Mahler's method in several variables

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$$
\begin{array}{r}
1\\
10\\
100\\
1000\\
10000\\
100000\\
1000000\\
10000000\\
\vdots\\
100\,000\,000\,000\,000\,000\,000\\
1\,000\,000\,000\,000\,000\,000\,000\\
\vdots
\end{array}
$$

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1
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1000
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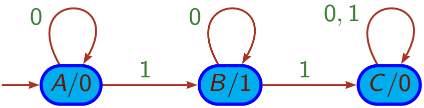
⋮
100 000 000 000 000 000 000
1 000 000 000 000 000 000 000
⋮

The set of powers of 2 is *recognizable by a finite automaton*: it is 2-automatic.



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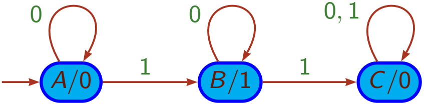

A set $\mathcal{E} \subset \mathbb{N}$ is q-automatic if there exists a finite automaton that takes as input the base-q expansion of n and outputs 1 if n belongs to $\mathcal{E}$ and 0 otherwise.

Though it is obvious to determine whether a binary natural number is a power of 2, it seems more difficult to identify this property from its decimal expansion.

1	2
10	4
100	8
1000	16
10000	32
100000	64
1000000	128
10000000	256
⋮	⋮
100 000 000 000 000 000 000	2 097 152
1 000 000 000 000 000 000 000	4 194 304
⋮	⋮

A first base change problem

In 1969, Cobham proved the following fundamental result.

Cobham's theorem. Let q_1 and q_2 be two multiplicatively independent natural numbers. A set $\mathcal{E} \subset \mathbb{N}$ is both q_1- and q_2-automatic if and only if it is a finite union of arithmetic progressions.

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In more algebraic terms, we expect that Cobham's theorem can be strengthened as follows.

Problem 1. Let $r \geq 1$ be an integer. Let $q_1, \ldots, q_r$ be pairwise multiplicatively independent natural numbers, and, for every i, $1 \leq i \leq r$, let $\mathcal{E}_i \subset \mathbb{N}$ be a q_i-automatic set that is not a finite union of arithmetic progressions.

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$$f_1(z) = \sum_{n \in \mathcal{E}_1} z^n, \ldots, f_r(z) = \sum_{n \in \mathcal{E}_r} z^n$$

are algebraically independent over $\overline{\mathbb{Q}}(z)$.

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A real number ξ is said to be *automatic* in base b if its base-b expansion can be generated by a finite automaton. This means that there exists a finite automaton that takes as input the expansion of n in some base and produces as output the nth digit of ξ.

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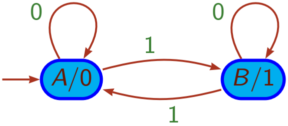
Example. The Thue–Morse number

$$\langle \tau \rangle_2 = 0.011\,010\,011\,001\,011\,010\,010\,110\,011\,010\,011\,001\,011 \cdots ,$$

is automatic in base 2.



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Theorem AB. The base-b expansion of an irrational algebraic number cannot be generated by a finite automaton.

A second base change problem

Again, though the Thue-Morse number τ has a simple binary expansion, its decimal expansion

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Problem 3. Let $r \geq 1$ be an integer. Let $b_1, \ldots, b_r$ be pairwise multiplicatively independent natural numbers, and, for every i, $1 \leq i \leq r$, let ξ_i be an irrational real number that is automatic in base b_i. Prove that the numbers $\xi_1, \ldots, \xi_r$ are algebraically independent over $\overline{\mathbb{Q}}$.

Let $q \geq 2$ be an integer. We say that $f(z) \in \overline{\mathbb{Q}}\{z\}$ is a *q-Mahler function* if there exist $p_0(z), \ldots, p_d(z) \in \overline{\mathbb{Q}}[z]$, not all zero, such that

$$p_0(z)f(z) + p_1(z)f(z^q) + \cdots + p_d(z)f(z^{q^d}) = 0\,.$$

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- If a sequence $\mathbf{a} = (a_n)$ with values in $\overline{\mathbb{Q}}$ is q-automatic, then the generating function

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Our main problems can thus be restated and extended as problems concerning the algebraic relations over $\overline{\mathbb{Q}}$ between the values of Mahler functions at algebraic points.

This leads us to study *linear Mahler systems*, that is systems of the form:

$$\begin{pmatrix} f_1(z^q) \\ \vdots \\ f_m(z^q) \end{pmatrix} = A(z) \begin{pmatrix} f_1(z) \\ \vdots \\ f_m(z) \end{pmatrix}, \tag{M}$$

where $A(z)$ belongs to $\mathrm{GL}_n(\overline{\mathbb{Q}}(z))$ and $f_i(z) \in \overline{\mathbb{Q}}\{z\}$.

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In 1990, Ku. Nishioka proved that, if $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$, is a regular point, then

$$\mathrm{degtr}_{\overline{\mathbb{Q}}}(f_1(\alpha), \ldots, f_m(\alpha)) = \mathrm{degtr}_{\overline{\mathbb{Q}}(z)}(f_1(z), \ldots, f_m(z)).$$

More recently, Philippon proved the following important refinement.

Theorem. Let $(f_1(z), \ldots, f_m(z)) \in \overline{\mathbb{Q}}\{z\}^m$ be a solution to (M). Let $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$, be a regular point. Then for all homogeneous $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_m]$ such that $P(f_1(\alpha), \ldots, f_m(\alpha)) = 0$,

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- Similar results were first obtained by Shidllovskii, Nesterenko and Shidlovskii, Beukers, and André in the framework of Siegel E-functions.

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Unfortunately, these results are of no help to solve our three problems!

Let $T = (t_{i,j})_{1 \leq i,j \leq n}$ be an $n \times n$ matrix with non-negative integer coefficients. We let T act on $\mathbb{C}^n$ by

$$T\boldsymbol{\alpha} = \left(\alpha_1^{t_{1,1}} \cdots \alpha_n^{t_{1,n}}, \ldots, \alpha_1^{t_{n,1}} \cdots \alpha_n^{t_{n,n}}\right).$$

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A *linear T-Mahler system* is a system of functional equations of the form

$$\begin{pmatrix} f_1(T\boldsymbol{z}) \\ \vdots \\ f_m(T\boldsymbol{z}) \end{pmatrix} = A(\boldsymbol{z}) \begin{pmatrix} f_1(\boldsymbol{z}) \\ \vdots \\ f_m(\boldsymbol{z}) \end{pmatrix}, \qquad \text{(M)}$$

where $A(\boldsymbol{z}) \in \mathrm{GL}_m(\overline{\mathbb{Q}}(\boldsymbol{z}))$ and $f_i(\boldsymbol{z}) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}$.

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In 1982, Loxton and van der Poorten claimed that, if the matrix $A(\mathbf{0})$ is well-defined and non-singular, and if T and $\boldsymbol{\alpha} \in \overline{\mathbb{Q}}^n$ satisfy some conditions, then

$$\mathrm{degtr}_{\overline{\mathbb{Q}}}(f_1(\boldsymbol{\alpha}), \ldots, f_m(\boldsymbol{\alpha})) = \mathrm{degtr}_{\overline{\mathbb{Q}}(\boldsymbol{z})}(f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})).$$

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• In fact, this has only been proved for *almost constant* matrices and *almost diagonal* matrices by Kubota and Ku. Nishioka.

The magic trick

The function $\mathfrak{f}(z) = \sum_{n=0}^{\infty} z^{2^n}$ satisfies the simple equation

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$$\begin{pmatrix} 1 \\ \mathfrak{f}(z_1^2) \\ \mathfrak{f}(z_2^2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -z_1 & 1 & 0 \\ -z_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \mathfrak{f}(z_1) \\ \mathfrak{f}(z_2) \end{pmatrix}.$$

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- The trick works because the point $\alpha = (1/2, 1/3)$ has sufficiently independent coordinates.

Our first main result is the following.

Theorem AF1. Let $(f_1(\boldsymbol{z}),\dots,f_m(\boldsymbol{z}))\in\overline{\mathbb{Q}}\{\boldsymbol{z}\}^m$ be a solution to a regular singular T-Mahler system. If $\boldsymbol{\alpha}\in\overline{\mathbb{Q}}^n$ is regular and that the pair $(T,\boldsymbol{\alpha})$ is admissible, then for all homogeneous $P\in\overline{\mathbb{Q}}[X_1,\dots,X_m]$ such that $P(f_1(\boldsymbol{\alpha}),\dots,f_m(\boldsymbol{\alpha}))=0$, there exists $Q\in\overline{\mathbb{Q}}[\boldsymbol{z},X_1,\dots,X_m]$, homogeneous in $X_1,\dots,X_n$, such that

$$Q(\boldsymbol{z},f_1(\boldsymbol{z}),\dots,f_m(\boldsymbol{z}))=0$$

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Theorem AF1. Let $(f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}^m$ be a solution to a regular singular T-Mahler system. If $\boldsymbol{\alpha} \in \overline{\mathbb{Q}}^n$ is regular and that the pair $(T, \boldsymbol{\alpha})$ is admissible, then for all homogeneous $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_m]$ such that $P(f_1(\boldsymbol{\alpha}), \ldots, f_m(\boldsymbol{\alpha})) = 0$, there exists $Q \in \overline{\mathbb{Q}}[\boldsymbol{z}, X_1, \ldots, X_m]$, homogeneous in $X_1, \ldots, X_n$, such that

$$Q(\boldsymbol{z}, f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})) = 0$$

and

$$Q(\boldsymbol{\alpha}, X_1, \ldots, X_m) = P(X_1, \ldots, X_m)\,.$$

Regular singular systems. There exists a gauge transform $\Phi(\boldsymbol{z}) \in \mathrm{GL}_m(\widehat{\mathbf{K}})$ such that $\Phi(T\boldsymbol{z})A(\boldsymbol{z})\Phi^{-1}(\boldsymbol{z}) \in \mathrm{GL}_m(\overline{\mathbb{Q}})$, where we let $\widehat{\mathbf{K}}$ denote the field of ramified generalized Laurent series.

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Remark. If the matrix $A(\mathbf{0})$ is well-defined and non-singular then the corresponding system is regular singular.

Pure algebraic relations

Given a finite set of complex numbers $\mathcal{E} := \{\zeta_1, \ldots, \zeta_m\}$, we let

$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}) := \left\{P(X_1, \ldots, X_m) \in \overline{\mathbb{Q}}[X_1, \ldots, X_m] : P(\zeta_1, \ldots, \zeta_m) = 0\right\}$$

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Now, let us consider several sets of complex numbers

$$\mathcal{E}_1 = \{\zeta_{1,1}, \ldots, \zeta_{1,m_1}\}, \ldots, \mathcal{E}_r = \{\zeta_{r,1}, \ldots, \zeta_{r,m_r}\},$$

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The elements of $\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E})$ are called the *pure algebraic relations* (with respect to $\mathcal{E}_i$).

The purity theorem

Let $r \geq 2$. For every integer i, $1 \leq i \leq r$, we consider a regular singular Mahler system

$$\begin{pmatrix} f_{i,1}(T_i \boldsymbol{z}_i) \\ \vdots \\ f_{i,m_i}(T_i \boldsymbol{z}_i) \end{pmatrix} = A_i(\boldsymbol{z}_i) \begin{pmatrix} f_{i,1}(\boldsymbol{z}_i) \\ \vdots \\ f_{i,m_i}(\boldsymbol{z}_i) \end{pmatrix}$$

and a point $\boldsymbol{\alpha}_i \in (\overline{\mathbb{Q}}^\star)^{n_i}$ where the lifting theorem applies.

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Theorem AF2. For every i, $1 \leq i \leq r$, let us consider

$$\mathcal{E}_i \subseteq \{f_{i,1}(\boldsymbol{\alpha}_i), \ldots, f_{i,m_i}(\boldsymbol{\alpha}_i)\}$$

and set $\mathcal{E} := \cup_{i=1}^r \mathcal{E}_i$. Then

$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{i=1}^r \mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E}) \,.$$

A conjecture on q-Mahler functions

Let $r \geq 2$ be an integer. For every integer i, $1 \leq i \leq r$, we let $q_i \geq 2$ be an integer, $f_i(z) \in \overline{\mathbb{Q}}\{z\}$ be a q_i-Mahler function, and $\alpha_i \in \overline{\mathbb{Q}}$ be such that $f_i(z)$ is well-defined at α_i.

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Main conjecture. The following properties hold.

(i) Let us assume that $\alpha_1, \ldots, \alpha_r$ are multiplicatively independent. Then the numbers $f_1(\alpha_1), f_2(\alpha_2), \ldots, f_r(\alpha_r)$ are algebraically independent over $\overline{\mathbb{Q}}$ if and only if they are all transcendental.

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(ii) Let us assume that $q_1, \ldots, q_r$ are pairwise multiplicatively independent. Then the numbers $f_1(\alpha_1), f_2(\alpha_2), \ldots, f_r(\alpha_r)$ are algebraically independent over $\overline{\mathbb{Q}}$ if and only if they are all transcendental.

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Remark. A solution to this conjecture would solve our three problems.

Combining the lifting theorem and the purity theorem, we can make significant progress towards the conjecture.

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Remark. If

$$p_0(z)f(z) + p_1(z)f(z^q) + \cdots + p_d(z)f_i(z^{q^d}) = 0\,,$$

with $p_0(0)p_d(0) \neq 0$, then $f(z)$ is a regular singular Mahler function.

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Unfortunately, generating functions of automatic sequences are not always regular singular.