The Lagrange and Markov Spectra of Pythagorean triples

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Approximation diophantienne et transcendance
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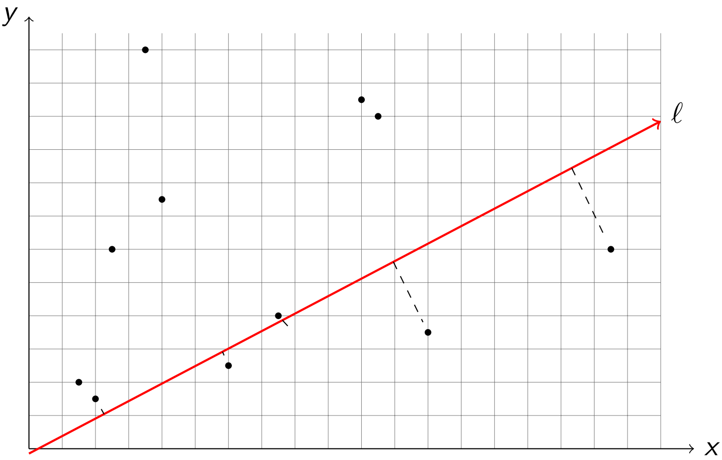
Let us call (a, b) a Pythagorean pair if $a^2 + b^2$ is a square.
How far can a half-line ℓ stay away from most Pythagorean pairs?

$$\delta(\ell) := \liminf_{(a,b)} (\text{the distance from } (a, b) \text{ to } \ell),$$



$P = (\alpha, \beta) \in \mathscr{Q}$ is a point in the unit quarter circle $\mathscr{Q}$

$Z = (\frac{a}{c}, \frac{b}{c}) \in \mathscr{Q}$ is a rational point in $\mathscr{Q}$ represented by an primitive Pythagorean integer triple (a, b, c).

We define

$$\delta(P, Z) = |c| \sqrt{\left(\alpha - \frac{a}{c}\right)^2 + \left(\beta - \frac{b}{c}\right)^2}$$

and

$$\delta(P) = \liminf_{|c| \to \infty} \delta(P, Z).$$

We define the Lagrange number to be

$$L(P) = \delta(P)^{-1} = \limsup_{|c| \to \infty} \delta(P, Z)^{-1}.$$

Classical Lagrange Spectrum

For an irrational number α, Lagrange number $L(\alpha)$ is

$$L(\alpha) = \limsup \left(q^2 \left| \alpha - \frac{p}{q} \right| \right)^{-1},$$

The set

$$\mathscr{L} = \{ L(\alpha) \mid \alpha \in \mathbb{R} - \mathbb{Q}, \quad L(\alpha) < \infty \} \subseteq \mathbb{R}$$

is called the Lagrange spectrum.

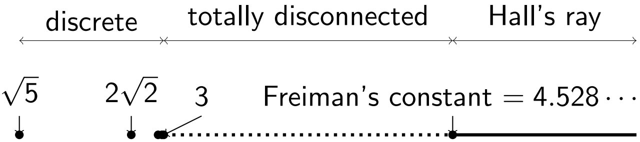
Closely related to $\mathscr{L}$ is the Markov spectrum

$$\mathscr{M} = \left\{ \frac{\sqrt{b^2 - 4ac}}{\inf\limits_{\substack{(x,y)\in\mathbb{Z}^2 \\ (x,y)\neq(0,0)}} |f(x,y)|} \;\middle|\; \begin{array}{l} f(x,y) = ax^2 + bxy + cy^2, \\ b^2 - 4ac > 0, \quad a, b, c \in \mathbb{R} \end{array} \right\}.$$

Classical Lagrange Spectrum

Markov showed

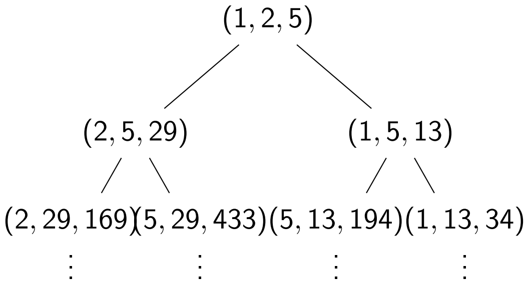
$$\mathscr{L}_{<3} = \left\{ \frac{\sqrt{9m^2 - 4}}{m} \middle| m = 1, 2, 5, 13, 29, \dots \right\} = \mathscr{M} \cap [0, 3).$$



Classical Lagrange Spectrum

Here, m is any one of a triple (m_1, m_2, m_3) of positive integers satisfying

$$m_1^2 + m_2^2 + m_3^2 = 3m_1 m_2 m_3.$$



Berggren Trees

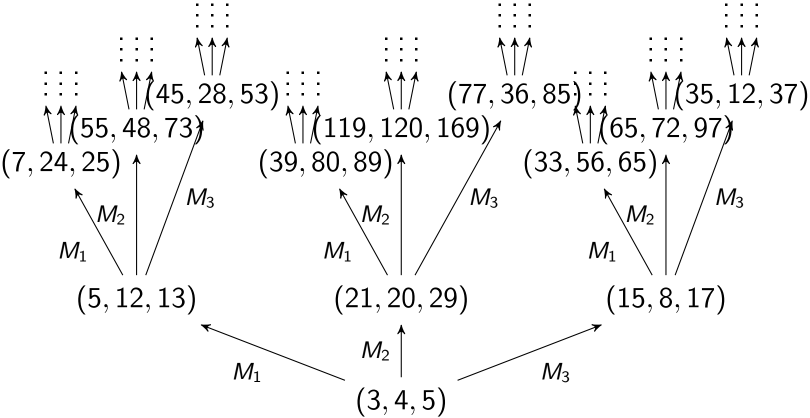
Theorem (Berggren (1934), Barning (1963), ...)

The matrices

$$M_1 = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}, M_3 = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix}$$

generate all (positive) primitive Pythagorean triples exactly once via repeated left-multiplication, beginning at $(3,4,5)$ *and* $(4,3,5)$.

Berggren Trees



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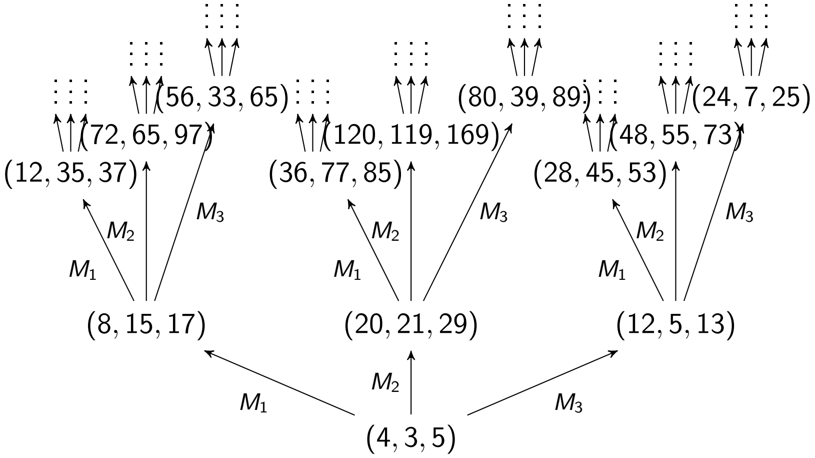


Romik System

Romik (2008) used Berggren trees (for the Pythagorean triples) to construct a dynamical system on $\mathscr{Q} = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0\}$:

$$T : \mathscr{Q} \longrightarrow \mathscr{Q}$$

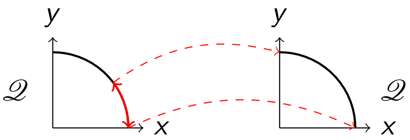
which is defined by

$$T(x, y) = \left(\frac{|2 - x - 2y|}{3 - 2x - 2y}, \frac{|2 - 2x - y|}{3 - 2x - 2y} \right).$$

For a rational point $(\frac{a}{c}, \frac{b}{c}) \in \mathscr{Q}$, T is a parent-finding map on its Berggren tree: $T(\frac{a}{c}, \frac{b}{c}) = (\frac{a'}{c'}, \frac{b'}{c'})$ exactly when (a', b', c') is the parent of (a, b, c) on the tree.

Romik System

In picture, T is a 3-1 map from $\mathscr{Q}$ onto itself:



Moreover, we define the digit $d(x, y)$ of $(x, y) \in \mathscr{Q}$ as follows:

$$\begin{cases} d(x,y) = 1 \text{ (the digit of } (x,y) \text{ is 1)} & \text{if } 4/5 \le x \le 1, \\ d(x,y) = 2 \text{ (the digit of } (x,y) \text{ is 2)} & \text{if } 3/5 < x < 4/5, \\ d(x,y) = 3 \text{ (the digit of } (x,y) \text{ is 3)} & \text{if } 0 \le x \le 3/5. \end{cases}$$

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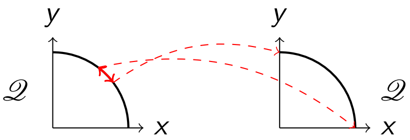

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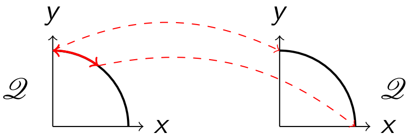

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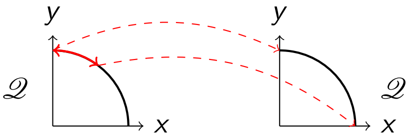

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Define $d_j := d_j(x, y) = d(T^{j-1}(x, y))$ and write

$$(x, y) := [d_1, d_2, \dots]_{\mathscr{Q}}.$$

Romik digit

A point (x, y) in $\mathscr{Q}$ is rational (over $\mathbb{Q}$) if and only if its digit expansion ends with infinite succession of 1's or 3's.

$$\left(\frac{3}{5}, \frac{4}{5}\right) = [3, 3, 3, 3, \dots]_{\mathscr{Q}}, \qquad \left(\frac{4}{5}, \frac{3}{5}\right) = [1, 1, 1, 1, \dots]_{\mathscr{Q}},$$

$$\left(\frac{21}{29}, \frac{20}{29}\right) = [2, 3, 3, 3, \dots]_{\mathscr{Q}}, \quad \left(\frac{15}{17}, \frac{8}{17}\right) = [1, 3, 3, 3, \dots]_{\mathscr{Q}}.$$

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = [2, 2, 2, \dots]_{\mathscr{Q}} := [\overline{2}]_{\mathscr{Q}},$$

$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = [1, 3, 1, 3, 1, \dots]_{\mathscr{Q}} := [\overline{1, 3}]_{\mathscr{Q}},$$

$$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = [3, 1, 3, 1, 3, \dots]_{\mathscr{Q}} := [\overline{3, 1}]_{\mathscr{Q}},$$

M matrices

We consider $Z = (\frac{a}{c}, \frac{b}{c}) \in \mathscr{Q}$ as $w = \frac{a+bi}{c}$. Then

$$M_1(w) = \frac{(1+i)w - 1}{w - 1 + i}, \quad M_3(w) = \frac{(1+i)w + 1}{w + 1 - i}$$

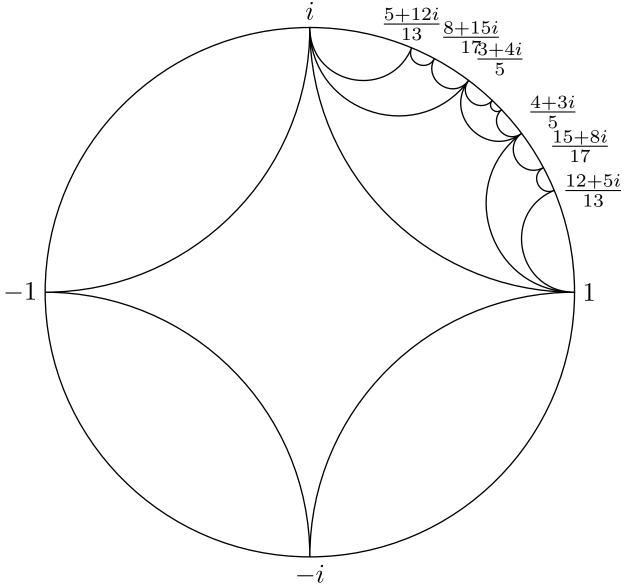
$$M_2(w) = \frac{(1+i)\bar{w} + 1}{\bar{w} + 1 - i}.$$

Note that

$$T(x+yi) = \begin{cases} M_1^{-1}(x+yi) & \text{if } 4/5 \le x \le 1, \\ M_2^{-1}(x+yi) & \text{if } 3/5 < x < 4/5, \\ M_3^{-1}(x+yi) & \text{if } 0 \le x \le 3/5. \end{cases}$$

Actions of M_1, M_2, M_3

M_1 sends the ideal quadrilateral $[1, i, -1, -i]$ to $[1, \frac{4+3i}{5}, \frac{3+4i}{5}, i]$.



The Hurwitz bound

Theorem (Moshchevitin)

For any irrational P,

$$L(P) \leq \sqrt{2}.$$

We define $C(Z;r)$ as the horocycle based on $Z = (a/c, b/c)$ with (Euclidean) radius r, i.e.,

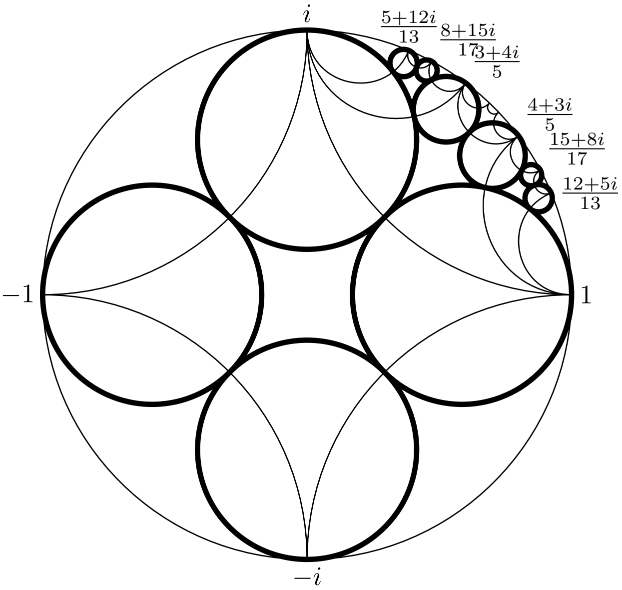
$$C(Z;r) = \left\{(x,y) : \left(x - \frac{(1-r)a}{c}\right)^2 + \left(y - \frac{(1-r)b}{c}\right)^2 = r^2\right\}.$$

Lemma

For each $Z = (a/c, b/c)$, $Z' = (a'/c',' b/c')$,

$$C\left(Z; \frac{1}{c\sqrt{2}+1}\right) \text{ and } C\left(Z'; \frac{1}{c'\sqrt{2}+1}\right) \text{ do not intersects.}$$

They are tangent if and only if $Q((a,b,c) \times_Q (a',b',c')) = 1$.


i
$\frac{5+12i}{13}$
$\frac{8+15i}{17}$
$\frac{3+4i}{5}$
$\frac{4+3i}{5}$
$\frac{15+8i}{17}$
$\frac{12+5i}{13}$
-1
1
$-i$

Formula for $L(P)$

For $P = [d_1, d_2, \ldots] \in \mathscr{Q}$, write

$$Z_k^{(1,0)} = [d_1, d_2, \ldots, d_k, 1, 1, 1, \ldots]_{\mathscr{Q}},$$
$$Z_k^{(0,1)} = [d_1, d_2, \ldots, d_k, 3, 3, 3, \ldots]_{\mathscr{Q}}.$$

Then we have

$$L(P) = \limsup_{k\to\infty} \left(\min\{\delta(P, Z_k^{(1,0)}), \delta(P, Z_k^{(0,1)})\}\right)^{-1}.$$

By the map from the unit disk to the upper half plane $\mathbb{H}$

$$z = f(w) = \frac{1 - iw}{w - i},$$

$$\mathbf{M}_1(z) = 2 - \frac{1}{z}, \quad \mathbf{M}_2(z) = 2 + \frac{1}{\bar{z}}, \quad \mathbf{M}_3(z) = z + 2$$

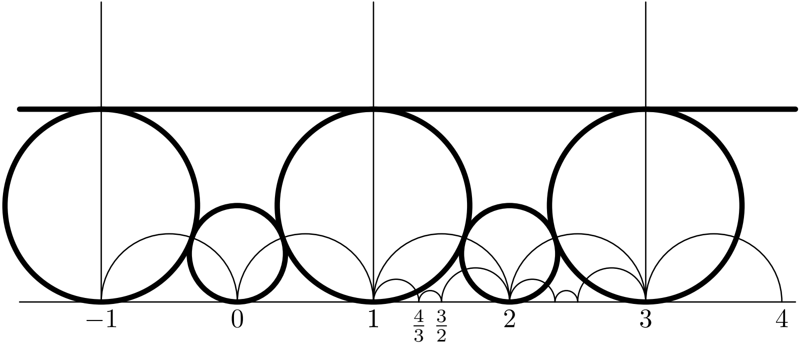
Let

$$M_1 = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \quad M_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Note that

$$(\mathbf{M}_2)^2(z) = \frac{5z + 2}{2z + 1}, \qquad (M_2)^2 = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

Upper half plane



M_1, M_2, M_3 send the ideal quadrilateral $[-1, 0, 1, \infty]$ to the ideal quadrilateral $[1, 2, 3, \infty]$.

For a Romik sequence P, we define the reverse P^* of P and the conjugate $P^\vee$ of P by $P^\vee = [p_1^\vee, p_2^\vee, \dots]$ where $p_j^\vee = 3 - p_j$.

Recall that, if $P = (\alpha, \beta)$ as a point in $\mathscr{Q}$, then $P^\vee = (\beta, \alpha)$.

Let $\|P\| = \frac{1+\beta}{\alpha} = f(\alpha + \beta i)$ for $P = (\alpha, \beta)$.

Define $L_1(P^*|Q)$ and $L_2(P^*|Q)$ to be

$$L_1(P^*|Q) = \frac{\|P\|^{-1} + \|Q\|^{-1} - 2}{2},$$

$$L_2(P^*|Q) = L_1((P^\vee)^*|Q^\vee) = \frac{\|P^\vee\|^{-1} + \|Q^\vee\|^{-1} - 2}{2}.$$

Also, we let

$$L(P^*|Q) = \max(L_1(P^*|Q), L_2(P^*|Q)).$$

Finally, we let $\quad L(T) = \sup L(P^*|Q).$

We define an ordering

$$P \le Q$$

between two Romik sequences P and Q whenever $\|P\| \le \|Q\|$.

Proposition

The followings are equivalent.
(i) $P \le Q$.
(ii) $[1, P] \le [1, Q]$.
(iii) $[2, Q] \le [2, P]$.
(iv) $[3, P] \le [3, Q]$.
(v) Write $P = [d_1, \ldots, d_k, r_1, r_2, \ldots]$, $Q = [d_1, \ldots, d_k, s_1, s_2, \ldots]$, *where* $r_1 \neq s_1$. *Then,* $(-1)^t(s_1 - r_1) > 0$ *where* t *is the number of 2's appearing in the sequence* $\{d_1, \ldots, d_k\}$.

For a doubly infinite Romik sequence T, we say T is *admissible* if $L(T) \le 2$ and *strongly admissible* if $L(T) < 2$.

Proposition

Suppose that a doubly infinite Romik sequence T is admissible. Then, the following sequences are forbidden in T:

$$\begin{cases} 2(13)^k12,\ 2(31)^k32, & \text{for } k \geq 0, \\ 12^k3,\ 32^k1, & \text{if } k \text{ is odd}, \\ 12^k1,\ 32^k3, & \text{if } k \text{ is even}. \end{cases}$$

Proposition

A doubly infinite Romik sequence T is admissible if and only if:

(i) The sequences $11, 33, 212, 232$ do not occur in T, and

(ii) Any section $P^*2|13Q$ of T, T^*, $T^\vee$ and $(T^*)^\vee$ satisfies $P \leq Q$.

Suppose that T is an admissible doubly infinite Romik sequence. Then T is written as a doubly infinite word B on three letters $a, a^{\vee}, b$ under the rule:

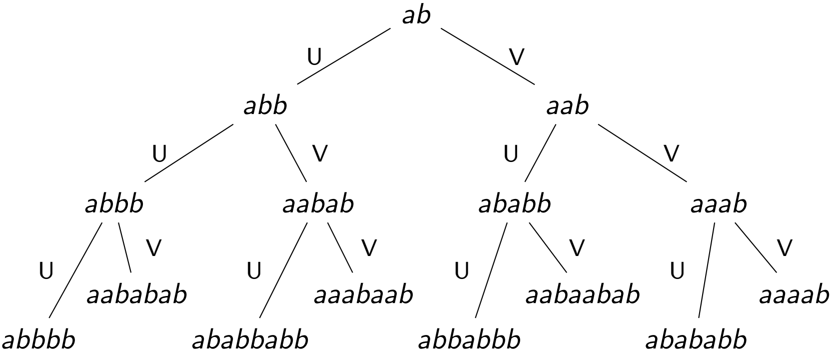
$$a = 2, \quad b = 13, \quad b^{\vee} = 31.$$

Furthermore, the letters are subject to the following rules:

$$\begin{cases} \cdots b \overbrace{a \cdots a}^{\text{even}} \boxed{b} \ldots, \\ \cdots b^{\vee} \overbrace{a \cdots a}^{\text{even}} \boxed{b^{\vee}} \ldots, \\ \cdots b \overbrace{a \cdots a}^{\text{odd}} \boxed{b^{\vee}} \ldots, \\ \cdots b^{\vee} \overbrace{a \cdots a}^{\text{odd}} \boxed{b} \ldots. \end{cases}$$

Doubly infinite admissible Romik sequence is a periodic word with a, b.

$$a^U = ab, \ b^U = b; \qquad a^V = a, \ b^V = ab$$



Let

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} = M_2 \in \Phi$$

and

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = M_1 M_3 \in \Phi.$$

Define

$$\Phi = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z}) \mid \begin{array}{c} a \equiv d \equiv 0 \pmod 2 \text{ or} \\ b \equiv c \equiv 0 \pmod 2 \end{array} \right\}$$
$$\bigcup \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid \begin{array}{c} ad - bc = 2, \\ a \equiv b \equiv c \equiv d \equiv 1 \pmod 2 \end{array} \right\}.$$

Proposition

For $M_W = \begin{bmatrix} p & p' \\ q & q' \end{bmatrix}$, *with discriminant* $\Delta = \mathrm{Tr}(M_W)^2 - 4$,

$$L(\cdots WW|WW\cdots) = \frac{\eta + (-\bar{\eta})}{2} = \frac{\sqrt{\Delta}}{2q}.$$

For $M_W = \frac{1}{\sqrt{2}}\begin{bmatrix} p & p' \\ q & q' \end{bmatrix}$, *with discriminant* $\Delta = \mathrm{Tr}(M_W)^2 - 4$,

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Here, $\eta, \bar{\eta}$ *are two end points of the axis of* M_W. $\left(\eta = \frac{p\eta + q'}{q\eta + q'}\right)$

Let $A^U = AB,\ B^U = B; \qquad A^V = A,\ B^V = AB.$

$$(A,B,AB) = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 7 & 11 \\ 3 & 5 \end{bmatrix} \right)$$

$$(A,B,AB)^U = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 7 & 11 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 25 & 43 \\ 11 & 19 \end{bmatrix} \right)$$

$$(A,B,AB)^V = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 7 & 11 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 12 & 19 \\ 5 & 8 \end{bmatrix} \right)$$

$$(A,B,AB)^{UU} = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 25 & 43 \\ 11 & 19 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 93 & 161 \\ 41 & 71 \end{bmatrix} \right)$$

$$(A,B,AB)^{UV} = \left(\begin{bmatrix} 12 & 19 \\ 5 & 8 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 7 & 11 \\ 3 & 5 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 141 & 227 \\ 59 & 95 \end{bmatrix} \right)$$

$$(A,B,AB)^{VU} = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 7 & 11 \\ 3 & 5 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 25 & 43 \\ 11 & 9 \end{bmatrix}, \begin{bmatrix} 148 & 255 \\ 65 & 112 \end{bmatrix} \right)$$

$$(A,B,AB)^{VV} = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 12 & 19 \\ 5 & 8 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 41 & 65 \\ 17 & 27 \end{bmatrix} \right)$$

The Cohn matrix M has the shape

$$M = \begin{bmatrix} 4q - q' & * \\ q & q' \end{bmatrix} \quad \text{or} \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 4q - q' & * \\ q & q' \end{bmatrix},$$

if the number of A in M is even, or odd, respectively

For each $(M, N, MN) = (A, B, AB)^w$,

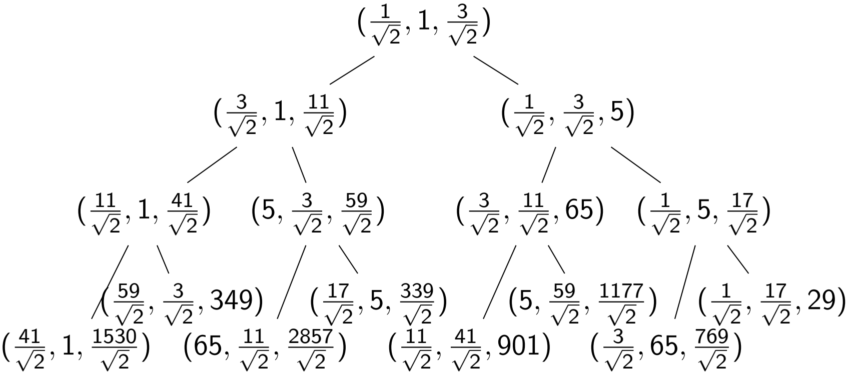
$$\mathrm{Tr}(M)^2 + \mathrm{Tr}(N)^2 + \mathrm{Tr}(MN)^2 = \mathrm{Tr}(M)\,\mathrm{Tr}(N)\,\mathrm{Tr}(MN).$$

.

$(q(M), q(N), q(MN))$ is an integral solution of the equation

$$2x^2 + y_1^2 + y_2^2 = 4xy_1y_2.$$

$(\mathrm{Tr}(M), \mathrm{Tr}(N), \mathrm{Tr}(MN))$



Theorem

For $M_W = \begin{bmatrix} 4x - x' & * \\ x & x' \end{bmatrix}$, *with discriminant* $\Delta = 16x^2 - 4$,

$$L(\cdots WW|WW\cdots) = \frac{\eta + (-\bar{\eta})}{2} = \frac{\sqrt{\Delta}}{2x} = \sqrt{4 - \frac{1}{x^2}}.$$

For $M_W = \frac{1}{\sqrt{2}} \begin{bmatrix} 4y - y' & * \\ y & y' \end{bmatrix}$, *with discriminant* $\Delta = 8y^2 - 4$,

$$L(\cdots WW|WW\cdots) = \frac{\eta + (-\bar{\eta})}{2} = \frac{\sqrt{\Delta}}{2y/\sqrt{2}} = \sqrt{4 - \frac{2}{y^2}}.$$

Here, $\eta, \bar{\eta}$ *are two end points of the axis of* M_W.

$$\mathscr{L}^P_{<2} = \left\{ \sqrt{4 - \frac{1}{x^2}} \,|\, x = 1, 5, 11, 29, 65, 349, \ldots \right\}$$
$$\cup \left\{ \sqrt{4 - \frac{2}{y^2}} \,|\, y = 1, 3, 11, 17, 41, 59, 99, \ldots \right\}.$$

Let

$$f(S) = f\left(S\begin{bmatrix}1\\0\end{bmatrix}\right).$$

Then

$$\inf_{S\in\Phi}|f(S)| = \inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{|f(x,y)|}{\text{parity}(x,y)}$$

where

$$\text{parity}(x,y) = \begin{cases}2 & \text{if both } x, y \text{ are odd,}\\ 1 & \text{otherwise.}\end{cases}$$

Let

$$\mathscr{M}^\Phi = \left\{\frac{\sqrt{b^2-4ac}}{\inf_{S\in\Phi}|f(S)|}\,\middle|\, \begin{array}{l} f(x,y) = ax^2+bxy+cy^2,\\ b^2-4ac>0, \quad a,b,c\in\mathbb{R}\end{array}\right\}.$$

Then

$$\frac{1}{2}\mathscr{M}^\Phi\cap[0,2) = \left\{\sqrt{4-\frac{1}{x^2}}\,\middle|\, x = 1,5,11,29,65,349,\ldots\right\}$$

$$\cup\left\{\sqrt{4-\frac{2}{y^2}}\,\middle|\, y = 1,3,11,17,41,59,99,\ldots\right\}.$$

The 10 smallest values of $L(P)$, together with corresponding Romik digits of P's, are

$L(P)$	Romik digits
$\sqrt{2} = 1.414213562\ldots$	$[\overline{2}]_{\mathscr{Q}}$
$\sqrt{3} = 1.732050808\ldots$	$[\overline{1,3}]_{\mathscr{Q}}$
$\frac{\sqrt{34}}{3} = 1.943650632\ldots$	$[\overline{2,1,3}]_{\mathscr{Q}}$
$\frac{3\sqrt{11}}{5} = 1.989974874\ldots$	$[\overline{2,2,1,3}]_{\mathscr{Q}}$
$\frac{\sqrt{482}}{11} = 1.995863491\ldots$	$[\overline{2,1,3,1,3}]_{\mathscr{Q}}$
$\frac{\sqrt{1154}}{17} = 1.998269147\ldots$	$[\overline{2,2,2,1,3}]_{\mathscr{Q}}$
$\frac{\sqrt{6722}}{41} = 1.999702536\ldots$	$[\overline{1,3,1,3,1,3,2,}]_{\mathscr{Q}}$
$\frac{\sqrt{3363}}{29} = 1.999702713\ldots$	$[\overline{2,2,2,2,1,3}]_{\mathscr{Q}}$
$\frac{\sqrt{13922}}{59} = 1.999856358\ldots$	$[\overline{2,1,3,2,2,1,3}]_{\mathscr{Q}}$
$\frac{\sqrt{16899}}{65} = 1.999940828\ldots$	$[\overline{2,1,3,1,3,2,1,3}]_{\mathscr{Q}}$

The 10 smallest values of $L(P)$, together with corresponding Romik digits of P's, are

$L(P)$	Romik digits
$\sqrt{2} = 1.414213562\ldots$	$[\overline{2}]_{\mathscr{Q}}$
$\sqrt{3} = 1.732050808\ldots$	$[\overline{1,3}]_{\mathscr{Q}}$
$\frac{\sqrt{34}}{3} = 1.943650632\ldots$	$[\overline{2,1,3,2,3,1}]_{\mathscr{Q}}$
$\frac{3\sqrt{11}}{5} = 1.989974874\ldots$	$[\overline{2,2,1,3}]_{\mathscr{Q}}$
$\frac{\sqrt{482}}{11} = 1.995863491\ldots$	$[\overline{2,1,3,1,3,2,3,1,3,1}]_{\mathscr{Q}}$
$\frac{\sqrt{1154}}{17} = 1.998269147\ldots$	$[\overline{2,2,2,1,3,2,2,2,3,1}]_{\mathscr{Q}}$
$\frac{\sqrt{6722}}{41} = 1.999702536\ldots$	$[\overline{1,3,1,3,1,3,2,3,1,3,1,3,1,2}]_{\mathscr{Q}}$
$\frac{\sqrt{3363}}{29} = 1.999702713\ldots$	$[\overline{2,2,2,2,1,3}]_{\mathscr{Q}}$
$\frac{\sqrt{13922}}{59} = 1.999856358\ldots$	$[\overline{2,1,3,2,2,1,3,2,3,1,2,2,3,1}]_{\mathscr{Q}}$
$\frac{\sqrt{16899}}{65} = 1.999940828\ldots$	$[\overline{2,1,3,1,3,2,1,3}]_{\mathscr{Q}}$