two regular problems

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Motivation

Definition (Mahler function)

A power series $F(z) \in \mathbb{C}[[z]]$ is *k-Mahler* for an integer $k \geqslant 2$ provided there is an integer $d \geqslant 1$ and polynomials $a_0(z), \ldots, a_d(z) \in \mathbb{C}[z]$ with $a_0(z)a_d(z) \neq 0$ such that

$$a_0(z)F(z) + a_1(z)F(z^k) + \cdots + a_d(z)F(z^{k^d}) = 0.$$

Mahler functions are coordinates of a vector $\mathbf{F}(z) := [F_1(z), \ldots, F_d(z)]^T$ such that there is a matrix of rational functions $\mathbf{A}(z)$ such that

$$\mathbf{F}(z) = \mathbf{A}(z)\mathbf{F}(z^k).$$

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So for every n, we have $\mathbf{F}(z) = \underbrace{\mathbf{A}(z)\mathbf{A}(z^k)\cdots\mathbf{A}(z^{k^{n-1}})}_{\text{matrix cocycle}}\mathbf{F}(z^{k^n})$.

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Two important classes of Mahler functions are the generating series of

- automatic sequences
- regular sequences

Regular sequences

Recall that a sequence is linearly recurrent if and only if there exist a positive integer d, a matrix $\mathbf{A} \in \mathbb{Z}^{d\times d}$, and vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^d$ such that

$$f(n) = \mathbf{w}^T \mathbf{A}^n \mathbf{v}.$$

Definition (Allouche and Shallit, 1992)

A sequence f is k-regular if and only if there exist a positive integer d, a finite set of matrices $\mathcal{A}_f = \{\mathbf{A}_0, \ldots, \mathbf{A}_{k-1}\} \subseteq \mathbb{Z}^{d\times d}$, and vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^d$ such that

$$f(n) = \mathbf{w}^T \mathbf{A}_{i_0} \cdots \mathbf{A}_{i_s} \mathbf{v},$$

where $(n)_k = i_s \cdots i_0$ is the base-k expansion of n.

We use the terminology 'k-regular' also for generating functions $F(x) := \sum_{n\geqslant 0} f(n)x^n$.

Theorem (Allouche and Shallit, 1992)

The set of k-regular functions form a ring under standard power series addition and multiplication.

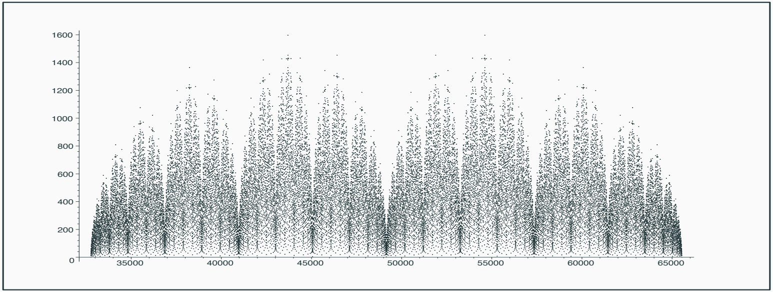
Example: Stern's diatomic sequence

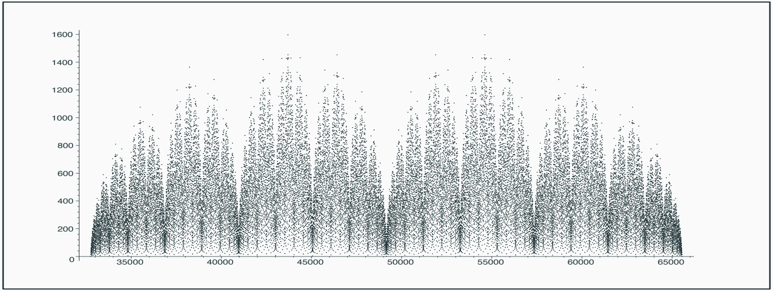
Let $\{s(n)\}_{n\geqslant 0}$ be *Stern's diatomic sequence*, which is determined by the relations $s(0)=0$, $s(1)=1$, and for $n \geqslant 0$, by

$$s(2n)=s(n), \quad \text{and} \quad s(2n+1)=s(n)+s(n+1).$$

One can show that the Stern sequence has linear representation

$$\mathbf{v}^T=\mathbf{w}^T=(1\ 0), \quad \mathbf{A}_0=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \quad \mathbf{A}_1=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}.$$





Question

What happens to this picture in the limit?

- Maximal growth?
- Almost everywhere?
- On average?

Stern on average

Set $S(z) := \sum_{n \geqslant 0} s(n) z^n$. Then $S(z)$ satisfies the functional equation

$$zS(z) - (z^2 + z + 1)S(z^2) = 0.$$

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Using the recurrence relations, one can show that

$$\sum_{m=2^n}^{2^{n+1}-1} s(m) = 3^n, \quad \text{and also that} \quad \frac{1}{N^{\log_2 3}} \sum_{n \leqslant N} s(n) \quad \text{is well-behaved.}$$

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This suggests that there is a "nice enough" function $C(z)$ such that as $z \to 1^-$ we have

$$S(z) \approx \frac{C(z)}{(1-z)^{\log_2 3}}.$$

Eigenvalue test for transcendence of Mahler functions

Let $k \geqslant 2$ and $d \geqslant 1$ be integers and $F(z)$ be a k-Mahler function converging inside the unit disc satisfying

$$a_0(z)F(z) + a_1(z)F(z^k) + \cdots + a_d(z)F(z^{k^d}) = 0,$$

for polynomials $a_0(z), \ldots, a_d(z) \in \mathbb{C}[z]$.

- Set $a_i := a_i(1)$ and form the polynomial

$$p_F(\lambda) := a_0\lambda^d + a_1\lambda^{d-1} + \cdots + a_{d-1}\lambda + a_d.$$

- If $a_0 a_d \neq 0$ and $p_F(\lambda)$ has distinct roots, then the function $F(z)$ is transcendental over $\mathbb{C}(z)$ provided
  - $p_F(k^n) \neq 0$ for all $n \in \mathbb{Z}$ or
  - the eigenvalue $\lambda_F \neq k^n$ for any $n \in \mathbb{Z}$.
- If $\lambda_F = k^n$ for some $n \in \mathbb{Z}$, the test is inconclusive.

An average result

Theorem (Bell and C. 2017)

Let $F(z)$ be a k-Mahler function whose characteristic polynomial $p_F(\lambda)$ has distinct roots. Then there is an eigenvalue λ_F with $p_F(\lambda_F) = 0$, such that as $z \to 1^-$

$$F(z) = \frac{C(z)}{(1-z)^{\log_k \lambda_F}}(1 + o(1)),$$

where $\log_k$ denotes the principal value of the base-k logarithm and $C(z)$ is a real-analytic nonzero oscillatory term, which on the interval $(0, 1)$ is bounded away from 0 and ∞, and satisfies $C(z) = C(z^k)$.

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Very similar methods allow one to show as well the other direction from a theorem like this. In particular if f is a k-regular sequence which is nonnegative, then there is an algebraic number α_f and a nonnegative integer m_f such that

$$\sum_{n\leqslant N} f(n)\asymp N^{\log_k \alpha_f}\log^{m_f} N.$$

Maximal growth

Theorem (Allouche and Shallit, 1992)

Let f be a k-regular sequence with values in $\mathbb{C}$. Then there is a constant c such that $f(n) = O(n^c)$.

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For the Stern sequence s, one has $c = \log_2 \varphi$, where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio. This follows from work of Reznick (1990).

Theorem (C. and Tyler, 2014)

Let $\{s(n)\}_{n \geq 0}$ denote the Stern sequence. Then

$$\limsup_{n \to \infty} \frac{s(n)}{n^{\log_2 \varphi}} = \frac{\varphi}{\sqrt{5}} \left(\frac{3}{2}\right)^{\log_2 \varphi} = \frac{\varphi^{\log_2 3}}{\sqrt{5}} = 0.9588541900 \cdots .$$

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Definition (Growth exponent)
Let $k \geqslant 1$ be an integer and $f : \mathbb{Z}_{\geqslant 0} \to \mathbb{C}$ be a (not eventually zero) k-regular sequence. We define the *growth exponent of f*, denoted $\mathrm{GrExp}(f)$, by

$$\mathrm{GrExp}(f) := \limsup_{\substack{n\to\infty \\ f(n)\neq 0}} \frac{\log |f(n)|}{\log n}.$$

Definition (Joint spectral radius)

The *joint spectral radius* of a finite set of matrices $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{k-1}\}$, denoted $\rho(\mathcal{A})$, is defined as the real number

$$\rho(\mathcal{A}) = \limsup_{n\to\infty} \max_{0\leqslant i_0, i_1, \ldots, i_{n-1} \leqslant k-1} \left\| \mathbf{A}_{i_0} \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_{n-1}} \right\|^{1/n},$$

where $\|\cdot\|$ is any (submultiplicative) matrix norm.

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Theorem (C., 2017)

Let $k \geqslant 1$ and $d \geqslant 1$ be integers and $f : \mathbb{Z}_{\geqslant 0} \to \mathbb{C}$ be a (not eventually zero) k-regular sequence. If $\mathcal{A}_f$ is any "minimal" collection of k integer matrices associated to f, then

$$\log_k \rho(\mathcal{A}_f) = \mathrm{GrExp}(f),$$

where $\log_k$ denotes the base-k logarithm.

Maximal growth

Recall the Stern sequence has linear representation

$$\mathbf{v}^T = \mathbf{w}^T = (1\ 0), \quad \mathbf{A}_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A}_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

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We have

$$\rho\left(\left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right\}\right) = \varphi = \rho\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right)^{1/2}.$$

One regular problem: the finiteness conjecture

The finiteness conjecture

Definition

A finite set of matrices $\mathcal{A}$ is said to have the *finiteness property* provided there is a specific finite product $\mathbf{A}_{i_0} \cdots \mathbf{A}_{i_{m-1}}$ of matrices from $\mathcal{A}$ such that $\rho(\mathbf{A}_{i_0} \cdots \mathbf{A}_{i_{m-1}})^{1/m} = \rho(\mathcal{A})$.

FINITENESS CONJECTURE. For each finite set Σ of $n \times n$ real matrices there is some finite k such that

$$\widehat{\rho}(\Sigma) = \overline{\rho}(\Sigma) = \overline{\rho}_k(\Sigma)^{1/k}. \tag{1.10}$$

This conjecture arose from work of Daubechies and Lagarias (1992a), in connection with the problem of whether there is an effectively computable procedure for deciding whether or not a finite set of matrices Σ with rational entries has joint spectral radius $\widehat{\rho}(\Sigma) < 1$. If the finiteness conjecture is true, then such an algorithm exists, namely, for $k = 1, 2, 3, \ldots$ compute $\widehat{\rho}_k(\Sigma)^{1/k}$ and $\widehat{\rho}_k(\Sigma, \|\cdot\|)^{1/k}$,

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- This was shown to be false by Bousch and Mairesse (2002).
- A constructive counterexample was recently given by Hare, Morris, Sidorov and Theys (2011).

Example (Hare, Morris, Sidorov and Theys, 2011)

Let τ denote the sequence of integers defined by $\tau_0 = 1$, $\tau_1, \tau_2 = 2$, and $\tau_{n+1} = \tau_n \tau_{n-1} - \tau_{n-2}$ for all $n \geqslant 2$, and let F_n be the nth Fibonacci number for $n \geqslant 0$. Define the real number $\alpha_* \in (0,1]$ by

$$\alpha_* := \prod_{n \geqslant 1} \left(1 - \frac{\tau_{n-1}}{\tau_n \tau_{n+1}}\right)^{(-1)^n F_{n+1}} = 0.749326546330367\ldots.$$

Then this infinite product converges unconditionally, and the set

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \alpha_* \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

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Question (Finiteness property for integer matrices)
Determine if finite sets of integer matrices satisfy the finiteness property.

Another regular problem: Lehmer's Mahler measure question

Definition (Mahler measure)

Let $p(z) = a_0 \prod_{i=0}^{s}(z - \alpha_i) \in \mathbb{C}[z]$. The *logarithmic Mahler measure* of $p(z)$ is given by

$$\mathfrak{m}(p) := \log|a_0| + \sum_{i=0}^{s} \log(\max\{|\alpha_i|, 1\}) = \int_0^1 \log|p(e^{2\pi i t})|\, dt.$$

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Lehmer found

$$\mathfrak{m}(1 + z - z^3 - z^4 - z^5 - z^6 - z^7 + z^9 + z^{10}) \approx \log(1.176281).$$

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For any integer polynomial p with $\mathfrak{m}(p) < \log 2$ there is an integer polynomial q such that pq has height 1.

Boyd observed that, in his experience, such a q can be taken to be cyclotomic of fairly small degree relative to the degree of p.

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We will show that Lehmer's question, restricted to height-one polynomials, is a property of a matrix cocycle associated to a binary automatic sequence.

Binary automatic sequence Example: Thue-Morse

The canonical example of an automatic sequence is the Thue-Morse sequence

$$\{t(n)\}_{n\geqslant 0} := 0110100110010110100101100110\cdots.$$

Here $t(n)$ takes the value 1 if the binary expansion of n has an odd number of ones, and the value 0 if the binary expansion has an even number of ones.

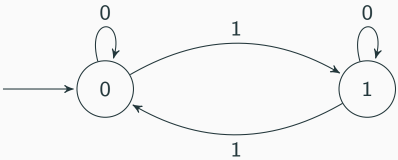

Figure 1: The 2-automaton that produces the Thue-Morse sequence.

The Thue–Morse sequence is given by the substitution

$$\varrho_{\mathrm{TM}} : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10. \end{cases}$$

Definition

A binary automatic sequence ϱ is given by by

$$\varrho : \begin{cases} 0 \mapsto w_0 \\ 1 \mapsto w_1 \, , \end{cases} \tag{1}$$

where w_0 and w_1 are finite words over $\{0, 1\}$ of equal length $|w_0| = |w_1| = L \geqslant 2$.

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The substitution matrix satisfies $M_{\varrho} = \big(\mathrm{card}(T_{ij})\big)_{0 \leqslant i,j \leqslant 1}$.

Binary automatic sequences and their Fourier matrices

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Definition (Fourier matrix of ϱ)

Using T, we build a matrix of pure point measures $\delta_T := (\delta_{T_{ij}})_{0 \leqslant i,j \leqslant 1}$, where $\delta_S := \sum_{x \in S} \delta_x$ with $\delta_\varnothing = 0$. This gives rise to an analytic matrix-valued function via

$$B(t) := \overline{\widehat{\delta_T}(t)},$$

which we call the *Fourier matrix* of ϱ.

Two paradigmatic examples, I

Example

Consider the Thue–Morse substitution, as given by

$$\varrho_{\mathrm{TM}} : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10. \end{cases}$$

Here, one has $T_{\mathrm{TM}} = \left(\begin{smallmatrix} \{0\} & \{1\} \\ \{1\} & \{0\} \end{smallmatrix}\right)$, *which gives*

$$\delta_{T_{\mathrm{TM}}} = \begin{pmatrix} \delta_0 & \delta_1 \\ \delta_1 & \delta_0 \end{pmatrix} \quad \textit{and} \quad B_{\mathrm{TM}}(t) = \begin{pmatrix} 1 & e^{2\pi i t} \\ e^{2\pi i t} & 1 \end{pmatrix}.$$

Two paradigmatic examples, II

Example

For the period doubling substitution,

$$\varrho_{pd} : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 00, \end{cases}$$

The corresponding matrices are $T_{\mathrm{pd}} = \left(\begin{smallmatrix} \{0\} & \{0,1\} \\ \{1\} & \varnothing \end{smallmatrix}\right)$ *together with*

$$\delta_{T_{\mathrm{pd}}} = \begin{pmatrix} \delta_0 & \delta_0 + \delta_1 \\ \delta_1 & 0 \end{pmatrix} \quad \textit{and} \quad B_{\mathrm{pd}}(t) = \begin{pmatrix} 1 & 1 + e^{2\pi i t} \\ e^{2\pi i t} & 0 \end{pmatrix}.$$

Lyapunov exponents and Fourier cocycles

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Due to the ergodicity of the transformation $t \mapsto Lt \mod 1$ relative to Lebesgue measure, Oseledec's multiplicative ergodic theorem ensures the existence of the Lyapunov exponents and the corresponding subspaces in which they represent the asymptotic exponential growth rate of the vector norms.

$$\{0\} =: \mathcal{V}_0 \subsetneq \mathcal{V}_1 \subsetneq \mathcal{V}_2 := \mathbb{C}^2$$

A vector v from the Oseledec subspace $\mathcal{V}_{i+1} \backslash \mathcal{V}_i$ satisfies the property that for almost every $t \in \mathbb{R}$, the norm $\|vB^{(n)}(t)\|$ grows like $e^{n\chi_{i+1}^B}$ as $n \to \infty$.

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- If $v \in \mathbb{C}^2$ is any (fixed) row vector, the values
$$\chi^B(v,t) := \lim_{n\to\infty} \frac{1}{n} \log \|vB^{(n)}(t)\|$$
exist for almost every $t \in \mathbb{R}$ and are constant on a set of full measure.
- For invertible cocycles, these exponents have v-independent forms
$$\chi^B_{\max}(t) := \lim_{n\to\infty} \frac{1}{n} \log \|B^{(n)}(t)\| \quad \text{and} \quad \chi^B_{\min}(t) := -\lim_{n\to\infty} \frac{1}{n} \log \|(B^{(n)}(t))^{-1}\|.$$
- Lyapunov regularity guarantees for almost every $t \in \mathbb{R}$ that
$$\chi^B_{\min}(t) + \chi^B_{\max}(t) = \lim_{n\to\infty} \frac{1}{n} \log |\det B^{(n)}(t)|.$$

Theorem (Baake, C. and Mañibo 2018)

For any primitive binary automatic sequence ϱ, the extremal Lyapunov exponents are explicitly given by

$$\chi^{B}_{\min} = 0 \quad \textit{and} \quad \chi^{B}_{\max} = \mathfrak{m}(Q - R).$$

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Important Point

Using Jenson's formula, we have

$$\lim_{n\to\infty} \frac{1}{n} \log|\det B^{(n)}(t)| = \lim_{n\to\infty} \frac{1}{n} \log|\det(\underbrace{B(t)B(Lt)\cdots B(L^{n-1}t)}_{\textit{matrix cocycle}})| = \mathfrak{m}(Q - R).$$

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Lemma

Let ϱ be a substitution as specified in Eq. (1)*. Consider the sets*

$$P_a := \left\{ m \mid \mathcal{C}_m = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \right\} \quad \textit{and} \quad P_b := \left\{ m \mid \mathcal{C}_m = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \right\},$$

which collect bijective positions of the same type. Further, let $z = e^{2\pi i t}$ and set

$$Q(z) := \overline{\widehat{\delta_{P_a}}(t)} \quad \textit{and} \quad R(z) := \overline{\widehat{\delta_{P_b}}(t)}.$$

Then, $\det B(t) = p_L(z) \cdot \big(Q - R\big)(z)$*, where* $p_L(z) = 1 + z + \cdots + z^{L-1}$.

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Example

Consider the Thue–Morse substitution, as given by

$$\varrho_{\mathrm{TM}} : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10. \end{cases}$$

Here $(Q - R)(z) = 1 - z$*, so that*

$$\chi^B_{\min} = \chi^B_{\max} = \mathfrak{m}(1 - z) = 0.$$

Example (Reverse direction)

Recall that

$$\ell(z) = 1 + z - z^3 - z^4 - z^5 - z^6 - z^7 + z^9 + z^{10}$$

is the polynomial with the smallest known positive logarithmic Mahler measure, $\mathfrak{m}(\ell) \approx \log(1.176281)$. *Here*

$$\varrho_\ell : \begin{cases} 0 \mapsto 00111111000 \\ 1 \mapsto 11100000011 \end{cases}$$

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Question (Automatic Lehmer)
Does there exist a constant $c > 0$ *such that, for any primitive binary automatic sequence with* $\chi^B_{\max} \neq 0$, *we have* $\chi^B_{\max} \geqslant c$ *?*

Thank you (Merci)