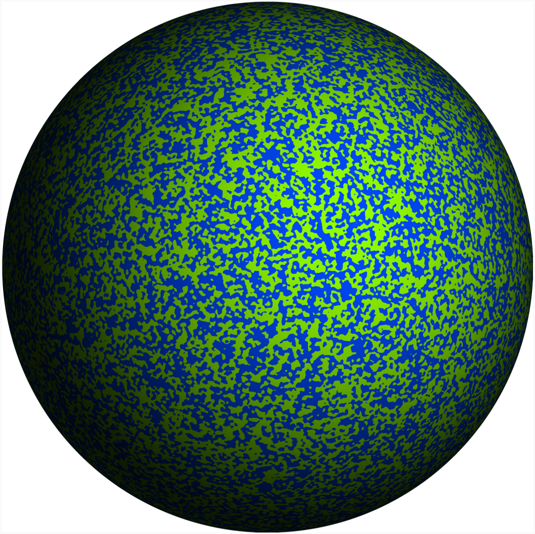
Restriction to the sphere (d=5000)



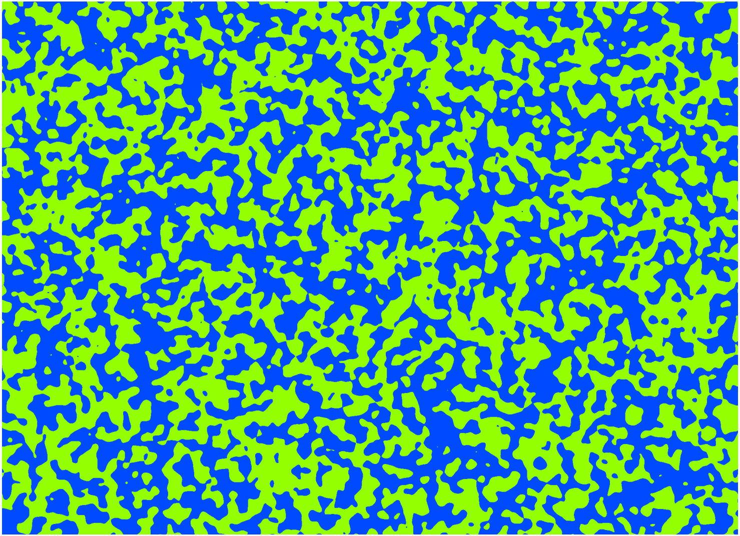
Restriction to the sphere (d=10000)



Restriction to the sphere (d=20000)



Local limit as $d \to \infty$



The limit as a Gaussian field

$$Q_d(x, y) = \sum_{i+j \leqslant d} a_{ij} \sqrt{\frac{(d+2)!}{i!j!(d-i-j)!}} x^i y^j$$

Rescale by a factor $\sqrt{d}$:

$$Q_d(x/\sqrt{d}, y/\sqrt{d}) \simeq \sum_{i+j \leqslant d} \frac{a_{ij}}{\sqrt{i!j!}} x^i y^j$$

In the limit $d \to \infty$:

$$\psi(x, y) = \sum_{i,j \geqslant 0} \frac{a_{ij}}{\sqrt{i!j!}} x^i y^j$$

The limit as a Gaussian field

$$Q_d(x,y) = \sum_{i+j\leqslant d} a_{ij}\sqrt{\frac{(d+2)!}{i!j!(d-i-j)!}}x^i y^j$$

Rescale by a factor $\sqrt{d}$:

$$Q_d(x/\sqrt{d}, y/\sqrt{d}) \simeq \sum_{i+j\leqslant d} \frac{a_{ij}}{\sqrt{i!j!}}x^i y^j$$

In the limit $d \to \infty$:

$$\psi(x,y) = e^{-(x^2+y^2)/2} \sum_{i,j\geqslant 0} \frac{a_{ij}}{\sqrt{i!j!}}x^i y^j$$

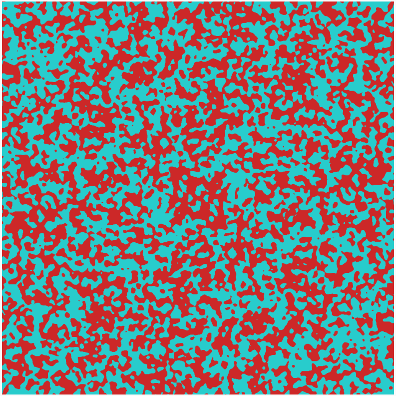
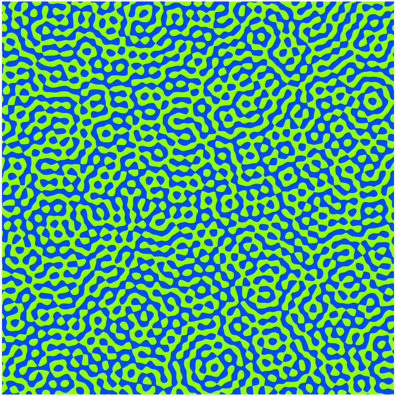
The limit as a Gaussian field

The limit is a stationary centered Gaussian field ψ on $\mathbb{R}^2$, with covariance given by

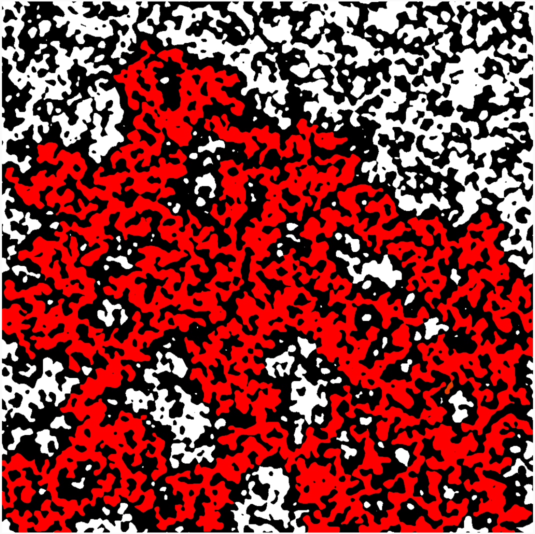
$$Cov[\psi(x), \psi(y)] = \exp(-\|y - x\|^2/2).$$

In particular, the covariance is positive and decays very fast.

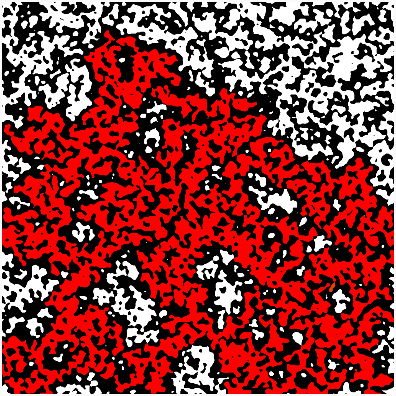
Comparison between the two models

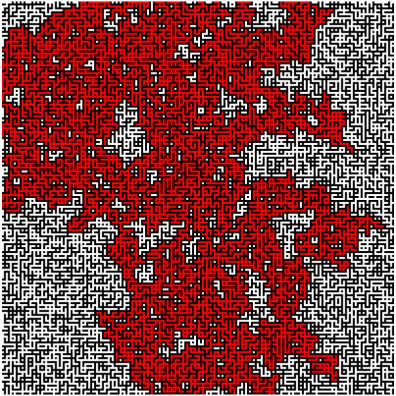
 

A large connected component in ψ



The same, and a critical percolation cluster





Percolation

Percolation : classical results

- Kesten (1980) : $p_c = 1/2$
- For $p < p_c$, sub-critical regime :
  - All clusters are a.s. finite
  - $P[0 \longleftrightarrow x] \approx \exp(-\lambda_p \|x\|)$
  - Largest cluster in Λ_n has diameter $\approx \log n$
- For $p > p_c$, super-critical regime :
  - There exists a.s. a unique infinite cluster
  - $P[0 \longleftrightarrow x, |C(x)| < \infty] \approx \exp(-\lambda_p \|x\|)$
  - Largest *finite* cluster in Λ_n has diameter $\approx \log n$
- At $p = p_c$, critical regime :
  - All clusters are a.s. finite
  - $P[0 \longleftrightarrow x] \approx \|x\|^{-5/24}$
  - Largest cluster in Λ_n has diameter $\approx n$

Russo-Seymour-Welsh

Russo-Seymour-Welsh for critical percolation

Theorem (RSW)

For every $\lambda > 0$ *there exists* $c \in (0,1)$ *such that for all* n *large enough,*

$$c \leqslant P_{p_c}[LR(\lambda n, n)] \leqslant 1 - c.$$

Russo-Seymour-Welsh for critical percolation

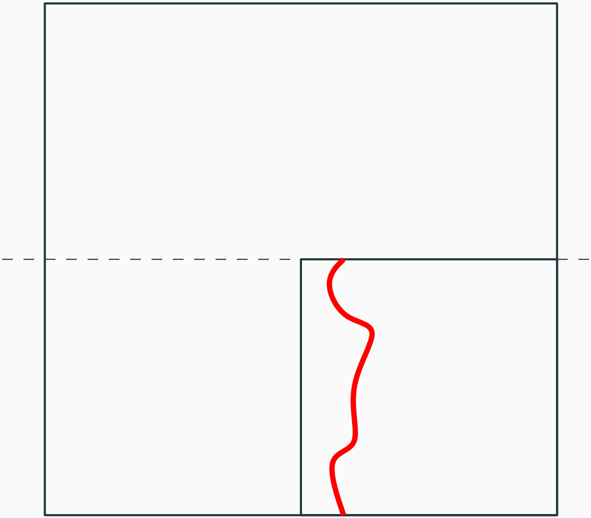
Theorem (RSW)

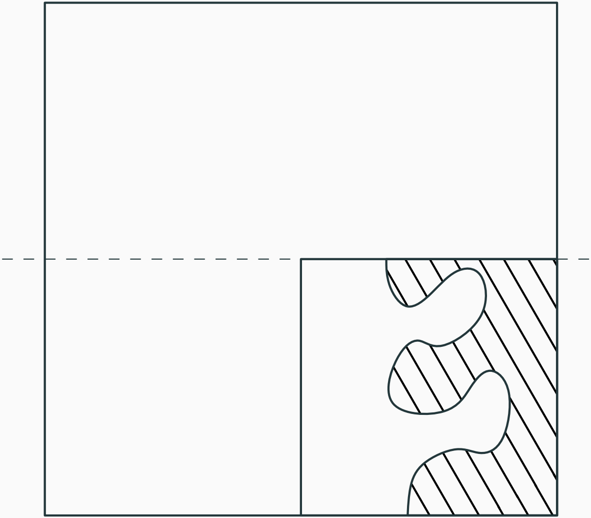
For every $\lambda > 0$ there exists $c \in (0,1)$ such that for all n large enough,

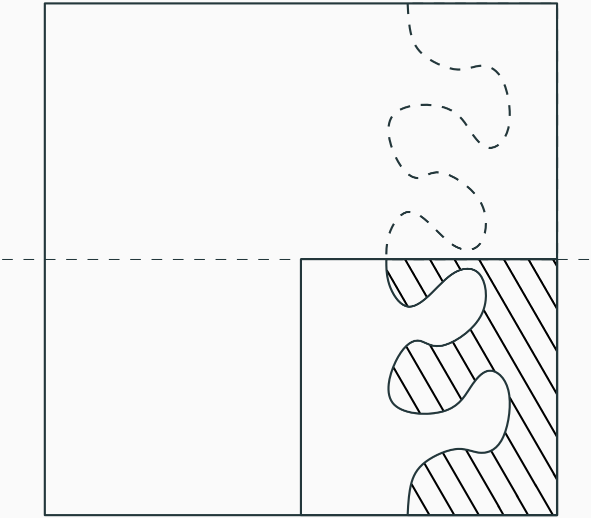
$$c \leqslant P_{p_c}[LR(\lambda n, n)] \leqslant 1 - c.$$

The case $\lambda = 1$ is easy by duality; it is enough to know how the estimate for one value of $\lambda > 1$ and then to glue the pieces.

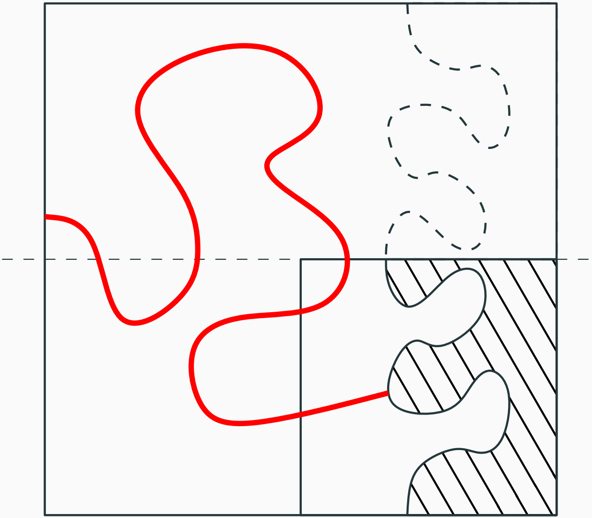
Russo-Seymour-Welsh : proof ($\lambda = 3/2$)



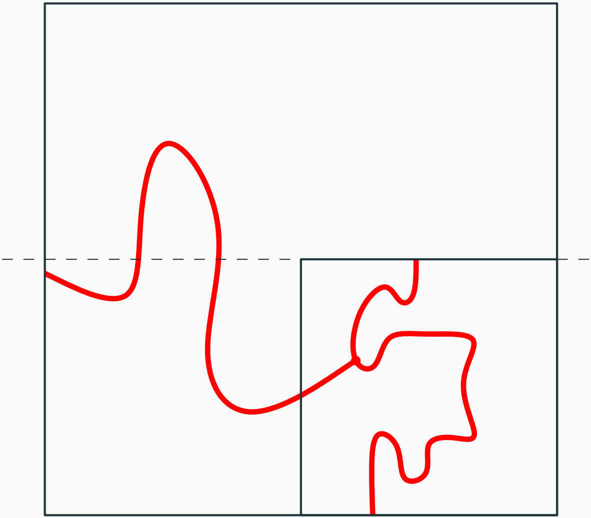




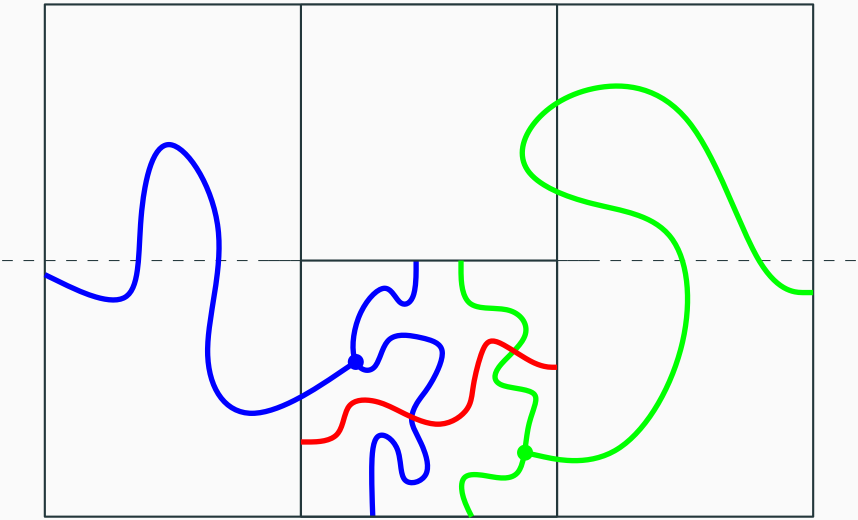
Russo-Seymour-Welsh : proof ($\lambda = 3/2$)



Russo-Seymour-Welsh : proof ($\lambda = 3/2$)



Russo-Seymour-Welsh : proof ($\lambda = 3/2$)



Russo-Seymour-Welsh for the field ψ

Main tools used were decorrelation and the FKG inequality.

Russo-Seymour-Welsh for the field ψ

Main tools used were decorrelation and the FKG inequality.

Theorem (B., Gayet — Publ. Math. IHES 2017, to appear)

The field ψ satisfies RSW.

Russo-Seymour-Welsh for the field ψ

Main tools used were decorrelation and the FKG inequality.

Theorem (B., Gayet — Publ. Math. IHES 2017, to appear)

The field ψ satisfies RSW.

A few consequences:

- The set $\{z : \psi(z) > 0\}$ has no unbounded component
- Neither do $\{z : \psi(z) < 0\}$ and $\{z : \psi(z) = 0\}$
- The universal critical exponents are the same as for percolation
- $\psi = 0$ is the critical level [Rivera-Vanneuville]

A few words about the proof

The main obstacle is the analyticity of the field ψ, which goes against independence of its behavior in distant regions.

A few words about the proof

The main obstacle is the analyticity of the field ψ, which goes against independence of its behavior in distant regions.

To go around it, we discretize the field on the vertices of a triangular lattice with a small mesh δ, and look only at its sign on it, to get a dependent, discrete percolation model. The choice of δ is crucial:

- If δ is too large, the discretization does not catch all the topology;

A few words about the proof

The main obstacle is the analyticity of the field ψ, which goes against independence of its behavior in distant regions.

To go around it, we discretize the field on the vertices of a triangular lattice with a small mesh δ, and look only at its sign on it, to get a dependent, discrete percolation model. The choice of δ is crucial:

- If δ is too large, the discretization does not catch all the topology;
- If δ is too small, we lose in the decorrelation.

A few words about the proof

The main obstacle is the analyticity of the field ψ, which goes against independence of its behavior in distant regions.

To go around it, we discretize the field on the vertices of a triangular lattice with a small mesh δ, and look only at its sign on it, to get a dependent, discrete percolation model. The choice of δ is crucial:

- If δ is too large, the discretization does not catch all the topology;
- If δ is too small, we lose in the decorrelation.

The case of the Laplacian eigenfunctions is bad on all respects: too slow decorrelation, no FKG inequality.

A decorrelation inequality

Theorem

Let X and Y be two Gaussian vectors in $\mathbb{R}^{m+n}$, of covariances

$$\Sigma_X = \begin{bmatrix} \Sigma_1 & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_2 \end{bmatrix} \quad \textit{and} \quad \Sigma_Y = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},$$

where $\Sigma_1 \in M_m(\mathbb{R})$ and $\Sigma_2 \in M_n(\mathbb{R})$ have all diagonal entries equal to 1. Denote by μ_X (resp. μ_Y) the law of the signs of the coordinates of X (resp. Y), and by η the largest absolute value of the entries of Σ_{12}. Then,

$$d_{TV}(\mu_X, \mu_Y) \leqslant C(m+n)^{8/5}\eta^{1/5}.$$

Another decorrelation inequality

Theorem

Let $X = (x_i)$ be a centered Gaussian vector in $\mathbb{R}^n$ with covariance matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ satisfying $\forall 1 \leq i \leq n$, $a_{ii} = 1$, and let $\delta \in (0, 1/n)$. Then, the shifted truncation

$$B = (b_{ij}) \quad \textit{where} \quad b_{ij} := a_{ij} 1_{|a_{ij}| > \delta} + (n\delta)^{3/5} 1_{i=j}$$

is a positive matrix, and there exists a coupling of X with another centered Gaussian vector $Y = (y_i)$ with covariance matrix B such that

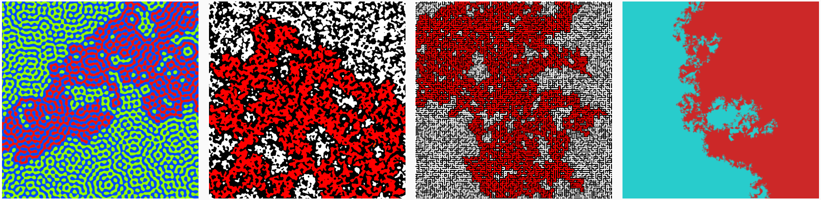
$$P\left[\forall 1 \leq i \leq n, \quad x_i y_i > 0\right] \geqslant 1 - 3n^{6/5}\delta^{1/5}.$$

Corollary: coupling with a finitely correlated field.

The Bogomolny-Schmidt conjecture

Conjecture

The nodal lines of ϕ (and ψ) converge, in the scaling limit, to the same conformally invariant object as interfaces of critical percolation; in particular, asymptotic crossing probabilities are given by Cardy's formula.



Negatively correlated fields

Russo-Seymour-Welsh along families of models

Theorem (B., Gayet, 2017, arXiv 1710.10644)

Let (P_u) be a one-parameter family of discrete site models satisfying the following assumptions:

- *symmetry and self-duality;*
- *uniformly good decorrelation;*
- *RSW estimates at parameter $u = 0$.*

Then, RSW estimates hold uniformly for all $u \in (-\varepsilon, \varepsilon)$.

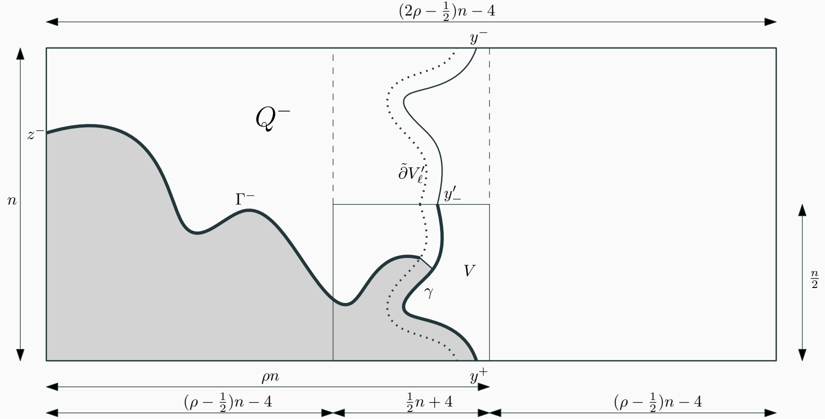
This applies in particular

- to the Ising model with possibly negative β
- to discrete Gaussian fields with possibly negative correlation

Main ideas of the argument:

- Follow the general proof from percolation
- Approaching within distance ℓ is free
- Then, need to glue both paths together

Russo-Seymour-Welsh for finitely correlated fields



Russo-Seymour-Welsh for finitely correlated fields

$$\beta(r,R) = \sup_{Q \in \mathcal{Q}(r,R)} P[Q \text{ is not glued}]$$

Lemma (Approximative statement)

There exist $\varepsilon > 0$, $L < \infty$ and $\lambda < \infty$ such that

$$\beta(\ell, L) < \varepsilon \quad \Longrightarrow \quad \forall r < R, \quad \beta(r,R) \leqslant \lambda (r/R)^{1/\lambda},$$

and moreover when this holds, RSW estimates are also valid.

That’s it

Happy birthday Anton!