Arithmetic equidistribution and elliptic curves

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CIRM, September, 2018
(joint work with Laura DeMarco)

Outline of the talk

- Unlikely intersections - Bogomolov's conjecture
- Our results (and work in progress)
  1. A generalization of Bogomolov's conjecture related to Masser-Zannier's theorem on torsion points in families of elliptic curves.
  2. A equidistribution theorem for 'small points' (2017)
  3. Its generalization to 'real' small points (2018).
  4. Applications related to Barroero-Capuano's theorem on simultaneous relations (in progress)
- Proof strategy
  1. A corollary of the equidistribution theorem
  2. Silverman's results on a variation of heights

Lang's Theorem

X : irreducible complex plane curve.
$\boldsymbol{\mu}$: set of roots of unity.

Theorem (Ihara-Serre-Tate 1965)

If X contains infinitely many points with both coordinates roots of unity, then X is given by an equation of the form $x^n y^m - \omega = 0$ for $(n, m) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ and $\omega \in \boldsymbol{\mu}$.

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$(\xi, \zeta) \in \boldsymbol{\mu} \times \boldsymbol{\mu}$	$\leftrightarrow$	torsion points of $(\mathbb{C}^*)^2$	$\leftrightarrow$	special points
Curves $x^n y^m - \omega = 0$	$\leftrightarrow$	translates of algebraic subgroups of $(\mathbb{C}^*)^2$ by a torsion point	$\leftrightarrow$	special curves

Bogomolov's Conjecture : Tori

Philosophical restatement

If a curve has an infinite (Zariski dense) set of special points, then it is a special curve.

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What if we replace the special points by small points, that is points of small logarithmic Weil height $h : \overline{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$*?*

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What if we replace the special points by small points, that is points of small logarithmic Weil height $h : \overline{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$?

For $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$, $h(\alpha) \geq 0$ and $h(\alpha) = 0 \Leftrightarrow \alpha$ is a root of unity.

Bogomolov's Conjecture : Tori

X: irreducible plane curve defined over number field K.

Theorem (Zhang 1992, Bombieri-Zannier 1995)

Assume that X is non-special. There is a constant $c(X) > 0$ such that

$$\{(x, y) \in X(\overline{K}) \ : \ h(x) + h(y) \leq c(X)\}$$

is finite.

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In other words, if $\exists$ infinitely many $(x_n, y_n) \in X(\overline{K})$ such that

$$h(x_n) + h(y_n) \to 0$$

as $n \to \infty$, then X is a special curve.

Bogomolov's Conjecture : Setting

A : abelian variety defined over K.
$\hat{h}_A$: Nèron-Tate height corresponding to an ample and symmetric divisor on A.

Example

- For an elliptic curve E over K and $P \in E(\overline{K})$, we have

$$\hat{h}_E(P) = \frac{1}{2} \lim_{n\to\infty} \frac{h(x([n]P))}{n^2}.$$

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- If $A = E_1 \times E_2$ for two elliptic curves E_i, we may take

$$\begin{aligned} \hat{h}_A : E_1(\overline{K}) \times E_2(\overline{K}) &\to \mathbb{R}_{\geq 0} \\ (P_1, P_2) &\mapsto \hat{h}_{E_1}(P_1) + \hat{h}_{E_2}(P_2). \end{aligned}$$

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$$\hat{h}_A : E_1(\overline{K}) \times E_2(\overline{K}) \to \mathbb{R}_{\geq 0}$$
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For $P \in A(\overline{K})$, we have $\hat{h}_A(P) = 0 \iff P \in A_{\mathrm{tors}}$.

Bogomolov's Conjecture : Abelian varieties

torsion points of A	$\leftrightarrow$	special points
translates of abelian subvarieties by a torsion point	$\leftrightarrow$	special subvarieties

Theorem (Zhang 1998, Ullmo 1998)

For each non-special subvariety X of A, there is a constant $c(X) > 0$ such that

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Remark

For a non-special X, the set $\{x \in X(\overline{K}) \ : \ \hat{h}_A(x) = 0\}$ is not Zariski dense in X by the Manin-Mumford Conjecture (Raynaud's theorem 1983).

Setting: An analog in families of abelian varieties

B smooth quasi-projective curve defined over a number field K.
For $i = 1, 2$ we consider

$\mathcal{E}_i \to B$ elliptic surfaces defined over $K \leftrightarrow E_i$ over $E(K(B))$.
e.g. $\mathcal{E}_{1,t} : y^2 = x(x-1)(x-t)$ and $\mathcal{E}_{2,t} : y^2 = x(x-1)(x+t)$.

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$P_i : B \to \mathcal{E}_i$ sections defined over $K \leftrightarrow P_i \in E_i(K(B))$.
e.g. $P_{1,t} = (2, \sqrt{2(2-t)}) \in \mathcal{E}_{1,t}$ and $P_{2,t} = (2, \sqrt{2(2+t)}) \in \mathcal{E}_{2,t}$.

$P = (P_1, P_2) : B \to \mathcal{E}_1 \times_B \mathcal{E}_2$ section of $\mathcal{E}_1 \times_B \mathcal{E}_2 \to B$.

A Bogomolov-type theorem in families of abelian varieties

Let $\mathcal{A} = \mathcal{E}_1 \times_B \mathcal{E}_2$ and

$$t \mapsto \hat{h}_{\mathcal{A}_t}(P_t) = \hat{h}_{\mathcal{E}_{1,t}}(P_{1,t}) + \hat{h}_{\mathcal{E}_{2,t}}(P_{2,t}).$$

Theorem (DeMarco, M. 2017)

For each non-special section $P : B \to \mathcal{A}$, there is a constant $c = c(P) > 0$ such that $\{t \in B(\overline{K}) : \hat{h}_{\mathcal{A}_t}(P_t) < c\}$, is finite.

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Remark

Let E_1 & E_2 be fixed elliptic curves over K. Assume that $\mathcal{A}_t = E_1 \times E_2$ for each t.

special sections $\leftrightarrow$ *special subvarieties of $\mathcal{A} = E_1 \times E_2$*

Our theorem then reduces to Zhang's theorem.

Special sections

Theorem (DeMarco, M. 2017)

If for a sequence $t_n \in B(\overline{K})$ we have

$$\lim_{n\to\infty} \hat{h}_{\mathcal{E}_{1,t_n}}(P_{1,t_n}) = 0 \ \ \& \ \ \lim_{n\to\infty} \hat{h}_{\mathcal{E}_{2,t_n}}(P_{2,t_n}) = 0,$$

then the section $P = (P_1, P_2)$ is special, i.e. one of the following holds.

- *P_1 is (identically) torsion in $\mathcal{E}_1$.*
- *P_2 is torsion in $\mathcal{E}_2$.*
- *There are isogenies $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ and $\psi : \mathcal{E}_2 \to \mathcal{E}_2$, so that $\phi(P_1) = \psi(P_2)$.*

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- *There are isogenies $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ and $\psi : \mathcal{E}_2 \to \mathcal{E}_2$, so that $\phi(P_1) = \psi(P_2)$. In particular, for each $\lambda \in B(\overline{K})$ we have*

$$P_{1,\lambda} \in (\mathcal{E}_{1,\lambda})_{\mathrm{tors}} \ \ \Leftrightarrow \ \ P_{2,\lambda} \in (\mathcal{E}_{2,\lambda})_{\mathrm{tors}}.$$

Masser and Zannier's theorems in unlikely intersections

Our theorem generalizes Masser-Zannier's theorem to 'small' points.

Theorem (Masser-Zannier 2010, 2012, 2014)

If for an infinite sequence $t_n \in B(\overline{K})$ we have

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If $P_{1,t_n} \in (\mathcal{E}_{1,t_n})_{\mathrm{tors}}$ & $P_{2,t_n} \in (\mathcal{E}_{2,t_n})_{\mathrm{tors}}$, then

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In fact, our proof uses Masser-Zannier's theorem!

Example: Special sections

Let $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E} \to B$ be the Legendre surface.

$$\mathcal{E}_t : y^2 = x(x-1)(x-t),$$
$$P_{1,t} = (2, \sqrt{2(2-t)})\ ,\ P_{2,t} = (3, \sqrt{6(3-t)}).$$

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Then (P_1, P_2) is a not special.

- Neither P_1 nor P_2 is identically torsion; and
- If for $n, m \in \mathbb{Z} \setminus \{0\}$ we have $[n]P_{1,t} = [m]P_{2,t}$ $\forall\ t \in B(\mathbb{C})$, then

$$P_{1,t} \in (\mathcal{E}_t)_{\text{ tors}} \ \Leftrightarrow\ P_{2,t} \in (\mathcal{E}_t)_{\text{ tors}}$$

for each t. However,

$$P_{2,3} = (3,0) \in (\mathcal{E}_3)_{\text{ tors}}\ \&\ P_{1,3} = (2, \sqrt{-2}) \notin (\mathcal{E}_3)_{\text{ tors}}.$$

Examples

$\exists\ c > 0$ such that $|\{t \in \overline{\mathbb{Q}} : \hat{h}_{\mathcal{E}_{1,t}}(P_{1,t}) + \hat{h}_{\mathcal{E}_{2,t}}(P_{2,t}) < c\}| < \infty$, when:

$$\mathcal{E}_{1,t} = \mathcal{E}_{2,t} : y^2 = x(x-1)(x-t),$$
$$P_{1,t} = \left(2t, \sqrt{2t^2(2t-1)}\right),\ P_{2,t} = \left(3t, \sqrt{6t^2(3t-1)}\right) \text{ or}$$

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$$\mathcal{E}_{1,t} : y^2 = x(x-1)(x-t) \qquad \mathcal{E}_{2,t} = \mathcal{E}_{1,-t} : y^2 = x(x-1)(x+t)$$

$$P_{1,t} = (2, \sqrt{2(2-t)}) \qquad P_{2,t} = (2, \sqrt{2(2+t)}).$$

The geometry of small points

E elliptic curve defined over $K(B)$
$P \in E(K(B))$ non-torsion.

Theorem (DeMarco, M. 2017)

Let $t_n \in B(\overline{K})$ be such that $\hat{h}_{\mathcal{E}_{t_n}}(P_{t_n}) \to 0$. There is a collection of probability measures

$$\mu_P = \{\mu_{P,v}\}_{v \in M_K}$$

on B_v^{an} such that for each $v \in M_K$ the discrete measures

$$\mu_{t_n} = \frac{1}{|\mathrm{Gal}(\overline{K}/K) \cdot t_n|} \sum_{t \in \mathrm{Gal}(\overline{K}/K) \cdot t_n} \delta_t$$

converge weakly to the measure $\mu_{P,v}$ on B_v^{an}.

Real equidistribution

Let E elliptic curve defined over $K(B)$ and
~~$P \in E(K(B))$~~ $P \in E(K(B)) \otimes \mathbb{R}$ non-trivial.

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Equidistribution

To get the equidistribution result, we have to show that the function

$$t \mapsto \hat{h}_{\mathcal{E}_t}(P_t),$$

is a ‘good’ height in the sense of the equidistribution theorem of Chambert-Loir, Thuillier and Yuan. This involves work of Silverman from 1992.

In the real case we also make use of work of Moriwaki.

Barroero-Capuano's theorem

$\mathcal{E} \to B$ a non-isotrivial elliptic surface defined over K.
$P_i : B \to \mathcal{E}$ sections defined over K, $i = 1, \ldots, m$, $m \geq 2$.

Theorem (Barroero-Capuano 2016)

Let $P_1, \ldots, P_m$ be $m \geq 2$ linearly independent sections. Then, there are at most finitely many $t \in B(\overline{K})$ such that

$P_{1,t}, \ldots, P_{m,t}$ satisty two independent linear relations in $\mathcal{E}_t$.

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- The constant case follows from work of Viada and Rémond (2003), Viada (2008) and Galateau (2010).

A corollary of Silverman's specialization theorem

For $i = 1, \ldots, m$, let $P_i : \mathcal{E} \to B$ be linearly independent sections.

Silverman's 'specialization theorem' implies that the set

$$\{t \in B(\overline{K}) \ : \ P_{1,t}, \ldots, P_{m,t} \text{ are linearly related in } \mathcal{E}_t\}$$

has bounded height.

In particular, it is a 'sparse' set (think Northcott property.)

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It is natural to expect that

Double sparseness $\Rightarrow$ finiteness.

Connection with our theorem - ‘Doubly small’ parameters?

Assume that for an infinite sequence $t_n \in B(\overline{K})$ we have

$$a_{1,n}P_{1,t_n} + \cdots + a_{m,n}P_{m,t_n} = \mathcal{O}.$$

Passing to a subsequence, we may assume that $a_{1,n} \neq 0$ and

$$\frac{a_{i,n}}{a_{1,n}} \to x_i \in \mathbb{R}.$$

Then, using Silverman’s specialization theorem and the bilinearity of the height pairing, we get

$$\hat{h}_{\mathcal{E}_{t_n}}(P_{1,t_n} + x_2P_{2,t_n} + \cdots + x_mP_{m,t_n}) \to 0.$$

Connection with our theorem - ‘Doubly small’ parameters?

If now for an infinite sequence $t_n \in B(\overline{K})$ we have

$$a_{1,n}P_{1,t_n} + \cdots + a_{m,n}P_{m,t_n} = \mathcal{O} \ \ \&$$
$$b_{1,n}P_{1,t_n} + \cdots + b_{m,n}P_{m,t_n} = \mathcal{O}$$

for linearly independent $(a_{1,n}, \ldots, a_{m,n}), (b_{1,n}, \ldots, b_{m,n}) \in \mathbb{Z}^m$, then

$$\hat{h}_{\mathcal{E}_{t_n}}(x_1 P_{1,t_n} + \cdots + x_m P_{m,t_n}) \to 0 \ \ \&$$
$$\hat{h}_{\mathcal{E}_{t_n}}(y_1 P_{1,t_n} + \cdots + y_m P_{m,t_n}) \to 0,$$

for linearly independent $\vec{x} = (x_1, \ldots, x_m), \vec{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m$.

So we have a ‘doubly small’ sequence for these ‘real’ heights.

A conjectural generalization

$P_i : \mathcal{E} \to B$ linearly independent sections, for $i = 1, \ldots, m$.

For $\vec{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ let $h_{\vec{x}} : B(\overline{K}) \to \mathbb{R}_{\geq 0}$

$$\begin{aligned} t \mapsto h_{\vec{x}}(t) &= \hat{h}_{\mathcal{E}_t}(x_1 P_{1,t} + \cdots + x_m P_{m,t}) \\ &= \sum_{1 \leq i,j \leq m} x_i x_j \langle P_{i,t}, P_{j,t} \rangle_{\mathcal{E}_t}. \end{aligned}$$

Conjecture (DeMarco-M.)

If $\vec{x}$, $\vec{y} \in \mathbb{R}^m$ are linearly independent, then there is a constant $c = c(P_1, \ldots, P_m, \vec{x}, \vec{y}) > 0$ such that the set

$$\{t \in B(\overline{K}) \ : \ h_{\vec{x}}(t) + h_{\vec{y}}(t) < c\},$$

is finite.

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Remark

- *The conjecture implies Baroerro-Capuano's theorem.*

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- *If $\vec{x}, \vec{y} \in \mathbb{Q}^m$, then we get our theorem (2017).*
- *When $m = 2$ the conjecture holds true by our theorem and the parallelogram law.*

Towards our Conjecture for ‘real’ points

So far, we recover Barroero-Capuano’s theorem for 3 sections $P_1, P_2, P_3 : B \to \mathcal{E}$ defined over K.

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Theorem (DeMarco-M.)

Let $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$ be linearly independent. Assume that

1. *$\exists$ an infinite sequence $t_n \in B(\overline{K})$ such that*

$$h_{\vec{x}}(t_n) \to 0 \;\&\; h_{\vec{y}}(t_n) \to 0,$$

and that

2. *$\exists\, \lambda \in B(\overline{K})$ such that $P_{1,\lambda}, P_{2,\lambda}, P_{3,\lambda}$ satisfy exactly two independent linear relations in $\mathcal{E}_\lambda$ (over $\mathbb{Z}$).*

Then P_1, P_2, P_3 are linearly related.

A reformulation of our conjecture: a height pairing

$$P, Q \in E(K(B)) \otimes \mathbb{R}$$

The Arakelov-Zhang-Moriwaki pairing for metrized line bundles induces a non-negative, symmetric 'pairing' between the 'heights'

$$h_P \cdot h_Q \in \mathbb{R}_{\geq 0}.$$

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The Arakelov-Zhang-Moriwaki pairing for metrized line bundles induces a non-negative, symmetric 'pairing' between the 'heights'

$$h_P \cdot h_Q \in \mathbb{R}_{\geq 0}.$$

$$h_P \cdot h_Q = 0 \iff \exists\, t_n \in B(\overline{K}) \text{ such that } \hat{h}_{\mathcal{E}_{t_n}}(P_{t_n}) + \hat{h}_{\mathcal{E}_{t_n}}(Q_{t_n}) \to 0.$$

A reformulation of our conjecture: a height 'pairing'

Let $\Lambda = E(K(B))$. The assignment

$$\begin{aligned} (\Lambda \otimes \mathbb{R}) \times (\Lambda \otimes \mathbb{R}) &\to \mathbb{R}_{\geq 0} \\ (P, Q) &\mapsto h_P \cdot h_Q \end{aligned}$$

is 'biquadratic', in the sense that it is a quadratic form if P (or Q) are fixed.

A reformulation of our conjecture: a height 'pairing'

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is 'biquadratic', in the sense that it is a quadratic form if P (or Q) are fixed.
By our theorem (2017), we know that it 'doesn't degenerate' in Λ in the sense that

$$h_P \cdot h_Q = 0 \Leftrightarrow P \;\&\; Q \text{ are linearly related.}$$

We conjecture that it also 'doesn't degenerate' in $\Lambda \otimes \mathbb{R}$.

A reformulation of our conjecture: a height 'pairing'

In other words, that our assignment can be compared with the 'biquadratic' assignment

$$\begin{aligned} (\Lambda \otimes \mathbb{R}) \times (\Lambda \otimes \mathbb{R}) &\to \mathbb{R}_{\geq 0} \\ (P, Q) &\mapsto \hat{h}_E(P)\hat{h}_E(Q) - \langle P, Q \rangle_E^2. \end{aligned}$$

Conjecture (DeMarco, M. - reformulation)

For $P, Q \in E(K(B)) \otimes \mathbb{R}$ the following are equivalent.

1. $h_P \cdot h_Q = 0$.
2. $\hat{h}_E(P)\hat{h}_E(Q) - \langle P, Q \rangle_E^2 = 0$.

Proof strategy

For $\vec{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$, $h_{\vec{x}}(t) = \hat{h}_{\mathcal{E}_t}(x_1 P_{1,t} + \cdots + x_m P_{m,t})$.

The 'real' equidistribution theorem yields

Proposition (DeMarco-M. 2017, 2018)

Assume that for infinitely many $t_n \in B(\overline{K})$ we have that

$$h_{\vec{x}}(t_n) \to 0 \;\; \& \;\; h_{\vec{y}}(t_n) \to 0.$$

Then for all $t \in B(\overline{K})$ we have

$$h_{\vec{x}}(t) = \alpha h_{\vec{y}}(t),$$

with $\alpha = \frac{\hat{h}_E(x_1 P_1 + \cdots + x_m P_m)}{\hat{h}_E(y_1 P_1 + \cdots + y_m P_m)}$.

Rational case - reduction to Masser-Zannier's theorem

Assume $\vec{x}, \ \vec{y} \in \mathbb{Q}^m$ are linearly independent.

$$P = x_1 P_1 + \cdots + x_m P_m \ \ \& \ \ Q = y_1 P_1 + \cdots + y_m P_m$$

such that $h_P \cdot h_Q = 0$. Then $\hat{h}_{\mathcal{E}_t}(P_t) = \alpha \hat{h}_{\mathcal{E}_t}(Q_t)$ for all $t \in B(\overline{K})$.

In particular, for each $t \in B(\overline{K})$ we have

$$\hat{h}_{\mathcal{E}_t}(P_t) = 0 \ \ \Leftrightarrow \ \ \hat{h}_{\mathcal{E}_t}(Q_t) = 0$$

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In particular, for each $t \in B(\overline{K})$ we have

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Then, we can find infinitely many $t'_n \in B(\overline{K})$ such that

$$\hat{h}_{\mathcal{E}_{t'_n}}(P_{t'_n}) = 0 \ \& \ \hat{h}_{\mathcal{E}_{t'_n}}(Q_{t'_n}) = 0.$$

Invoking Masser-Zannier's theorem we get that

$$(P, Q) : B \to \mathcal{A} \text{ is a special section.}$$

$\square$

A 'good' height: Variation of the canonical height (VCH)

h_B: Weil height on B relative to a divisor of degree 1.
$\hat{h}_E(P)$: 'geometric' Néron-Tate height of $P \in E(k)$.

Remark

$\hat{h}_E(P) = 0 \Leftrightarrow P$ *is a torsion section.*

Theorem (Silverman 1983)

$$\lim_{t \in B(\overline{K}),\ h_B(t) \to \infty} \frac{\hat{h}_{\mathcal{E}_t}(P_t)}{h_B(t)} = \hat{h}_E(P).$$

A 'good' height: VCH

Theorem (Tate 1983)

There is a divisor $D = D(E, P) \in \mathrm{Pic}(C) \otimes \mathbb{Q}$ *of degree* $\hat{h}_E(P)$ *such that*

$$\hat{h}_{\mathcal{E}_t}(P_t) = h_D(t) + O_P(1),$$

as $t \in C(\overline{K})$ *varies.*

In particular, if $C = \mathbb{P}^1$ we have

$$\hat{h}_{\mathcal{E}_t}(P_t) = \hat{h}_E(P)h(t) + O_P(1).$$

The variation of local heights

Let $v \in M_K$. For $t_0 \in C(\mathbb{C}_v)$, fix a uniformizer u at t_0.

To describe the variation of $t \mapsto \hat{h}_{\mathcal{E}_t}(P_t)$ in a more precise way, Silverman considered the 'local components' of VCH

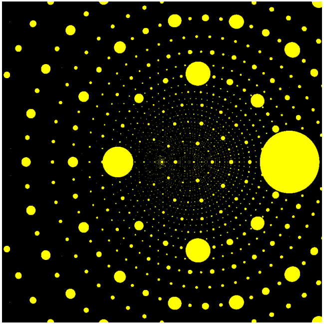
$$V_{P,t_0,v}(t) := \hat{\lambda}_{\mathcal{E}_t}(P_t; v) + \hat{\lambda}_E(P; \mathrm{ord}_{t_0}) \log |u(t)|_v.$$

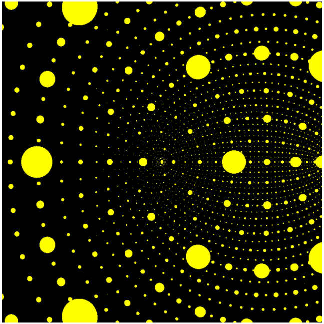
Theorem (Silverman 1992)

1. *$V_{P,t_0,v}(t)$ extends to a continuous function in a neighborhood of t_0.*
2. *$V_{P,t_0,v}(t) \equiv 0$ for all but finitely many $v \in M_K$ in a v-adic neighborhood of t_0.*

Silverman's results + dynamical perspective + ingredients from Silverman's proof $\Rightarrow \hat{h}_{\mathcal{E}_t}(P_t)$ is a 'good height' for equidistribution.

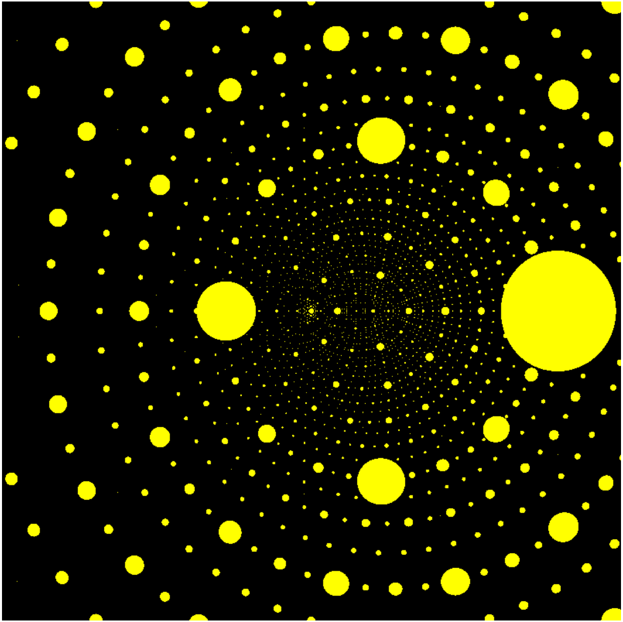
The end



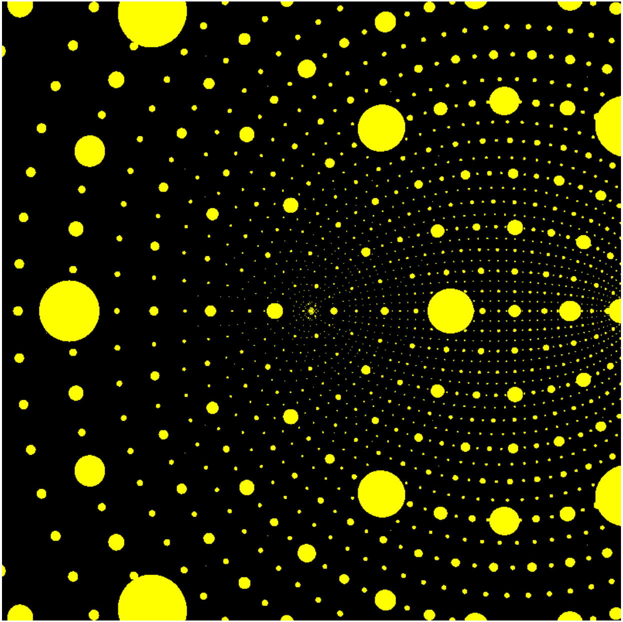


Thank you!

Torsion parameters for $P_2 = (2, \sqrt{2(2-t)})$



Torsion parameters for $P_5 = (5, \sqrt{20(5-t)})$



A 'good' metrized line bundle

Assume $\hat{h}_E(P) \neq 0$. We want to show that $t \mapsto \hat{h}_{\mathcal{E}_t}(P_t)$ comes from a 'good' metric in the sense of equidistribution.
Let $D_E(P) = \sum_{\gamma \in C(\overline{K})} \hat{\lambda}_{E,\mathrm{ord}_\gamma}(P) \cdot (\gamma) \in \mathrm{Div}(C) \otimes \mathbb{Q}$.
$\mathcal{L}_P$: the line bundle on C corresponding to $mD_E(P) \in \mathrm{Div}(C)$.

We give a collection of metrics $\| \cdot \| = \{\| \cdot \|_v\}_{v \in M_K}$ on $\mathcal{L}_Q$.

Let $U \subset C_v^{an}$ open. Each section $s \in \mathcal{L}_P(U)$ is identified with a meromorphic function f on U such that $(f) \geq -mD_E(P)$. We set

$$\|s(t)\|_v = \begin{cases} e^{-m\hat{\lambda}_{E_t,v}(P_t)}|f(t)|_v & \text{if } f(t) \text{ is finite and nonzero} \\ 0 & \text{if } \mathrm{ord}_t f > -m\,\hat{\lambda}_{E,\mathrm{ord}_t}(P) \\ e^{-m\,V_{P,t,v}(t)} & \text{otherwise.} \end{cases}$$

taking the locally-defined uniformizer $u = f^{1/\mathrm{ord}_t f}$ at t in the definition of $V_{P,t,v}$.

Parameters yielding small height

$$\mathcal{E}_t : y^2 = x(x-1)(x-t),$$
$$P_t = (2, \sqrt{2(2-t)})\ ,\ Q_t = (3, \sqrt{6(3-t)})\ ;\ \ t \in \mathbb{C} \setminus \{0,1\}.$$

Claim: If $t_n \in B(\overline{K})$ is such that $[n]P_{t_n} - Q_{t_n} = \mathcal{O}$, then

$$\hat{h}_{\mathcal{E}_{t_n}}(P_{t_n}) \to 0.$$

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To see this note that

$$[n]P_{t_n} = Q_{t_n} \ \ \Rightarrow \hat{h}_{\mathcal{E}_{t_n}}(P_{t_n}) = \frac{\hat{h}_{\mathcal{E}_{t_n}}(Q_{t_n})}{n^2}.$$

Parameters yielding small height

$$\hat{h}_{\mathcal{E}_{t_n}}(P_{t_n}) = \frac{\hat{h}_{\mathcal{E}_{t_n}}(Q_{t_n})}{n^2}.$$

By Silverman's specialization theorem we know that

$$\{h(t_n)\} \text{ is bounded.}$$

Moreover, by Tate's theorem we get that

$$\{\hat{h}_{\mathcal{E}_{t_n}}(Q_{t_n})\}_{n\in\mathbb{N}} \text{ is bounded.}$$

Hence,

$$\hat{h}_{\mathcal{E}_{t_n}}(P_{t_n}) \to 0 \text{ as } n \to \infty.$$

‘Pairing’

By work of Chambert-Loir, Thuillier and Moriwaki, we know that if $t_n \in B(\overline{K})$ is such that

$$\hat{h}_{\mathcal{E}_{t_n}}(P_{t_n}) \to 0,$$

then

$$\hat{h}_{\mathcal{E}_{t_n}}(Q_{t_n}) \to \frac{h_P \cdot h_Q}{\hat{h}_E(P)}.$$

So the assignment $(P, Q) \mapsto h_P \cdot h_Q$ inherites properties of the canonical heights.

Unlikely intersections: A conjecture

C smooth projective curve defined over a number field K
$k = K(C)$

Conjecture (Baker-DeMarco, Ghioca-Hsia-Tucker)

Consider $\mathbf{f} \in K(z)$ *and* $\mathbf{c}_1, \mathbf{c}_2 \in K$. *Assume that for an infinite sequence* $t_n \in C(\overline{K})$ *we have*

$$\hat{h}_{f_{t_n}}(c_1(t_n)) + \hat{h}_{f_{t_n}}(c_2(t_n)) = 0.$$

Then one of the following is true;

1. $\exists\ i \in \{1, 2\}$ *such that* $\mathbf{c}_i$ *is preperiodic for* $\mathbf{f}$.
2. $\exists$ *a Zariski open* $Y \subset X$ *such that* $\forall\ t \in Y(\overline{K})$ *we have* $c_1(t)$ *is preperiodic for* $f_t \Leftrightarrow c_2(t)$ *is preperiodic for* f_t. *Moreover,* $\mathbf{c}_1$ *and* $\mathbf{c}_2$ *are 'dynamically related'.*
   (e.g. for $n, m \in \mathbb{N}$, *we have* $\mathbf{f}^n(\mathbf{c}_1) = \mathbf{f}^m(\mathbf{c}_2)$.*)*