On the correlations in the $2D$ Random Field Ising Model

Michael Aizenman

Princeton Univ.

"Advances in Statistical Mechanics"

Conference in celebration of Anton Bovier

CIRM, Luminy
30 Aug. 2018

As an example of the Imry- Ma phenomenon, the famed discontinuity of the magnetization in the two dimensional Ising model is unstable to the addition of quenched random magnetic field of uniform variance, even if that is small.

The talk will focus on a quantitative version of the statement, yielding a power-law upper bound on the decay rate of the effect of boundary conditions on the magnetization in finite systems, as function of the distance to the boundary.

Unlike exponential decay which is only proven for strong disorder or high temperature, the power-law upper bound is now established for all field strengths and at all temperatures, including $T = 0$, for the case of independent Gaussian random field.

The analysis proceeds through a better quantified variant of the Aiz.-Wehr proof of the Imry-Ma rounding effect.

The talk is based on a joint work with Ron Peled (TAU).

Collaborators on related past works: Jan Wehr,
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Quenched disorder, and its effects on phase transitions

Models modified by quenched disorder (denoted η, of strength ε)

1) Random Field Ising Model (RFIM): $\sigma_v = \pm 1$ for $v \in \mathbb{Z}^d$

$$H_\eta(\sigma) := -J \sum_{\substack{\{u,v\}\subset\mathbb{Z}^d \\ u\sim v}} \sigma_u \sigma_v - \sum_{v\in\mathbb{Z}^d} (h + \varepsilon\, \eta_v)\sigma_v \tag{1}$$

with $\{\eta_v\}_{v\in\mathbb{Z}^d}$ independent random variables, e.g. iid normal gaussian, $N(0,1)$.

2) Random Field O(N) Model
$\{\sigma_v, h, \eta_v\} \Rightarrow \{\vec{\sigma}_v, \vec{h}, \vec{\eta}_v\}$, N component vectors; $\sigma_u\sigma_v \Rightarrow \vec{\sigma}_u \cdot \vec{\sigma}_v$, etc.
with $\vec{\eta}_v$ given by independent random variables of rotation invariant distribution.

3) Q-state Potts model ($\sigma_v \in \{1, ..., Q\}$) with randomized couplings

$$H_\eta(\sigma) = - \sum_{\substack{\{u,v\}\subset\mathbb{Z}^2 \\ u\sim v}} (J + \varepsilon\, \eta_{x,y})\, \mathbb{1}[\sigma_x = \sigma_y] - \sum_{x\in\mathbb{Z}^2} h\, \mathbb{1}[\sigma_x = 1] \tag{2}$$

with $\eta_{x,y}$ uniformly distributed over $[-1, 1]$ (w.r.t. Lebesgue measure).

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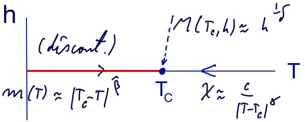
Quenched Disorder's effects on phase transitions

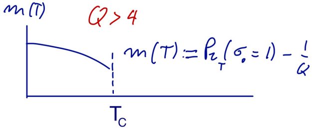
Without the disorder, in d-dimensions the above models exhibit 1^{st} order phase transitions, provided:

Ising	$d > 1$
$O(N)$	$d > 2$
Q-state Potts	$d = 2, \quad Q > 4$



Ising order parameter $m(T)$ defined as

$$m(T) = \lim_{h \downarrow 0} M(T,h)$$
$$M(T,h) := \langle \sigma_0 \rangle_{T,h}$$



$\widehat{\beta}$, γ, δ – critical exponents

Initial questions:

Q1) Does the first order transition persist under the quenched disorder?

Q2) If so: does the disorder affect the critical exponents?
(Q2 will not be discussed here - Harris criterion & all that.)

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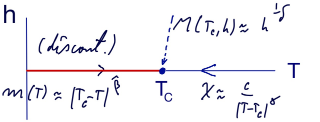

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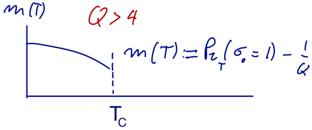

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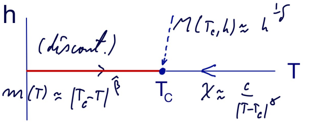

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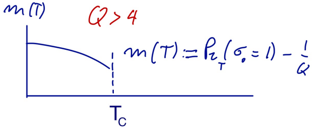

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Concerning $Q1$:

The *Imry-Ma argument* won over *dimensional reduction*

I) Imry-Ma prediction:

$$1^{st} \text{ order discontinuity persists iff } \begin{cases} d > 2 & \text{discrete systems} & (L^{d/2} \geq L^{d-1}) \\ d > 4 & \text{cont. symm.} & (L^{d/2} \geq L^{d-2}) \end{cases}$$

Y. Imry and S.K. Ma, PRL **35** (1975).

II) An alternative "dimensional reduction proposal" (disproved for Ising model):

$$d_{lc}(disord.) = d_{lc}(homog.) - 2\,.$$

and also in terms of the critical exponents

$$\textit{disordered systems in dim. } d \;\approx\; \textit{homogen. systems in dim. } (d-2).$$

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*Q*1 is equivalent to: does the free energy $\mathcal{F}(h, \beta, \varepsilon)$ retain its kink singularity in *h* under quenched disorder, i.e. for all $\varepsilon > 0$?

Quenched free energy: For random fields $\eta = (\eta_v)$ of finite variance, forming a translation invariant and ergodic process: $\forall \beta \in [0, \infty]$ there is a full measure set of configurations η for which

$$\mathcal{F}(h, \beta, \epsilon) := \lim_{L\to\infty} \frac{-1}{\beta\,|\Lambda_L|} \log Z_{\Lambda_L, B}(h, \beta, \epsilon; \eta) \tag{3}$$

exists and its value is independent of η and of the boundary conditions *B*.

Furthermore

i. $\mathcal{F}(h, \beta, \epsilon)$ is concave as a function of *h*.

ii. $\lim_{\beta\to\infty} \mathcal{F}(h, \beta, \epsilon)$ gives the (a.s.) ground state energy *density*

($\beta \to \infty$ and $L \to \infty$ are interchangeable for $\mathcal{F}$.)

iii. Uniqueness of the limit does not extend to uniqueness of the Gibbs states.

However: for any (β, h, ε) at which $\mathcal{F}$ is differentiable in *h*

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} \left\langle \frac{\partial H}{\partial h_x} \right\rangle^{h}_{\Lambda_L, B} (\eta) = \frac{\partial \mathcal{F}}{\partial h} \tag{4}$$

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Rigorous results concerning Q1 - (η of cont. distr. $\mathbb{E}(\eta^4) < \infty$)

Thm 1 (1st order phase transition persists in 3D)
In $d = 3$ dimensions, at sufficiently small ε the RFIM continues to display a 1st order transition in h, both at $T = 0$ and at small enough temperatures.

T=0: J.Z. Imbrie, PRL '84 / CMP '85
T> 0: J. Bricmont, A. Kupiainen, PRL '87 / CMP '88

Thm 2 (Rounding of the phase transition in 2D)
In $d = 2$ dimensions, at any $\varepsilon > 0$: the RFIM has almost surely a unique ground state, and a unique Gibbs states at any (β, h).

M. Aiz., J. Wehr, PRL '89/ CMP '90).

Thm 2' (A more general statement [AW])
In $d = 2$ dimensions, at any $\varepsilon > 0$ the free energy is differentiable in the parameter to which disorder was added (such as h in RFIM).

- For systems with continuous symmetry Thm 2' extends to in $d \leq 4$ dims (free energy is diff. at $\vec{h} = \vec{0}$, provided also the distribution of $\vec{\eta}$ is rot. inv.)

- Thm 2' holds also for Quantum Systems.

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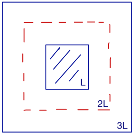
A handy tool for the non-perturbative argument ([AW])

For the RFIM at $T = 0$ and $h = 0$, and volumes $\Lambda(L) = [-L, L]^2$, consider:

$$G_L(\eta) := \left[\min_\sigma H^+_{\Lambda(3L),\eta}(\sigma) - \min_\sigma H^-_{\Lambda(3L),\eta}(\sigma)\right] - \left[\min_\sigma H^+_{\Lambda(3L),R_L\eta}(\sigma) - \min_\sigma H^-_{\Lambda(3L),R_L\eta}(\sigma)\right]$$



where $\pm$ denotes the boundary conditions on the outer box, and $R_L\eta$ is the field obtained by setting $\eta_x = 0$ within the inner box.

This quantity obeys:

1) the uniform bound: $|G_L(\eta)| \leq 4J\,|\partial\Lambda(2L)| = \text{Const.}L^{d-1} \quad (\forall \eta)$

2) for each $x \in \Lambda(L)$:

$$\frac{\partial}{\partial \eta_x} G_L(\eta) = -\left[\widehat{\sigma}^+_x(\eta) - \widehat{\sigma}^-_x(\eta)\right] = -2\mathbb{1}\left[\widehat{\sigma}^+_x(\eta) \neq \widehat{\sigma}^-_x(\eta)\right]$$

From (2) (+ ergodicity) one can deduce the anti-concentration bound:

if $\Pr\left(\sigma^+_0 \neq \widehat{\sigma}^-_0\right) = m(\varepsilon) \neq 0$ then $\mathcal{D} - \lim_{L\to\infty} G_L = \theta(m(\varepsilon))\, L^{d/2}\, N(0,1)$

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For $d = 2$ that's a contradiction $\Longrightarrow$ $m(\varepsilon) = 0$.

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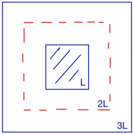

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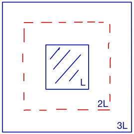

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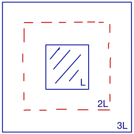

where $\pm$ denotes the boundary conditions on the outer box,
and $R_L\eta$ is the field obtained by setting $\eta_x = 0$ within the inner box.

This quantity obeys:

1) the uniform bound: $\boxed{|G_L(\eta)| \le 4J\,|\partial\Lambda(2L)| = \text{Const.}L^{d-1} \quad (\forall \eta)}$

2) for each $x \in \Lambda(L)$:

$$\frac{\partial}{\partial \eta_x} G_L(\eta) = -\left[\widehat{\sigma}^+_x(\eta) - \widehat{\sigma}^-_x(\eta)\right] = -2\mathbb{1}\left[\widehat{\sigma}^+_x(\eta) \neq \widehat{\sigma}^-_x(\eta)\right]$$

From (2) (+ ergodicity) one can deduce the anti-concentration bound:

if $\Pr\left(\sigma^+_0 \neq \widehat{\sigma}^-_0\right) = m(\varepsilon) \neq 0$ then $\boxed{\mathcal{D} - \lim_{L\to\infty} G_L = \theta(m(\varepsilon))\,L^{d/2}\,N(0,1)}$

i.e., $G_L(\eta) \approx \theta(m)\times$ norm. Gaussian var..

For $d = 2$ that's a contradiction $\Longrightarrow$ $\boxed{m(\varepsilon) = 0}$.

Next question – influence percolation decay rate

Focusing now on the critical dimensions $d = 2$, and $d = 4$ (for the case of continuous symmetry), we have the following *influence percolation* question

Q3 : At what rate does the unique (!) ground state localize, i.e. decouple from the boundary conditions and disorder, at distance L?

More explicilty, let

$$M_L(\varepsilon) := \frac{1}{2}\,\mathrm{Av}\left(\sigma_0^{\Lambda(L),+} - \widehat{\sigma}_0^{\Lambda(L),-}\right) \stackrel{Ising}{=} \mathrm{Pr}\left(\sigma_0^{\Lambda(L),+} \neq \widehat{\sigma}_0^{\Lambda(L),-}\right) .$$

A percolation argument $\Longrightarrow$ in any dimension at strong disorder,

$$\boxed{M_L(\varepsilon) \leq C(\varepsilon) e^{-\mu(\varepsilon) L} \qquad \text{(with } \mu(\varepsilon) > 0 \text{ for } \varepsilon > \varepsilon_d\text{).}}$$

(cf. A,-Wehr, A.-Peled, Camia-Jiang-Newman arXiv 2018)

Does this persists to weak disorder, or is there a transition to a phase with a slower decay (still at $T = 0$)? And how slow can the decay be?

The first question was considered early on in

B. Derida and Y. Shnidman,"Possible line of critical points for [RFIM] in dimension 2", J. Phys. Lett. '84.

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Correlation function bounds

Related, but rather weak, recent bound

S. Chatterjee "On the decay of correlations in the [RFIM]", CMP '2018

where it is shown that ($\forall \varepsilon$): $\boxed{M_L \le C(\varepsilon)/\sqrt{\log\log L}}$

A better bound, derived through a significantly improved version of the argument outlined above, is

$\boxed{M_L \le C(\varepsilon)/L^{\alpha(\varepsilon)}}$ (joint work with Ron Peled, arXiv 2018)

Our current guess is that the actual behavior may be exponential decay at all $eps > 0$. However, it may well be that the correlation length satisfies

$$\ell(\varepsilon) \ \ge \ e^{-C/\varepsilon^2} .$$

• The arguments used above were not (yet ?) extended to the case of continuous symmetry breaking under quenched disorder.

Question: Is there a K-T like line of critical points in $4D$, at $T = 0$?
(This would also be in line with the dimensional reduction picture..)

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A tempting, but potentially misleading picture - Mandelbrot percolation

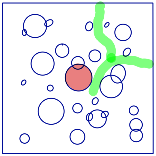
Of possible relevance for an intuitive picture are the scale-invariant *swiss cheese* and the related Mandelbrot's "canonical curdling" models:

J.T. Chayes, L. Chayes, R. Durrett, PTRF '88.

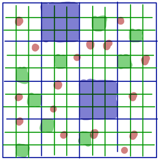
- Consider the Poisson process of spheres $B(x, r)$ in $\mathbb{R}^\nu$ with density

$$\lambda \frac{dr\, d^\nu x}{r^{1+\nu}} \quad \text{over } [0, 1] \times \mathbb{R}^\nu \qquad (\text{with } \lambda = e^{-C/\varepsilon^2})$$

The cheese is the set left uncovered. For $\varepsilon < \varepsilon_0$ the set is not empty, though of zero Lebesgue measure. Its dimension, $\dim_{\mathcal{H}}(\text{cheese})$, increases as $\varepsilon \downarrow 0$, and at small enough ε it percolates



The picture may initially suggests that at weak enough disorder the influence percolation may decay by a power law. But it also leaves room for improvement.



Congratulations Anton

and best wishes!

& Thank you for your attention.