Data Assimilation: A Deterministic Vision – Theory and Applications – Asymptotic observers

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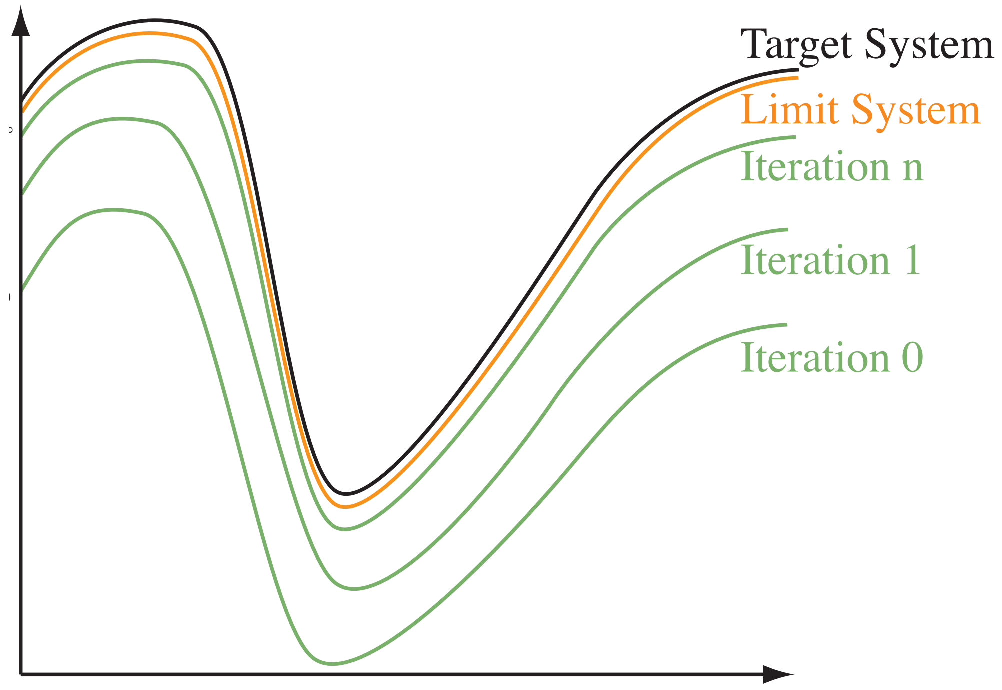
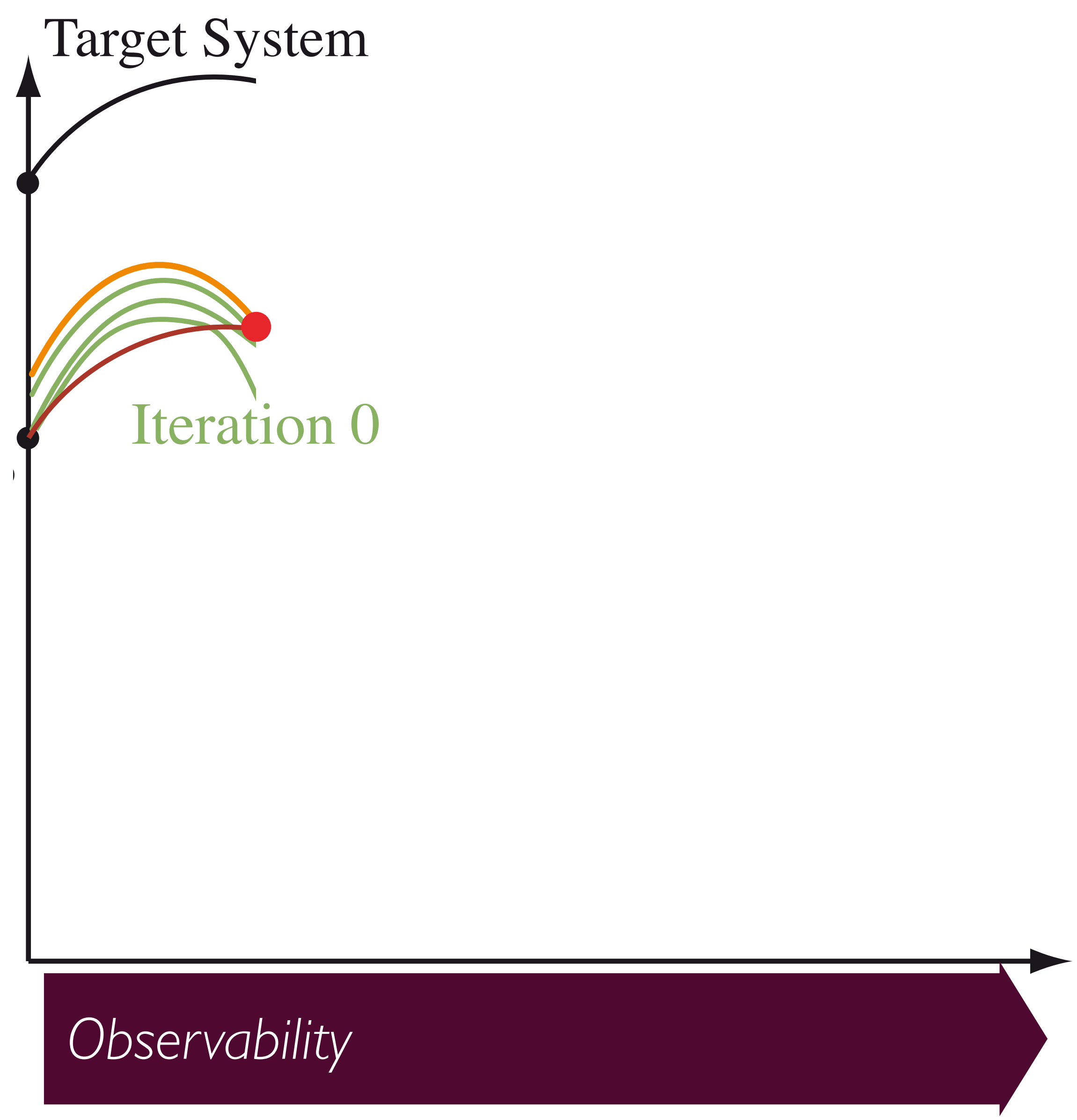
Outline

Asymptotic observer - motivations and definition

Summary


Target System
Limit System
Iteration n
Iteration 1
Iteration 0

Summary



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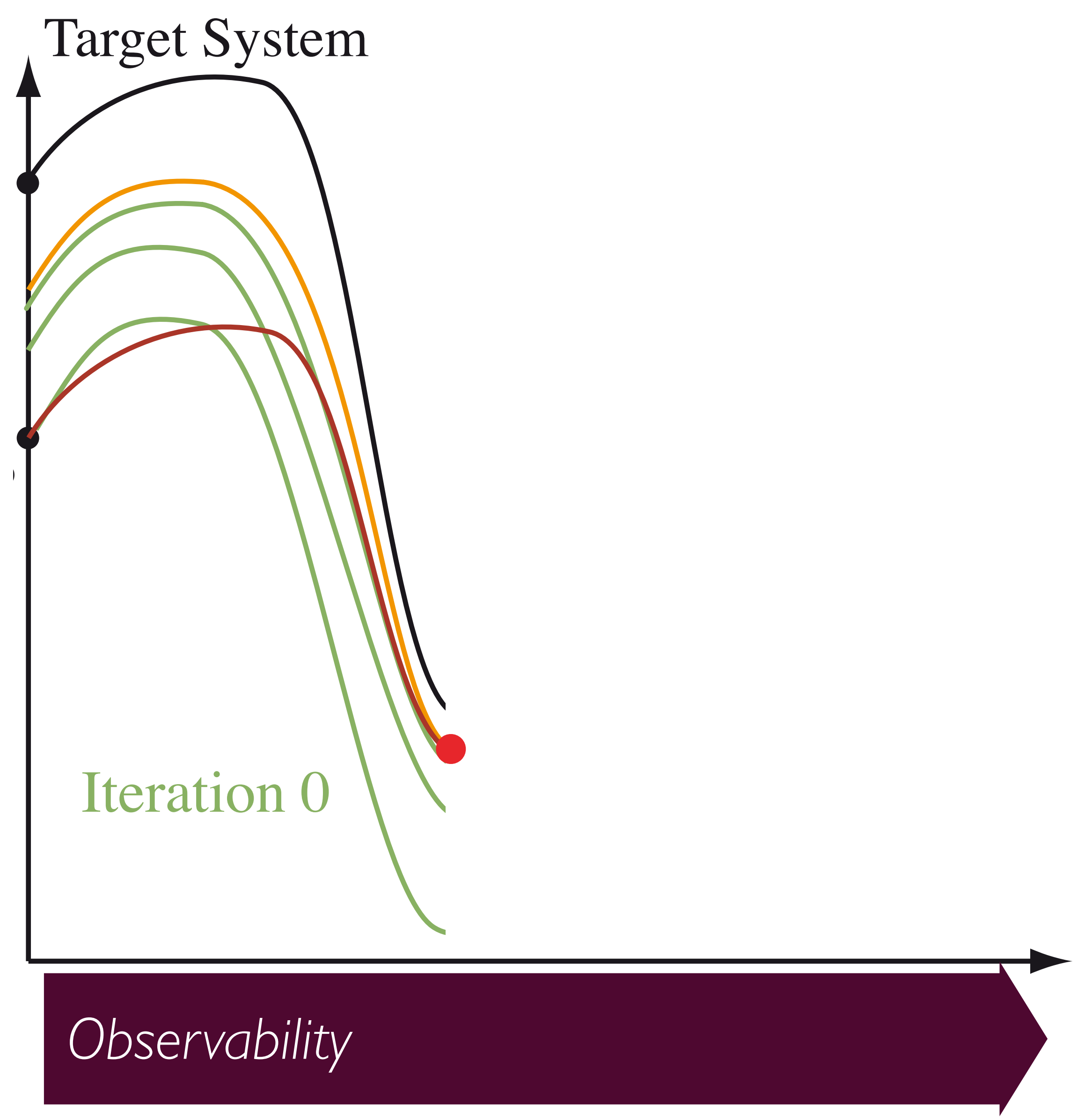

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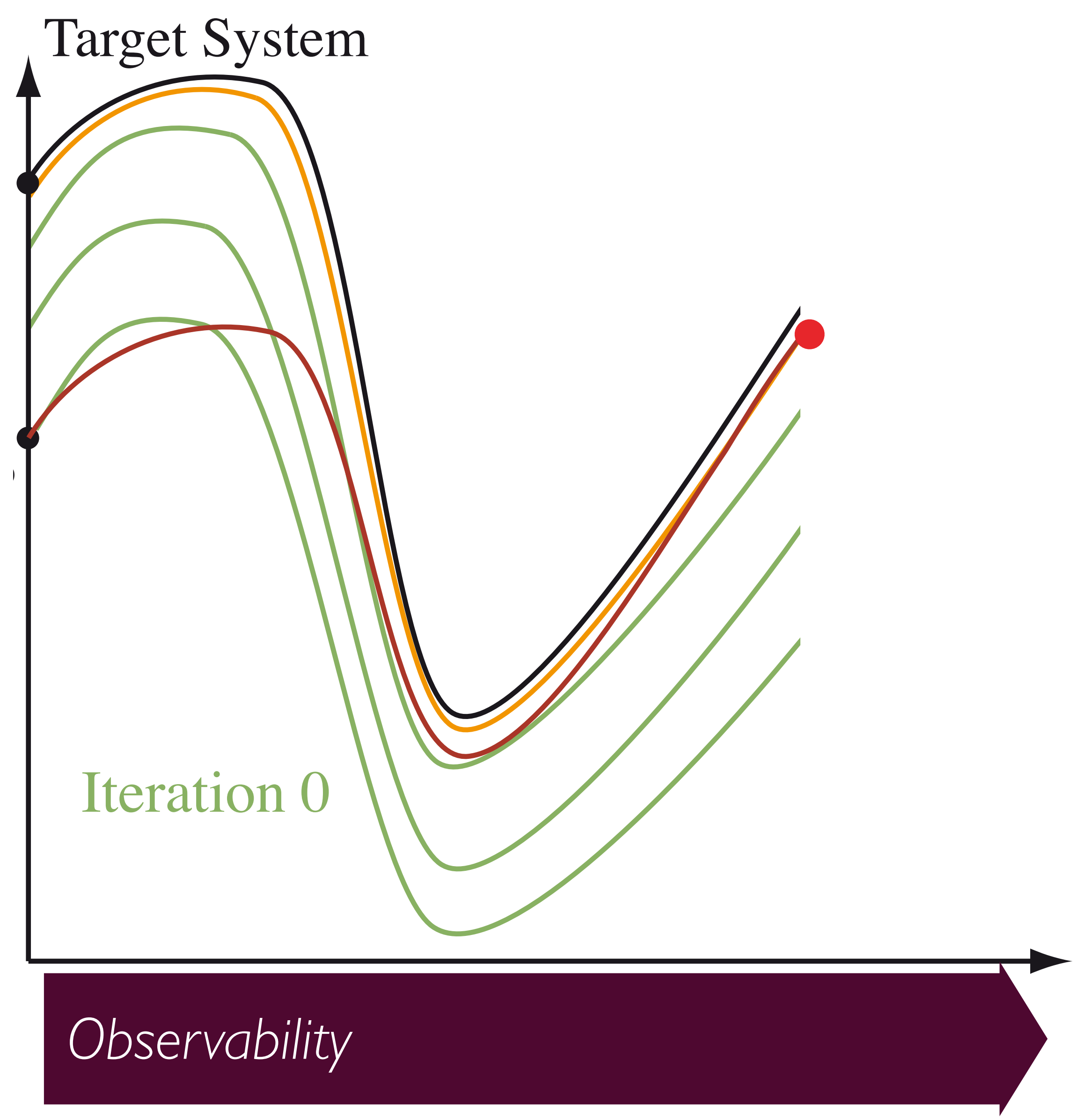

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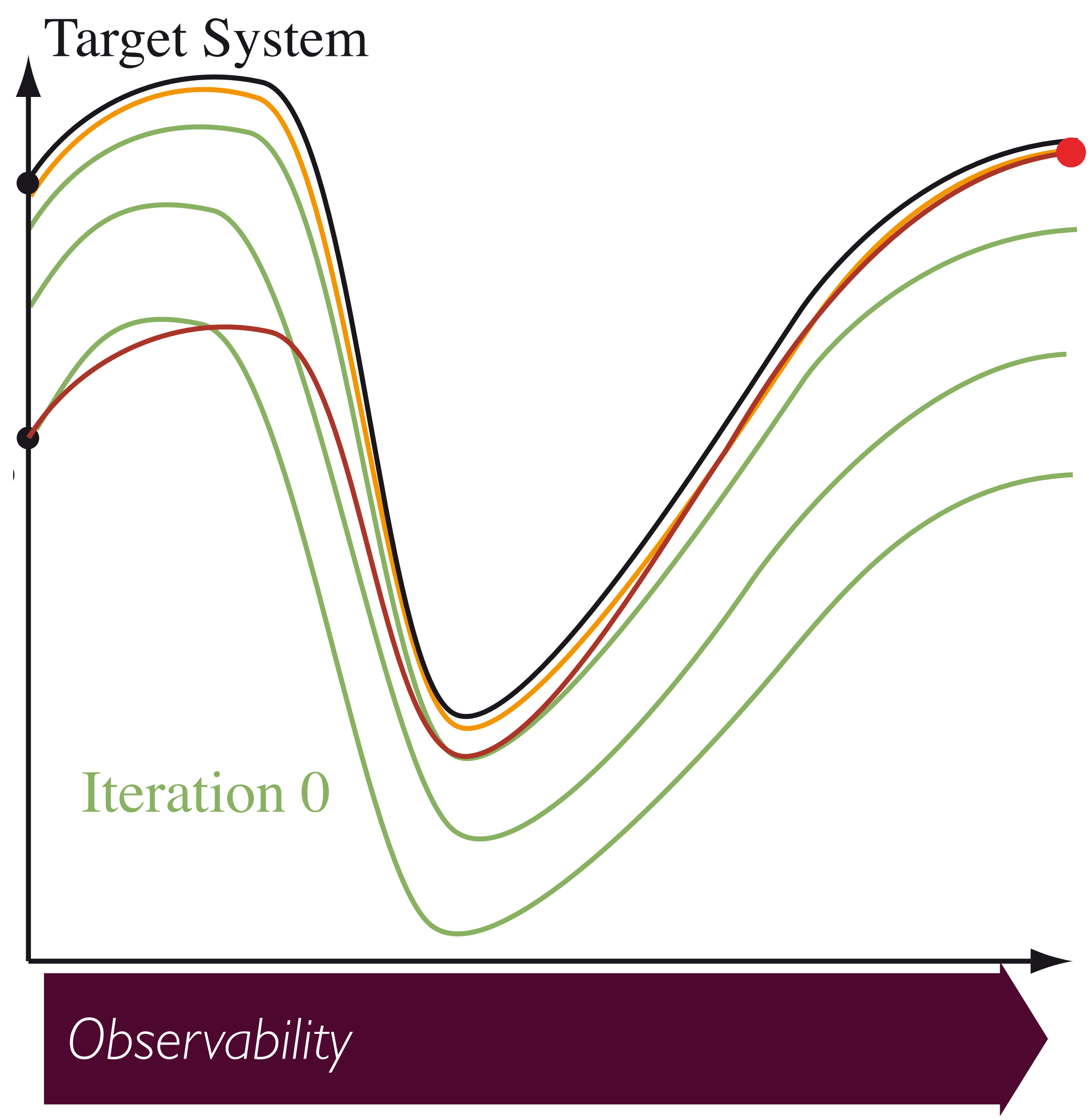

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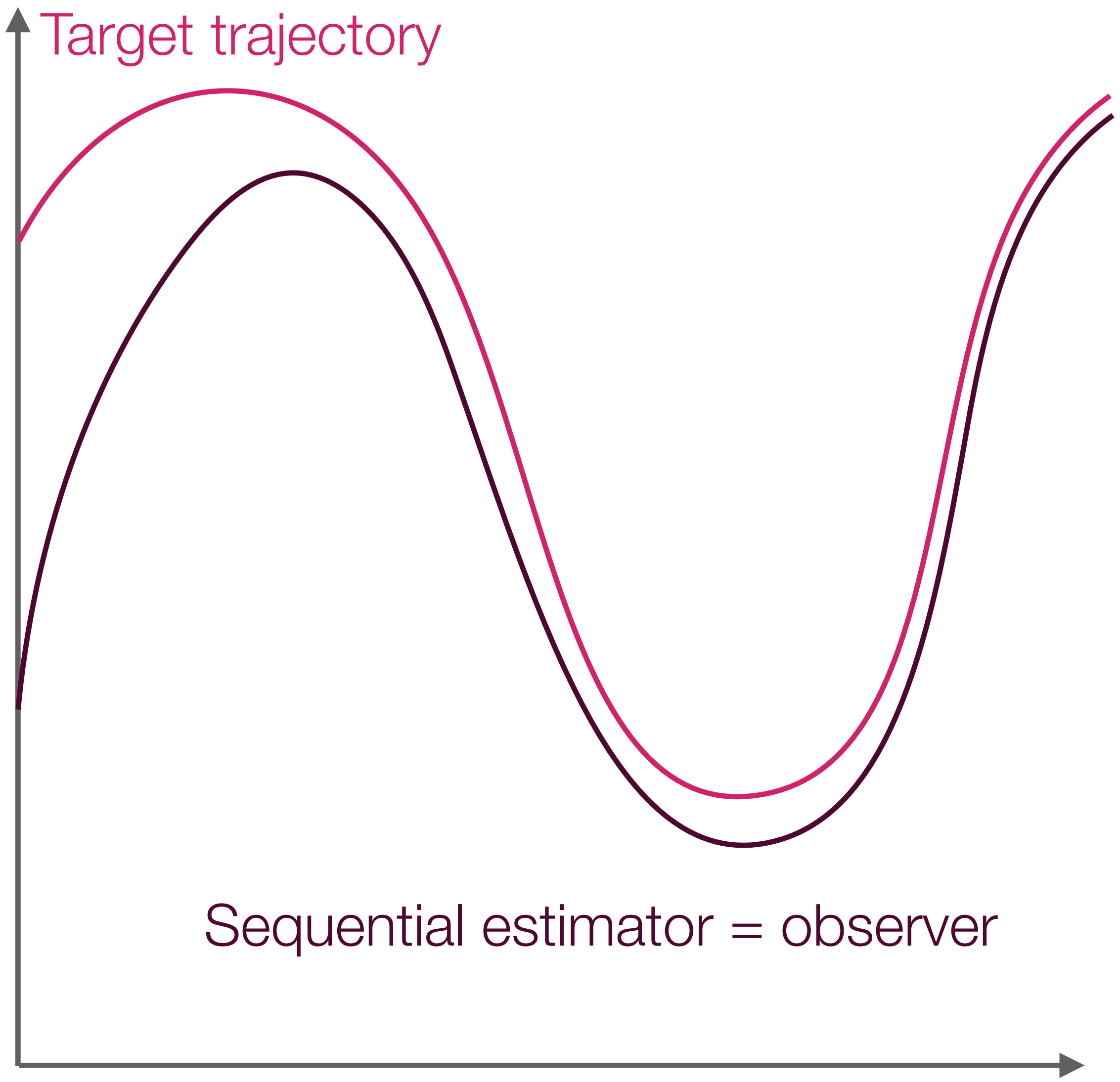
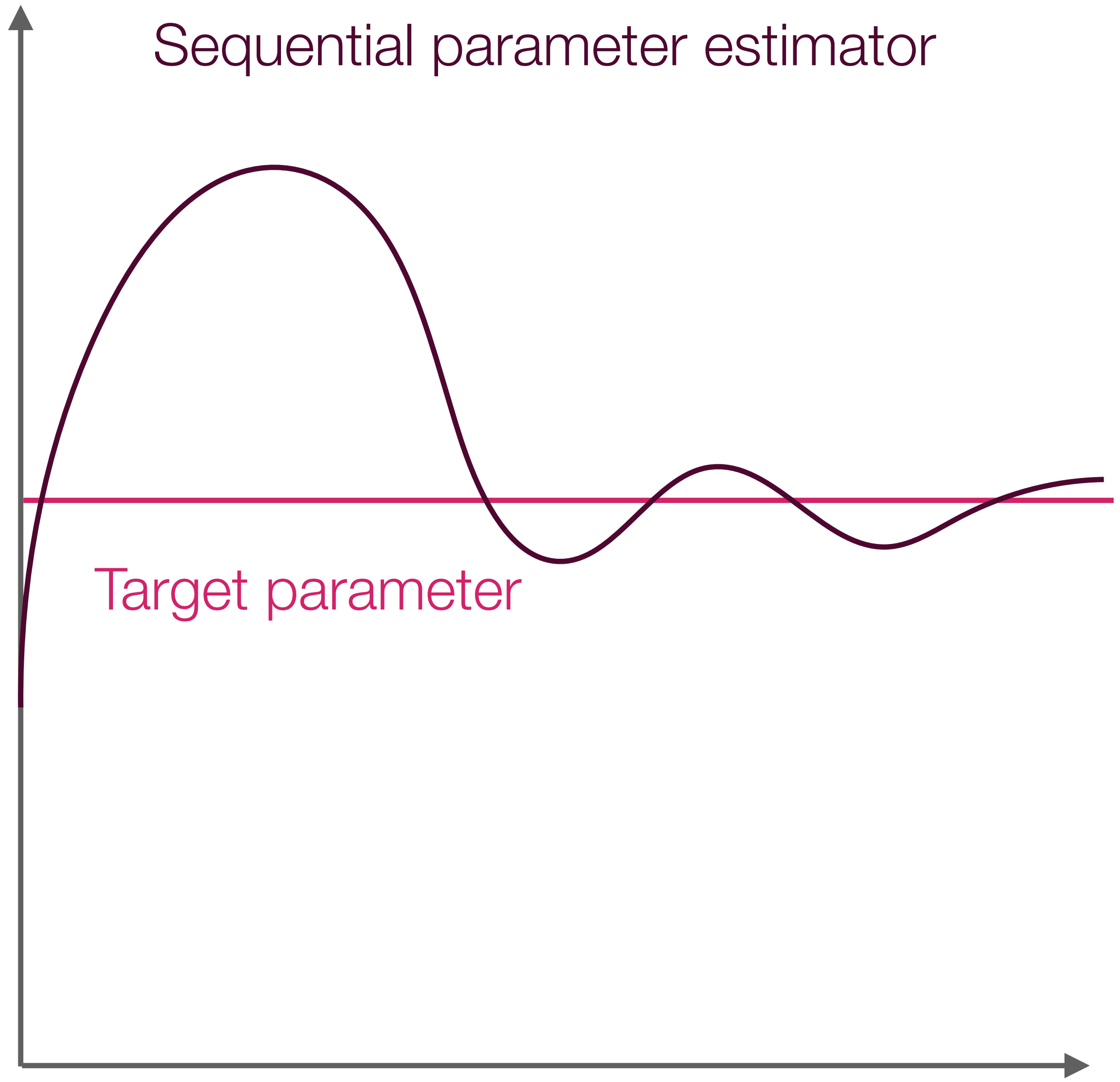

Target trajectory
Sequential estimator = observer


Sequential parameter estimator
Target parameter

Asymptotic observer definition (Krener, 1998)

Definition

An observer is a causal functional $\mathbb{R}^+ \ni t \mapsto \hat{y}(y_\diamond, z^\delta_{|[0,t]})$, s.t. $\forall 0 \le r \le s \le t$,

- For any $\epsilon > 0$, there exist α, β δ K^∞ functions[1] such that

  $$\|\check{y}(r)-\hat{y}(r)\|_\mathcal{Y} \le \alpha(\epsilon), \|\check{\nu}\|_{\mathcal{U}_{[r,s]}} \le \beta(\epsilon), \|\check{\eta}\|_{\mathcal{Z}_{[r,s]}} \le \delta(\epsilon) \Rightarrow \|\check{y}(t)-\hat{y}(t)\| \le \epsilon$$

- For any $\epsilon, \alpha, \beta, \delta > 0$, there exists $\tau(\epsilon, \alpha, \beta, \delta)$ such that if $t - s \ge \tau$ then

  $$\|\check{y}(r) - \hat{y}(r)\|_\mathcal{Y} \le \alpha, \|\check{\nu}\|_{\mathcal{U}_{[r,s]}} \le \beta, \|\check{\eta}\|_{\mathcal{Z}_{[r,s]}} \le \delta \Rightarrow \|\check{y}(t) - \hat{y}(t)\| \le \epsilon$$

[1] a continuous strictly increasing function f on $\mathbb{R}^+$ such that $f(0) = 0$ and $\lim_{+\infty} f = +\infty$

Asymptotic observer definition interpretation

Definition

An observer is a causal functional

- The state estimation is uniformly continuous with respect to the uncertainties

$$\|\check{\zeta}\|_{\mathcal{Y}},\ \|\check{\nu}\|_{L^2([0,T];\mathcal{U})},\ \|\check{\eta}\|_{L^2([0,T];\mathcal{Z})}$$

- Asymptotic stability of the noise free estimator, i.e. without noise we have

$$\hat{y}(s) = \check{y}(s) \Rightarrow \hat{y}(t) = \check{y}(t), t \geq s$$

The error system

We define $\tilde{y} = \check{y} - \hat{y}$

- In a linear context

$$
\begin{array}{rccccccl}
 & \dot{\check{y}} & = & A\check{y} & & & + & B\check{\nu} \\
- & \dot{\hat{y}} & = & A\hat{y} & + & G(z^\delta - C\hat{y}) & & \text{with } z^\delta = C\check{y} + \eta \\
\hline
 & \dot{\tilde{y}} & = & (A - GC)\tilde{y} & + & G\eta & + & B\check{\nu}
\end{array}
$$

  So the question is can we build G so that the error is (exponentially) stable to 0

- In a non-linear context: the error dynamics is no more autonomous
- Sufficient condition: Linearization around the target trajectory: $\mathrm{d}A(\check{y}),\ \mathrm{d}C(\check{y})$ to find a linearized error and proved its (exponential) stability

Detectability and Observability

Definition (Observability)

A system is observable if

$$\check{z}_1 \equiv \check{z}_2 \Rightarrow \check{y}_{|\zeta_1,0} \equiv \check{y}_{|\zeta_2,0}.$$

It means that $\operatorname{Ker} \Psi_T = \{0\}$ which in infinite dimension can be associated with the approximate observability (Tucsnak and Weiss, 2009).

Definition (Detectability)

A system is detectable if

$$\check{z}_1 \equiv \check{z}_2 \Rightarrow \lim_{t \to +\infty} \|\check{y}_{|\zeta_1,0} - \check{y}_{|\zeta_2,0}\|_{\mathcal{Y}} = 0.$$

Observability in a Linear Context

- Grammian of observability

$$\Upsilon_T = \int_0^T \Phi(t,0)^* C^* C \Phi(t,0)\,\mathrm{d}t$$

Observability $\Leftrightarrow$ Grammian coercivity. Remember $(y, \Upsilon_T y) = \|\Psi_T y\|^2$.

- For autonomous system in finite dimension ($\dim = \mathrm{N}_y$), it is equivalent to the Kalman Criterion

$$rank \begin{pmatrix} C & CA & \cdots & CA^n \end{pmatrix}^\intercal = \mathrm{N}_y$$

- The Hautus test: Observable $\Leftrightarrow$ for all modes φ, $C\varphi = 0 \to \varphi = 0$

in dim ∞ as a system may not have a complete basis of eigenmodes.

Detectability in a Linear Context

Definition (Detectability)

The pair (A, C) is (exponentially) detectable if there exists G such that $A - GC$ is (exponentially) stable

Interpretation in finite dimension: The modes that are not observable are already stable. But again ☡ in dim ∞ as a system may not have a complete basis of eigenmodes.

Definition (Stabilizability)

The pair (A, B) is (exponentially) stabilizable if there exists G such that $A - BG$ is (exponentially) stable

The kalman estimator as an asymptotic observer: the finite dimension linear case

- The estimation error: $\tilde{y} = \hat{y} - \check{y}$
- A Liapunov functional: $\tilde{\mathscr{V}}(y, t) = \frac{1}{2}(y, \Pi^{-1} y)_{\mathcal{Y}}$
- A Liapunov result $\frac{\mathrm{d}}{\mathrm{dt}}\tilde{\mathscr{V}}(\tilde{y}(t), t) \leq -c_{\mathrm{st}}\|y\|^2_{\mathcal{Y}}$

Theorem

Let $\tilde{y} = 0$ be an equilibrium point and $D \subset \mathcal{Y}$ be a domain containing $x = 0$. Let $\tilde{\mathscr{V}}$ be a continuously differentiable on $D \times \mathbb{R}^+$ function such that $W_1 \leq \tilde{\mathscr{V}}(\tilde{x}, t) \leq W_2(x)$ and $\frac{\mathrm{d}}{\mathrm{dt}}\tilde{\mathscr{V}} + (\nabla_y \tilde{\mathscr{V}}, (Ay))_{\mathcal{Y}} \leq -W_3(x)$ where $W_1(X)$, $W_2(x)$, and $W_3(x)$ are continuous positive definite functions on D. Then, $y = 0$ is uniformly asymptotically stable.

- Exists also for discrete-time $\tilde{\mathscr{V}}_n(y) = \frac{1}{2}(y, \Pi_n^{-1} y)_{\mathcal{Y}}$

The optimal estimator as an asymptotic observer: other cases

- Managing the robustness with respect to errors (Krener, 1998)
- The non-linear case :

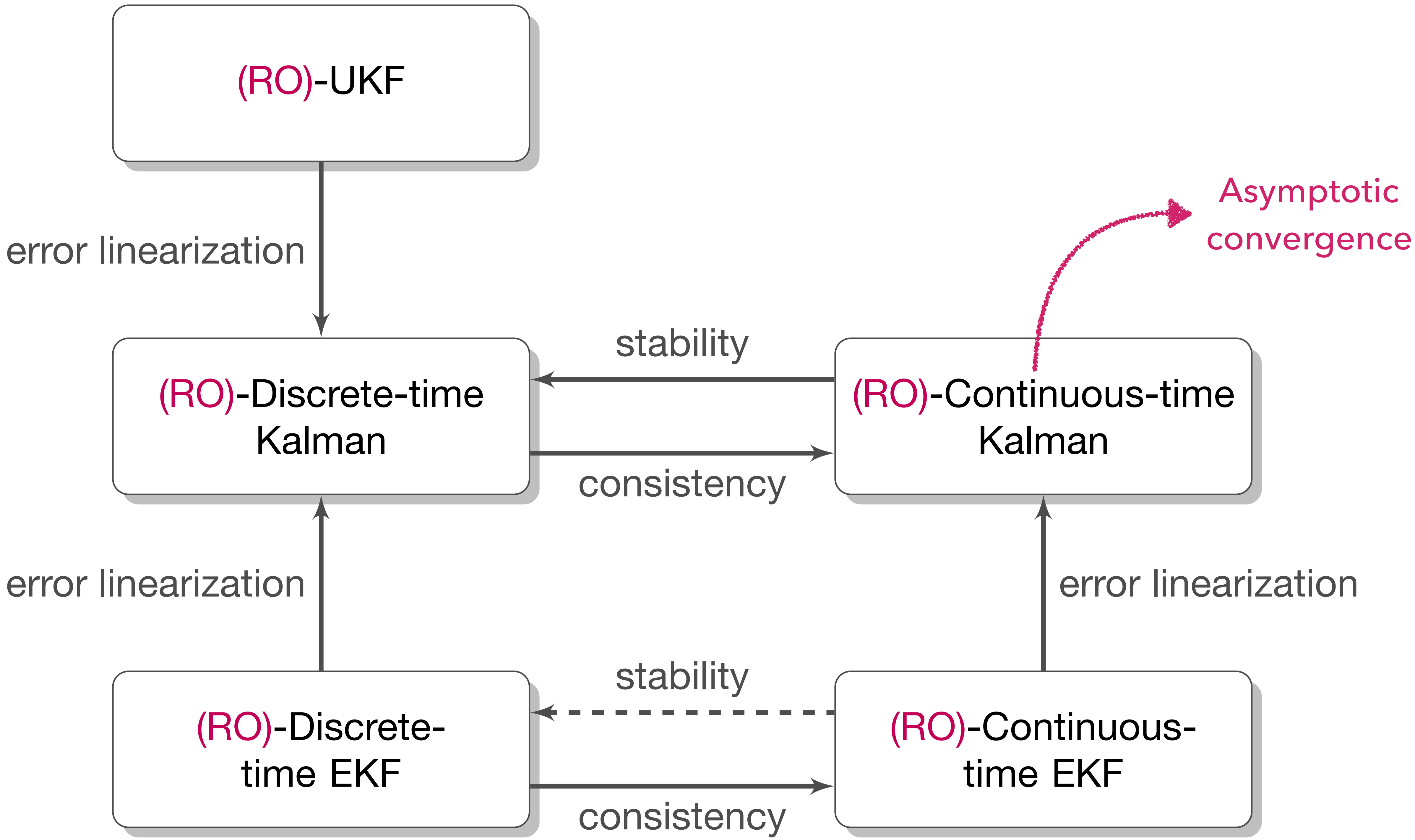
$$\forall y \in \mathcal{Y}, \quad \tilde{\mathscr{V}}(y,t) = \mathscr{V}(y+\hat{y}(t),t) - \mathscr{V}(\hat{y}(t),t),$$

which satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\tilde{\mathscr{V}}(\tilde{y}(t),t) &= \partial_t \mathscr{V}(\check{y}(t),t) + \nabla\mathscr{V}(\check{y}(t),t)^\intercal \dot{\check{y}}(t) - \partial_t \mathscr{V}(\hat{y}(t),t) - \nabla\mathscr{V}(\hat{y}(t),t)^\intercal \dot{\hat{y}}(t)\\ &= -\frac{1}{2}\nabla\mathscr{V}(\check{y}(t),t)^\intercal BQB^\intercal \nabla\mathscr{V}(\check{y}(t),t) + \underbrace{\frac{1}{2}\|J(\check{y},t)\|_R^2}_{=0}\\ &\quad + \underbrace{\frac{1}{2}\nabla\mathscr{V}(\hat{y}(t),t)^\intercal BQB^\intercal \nabla\mathscr{V}(\hat{y}(t),t)}_{=0} - \frac{1}{2}\|J(\hat{y}(t),t)\|_R^2 \leq 0.\end{aligned}$$

- EKF convergence: continuous (Krener, 2003) or discrete (Boutayeb et al., 1997)
- In dim ∞ no dynamics for Π^{-1} in general but possible alternatives

Summary



(RO)-UKF
error linearization
Asymptotic
convergence
stability
(RO)-Discrete-time
Kalman
(RO)-Continuous-time
Kalman
consistency
error linearization
error linearization
stability
(RO)-Discrete-
time EKF
(RO)-Continuous-
time EKF
consistency

Asymptotic Riccati

- In finite dimension

Theorem

The steady-state Riccati equation $A\Pi + \Pi A^* - \Pi C^* R C \Pi + BQB^* = 0$ *admits one a only one solution in* $S(\mathcal{Y})^+$ *if (1)* (A, C) *is detectable and (2)* (A, B) *is stabilizable.*

- Generalizes in dim ∞ (Bensoussan et al., 2007)

Pole Shifting in finite dimension

Theorem (The pole shifting theorem)

Let (A, C) satisfying the Kalman condition, then there exists G such that for all monic polynomial μ – i.e. with a nonzero coefficient of highest degree equal to 1 – of degree N_y,

$$\det(\lambda \mathbb{1} - A - GC) = \mu(\lambda).$$

- In practice (and proof) (Kraus and Kučera, 1999)
- Various cases to be considered + iterative design: Relocation of (1) real simple eigenvalue, (2)real multiple eigenvalues, (3) Complex conjugate pair of eigenvalues

Pole Shifting example

$$A = \begin{bmatrix} v_1 \Big| & \cdots & \Big| v_n \end{bmatrix} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{bmatrix} w_1 \\ \hline \vdots \\ \hline w_n \end{bmatrix} \text{ with } w_j^\intercal v_j = 0$$

Let $BQB^\intercal = v_1 q_1 v_1^\intercal$ hence $\Pi_\infty = v_1 p_1 v_1^\intercal$, with
$2\lambda_1 p_1 - (Cv_1, RCv_1)p_1^2 + q_1 = 0 \Rightarrow p_1 = \frac{\lambda_1 + \sqrt{\lambda_1^2 + \gamma_1 q_1}}{\gamma_1}$ where $\gamma_1 = (Cv_1, RCv_1)$
Let now define $(\mu_1, \ldots, \mu_n)$ the eigenvalues of
$A - \Pi_\infty C^\intercal RC = A - v_1 p_1 v_1^\intercal C^\intercal RC$.
if $i \neq 1$, $(A^\intercal - C^\intercal RCv_1 p_1 v_1^\intercal) w_i = A^\intercal w_i = \lambda_i w_i$
Let $u_1 = \sum \alpha_i w_i$, we have $\sum \alpha_i \lambda_i w_i - \alpha_1 C^\intercal RCv_1 p_1 v_1^\intercal w_1 = \mu_1 \sum \alpha_i w_i$
Hence by taking the scalar product w.r.t w_1, we get
$\alpha_1 \lambda_1 - \alpha_1 p_1 v_1^\intercal C^\intercal RCv_1 = \alpha_1 \mu_1$.
So Finally $\mu_1 = \lambda_1 - p_1 \gamma_1 = -\sqrt{\lambda_1 + \gamma_1 q_1} \leq -|\lambda_1|$

Combining several concepts

- We decompose the initial condition $y = y_\diamond + \Pi_\mathrm{r}^* \zeta_\mathrm{r}$.
- Estimator Dynamics

$$\begin{cases} \dot{\hat{y}} = Ay + f + LU^{-1}L^*C^*(z^\delta - Cy), & \hat{y}(0) = y_\diamond \\ \dot{L} = AL, & L(0) = \Pi_\mathrm{r}^* \\ \dot{U} = L^*\mathrm{C}^*R\mathrm{C}L, & U(0) = U_0 \end{cases}$$

- Associated to the minimization

$$\min_{\zeta_\mathrm{r}} \left\{ \mathscr{J}(\zeta_\mathrm{r}, t) = \frac{1}{2}\|\zeta_\mathrm{r}\|_{U_0}^2 + \frac{1}{2}\int_0^t \left(\|z^\delta(s) - C(y_{|\zeta_\mathrm{r}}(s))\|_R^2 \right) \mathrm{d}s \right\}.$$

- Error Analysis $(\tilde{y}_\perp, \tilde{y}_\mathrm{r}) \mapsto (\lambda, \mu) = (\tilde{y}_\perp - L_\perp L_\mathrm{r}^{-1}\tilde{y}_\mathrm{r}, (L_\mathrm{r})^{-1}\tilde{y}_\mathrm{r})$

$$\begin{cases} \dot{\lambda} = (A_{\perp\perp} - L_\perp L_\mathrm{r}^{-1} A_{\mathrm{r}\perp})\lambda \\ \dot{\mu} = -U^{-1}L^*C^*RCL\mu - U^{-1}L^*C^*RC(\mathbb{1} - \Pi_\mathrm{r})(\lambda + \eta), \end{cases}$$

Luenberger observers

The case of conservative systems (transport, waves, elastodynamics)

- A linear conservative system $\dot{y} = Ay$ with A is skew adjoint – *i.e.* $A^* = A$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2}\|y\|^2\right) = (y, Ay) = 0$$

- Find G so that $A - GC$ is exponentially stable

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2}\|\tilde{y}\|^2\right) = -(\tilde{y}, GC\tilde{y})$$

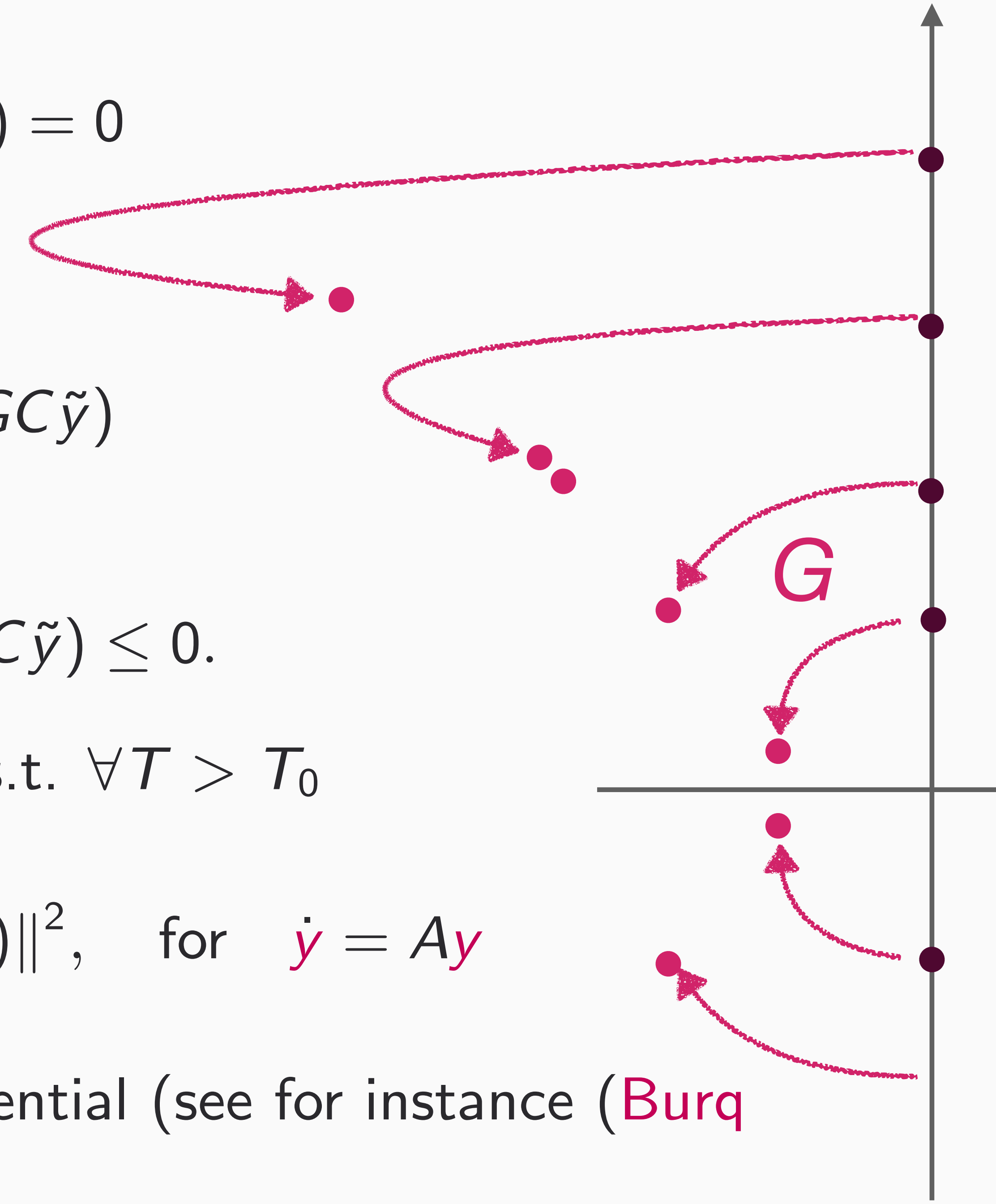
- With $G = C^* R$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2}\|\tilde{y}\|^2\right) = -(C^*\tilde{y}, RC\tilde{y}) \leq 0.$$

- (Haraux, 1989) $\tilde{y}$ is exponentially stable if $\exists T_0$ s.t. $\forall T > T_0$

$$\exists c_{\mathrm{st}} \quad | \quad \int_0^T (C^* y, RCy) \geq c_{\mathrm{st}} \|y(0)\|^2, \quad \text{for} \quad \dot{y} = Ay$$

- if only detectable then the stability is not exponential (see for instance (Burq and Gérard, 2002))



Effect of G on the poles

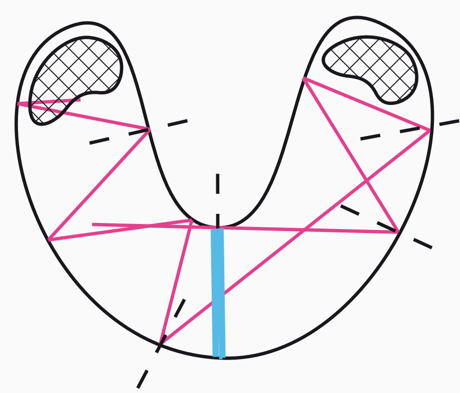
Example : The wave equation (i)

- Model: $\partial^2_{tt} u - \Delta u = f$ in Ω, with adequate B.C. ex: $u = 0$ on $\partial\Omega$
- First order system with $y = \begin{pmatrix} u \\ v \end{pmatrix}$, and $A = \begin{pmatrix} 0 & \mathbb{1} \\ \Delta & 0 \end{pmatrix}$
- Case 1: Observation $z = \dot{u}_{|\omega}$, namely $C = \begin{pmatrix} 0 & \mathbb{1}_{|\omega} \end{pmatrix}$
- Observer: $\dot{\hat{y}} = A\hat{y} + \gamma C^*(z^\delta - C\hat{y})$, we get the so-called Direct Velocity Feedback

$$\partial^2_{tt} u - \Delta u = f + \gamma \mathbb{1}_\omega (z^\delta - \partial_t u_\omega)$$

- Detectability: Holmgren theorem: modes can not canceled in the observation zone (see for instance Burq and Gérard (2002))
- Observability: Geometric Control Condition (GCC, Bardos et al. (1987)) : $\exists T_0$ s.t. $\forall T > T_0$

$$\exists c_{st} \quad | \quad \int_0^T \int_\omega |\partial u(x,t)|^2 \geq c_{st}(\|u(0)\|^2_{H^1_0(\Omega)} + \|\partial_t u(0)\|^2_{L^2(\Omega)})$$



Example : The wave equation (ii)

- Case 2: Observation $z = u_{|\omega}$, namely $C = \begin{pmatrix} \mathbb{1}_{|\omega} & 0 \end{pmatrix}$
- Observer: $\dot{\hat{y}} = A\hat{y} + \gamma C^*(z^\delta - C\hat{y})$, we get the so-called Direct Velocity Feedback

$$\begin{cases} \partial_t u = v + R_\omega(z^\delta - u_\omega) \\ \partial_t v - \Delta u = f \end{cases} \text{with } R_\omega : \varphi \mapsto \psi \text{ s.t.} \left| \begin{aligned} -\Delta\psi &= 0 \text{ in } \Omega \\ \psi &= \varphi \text{ in } \omega \\ \psi &= 0 \text{ on } \partial\Omega \end{aligned} \right.$$

- Detectability: Holmgren theorem: modes can not canceled in the observation zone
- Observability: Geometric Control Condition (Chapelle et al. (2012))

$$\exists c_{st} \quad | \quad \int_0^T \int_\omega |\nabla u(x,t)|^2 \geq c_{st}(\|u(0)\|^2_{H^1_0(\Omega)} + \|\partial_t u(0)\|^2_{L^2(\Omega)})$$

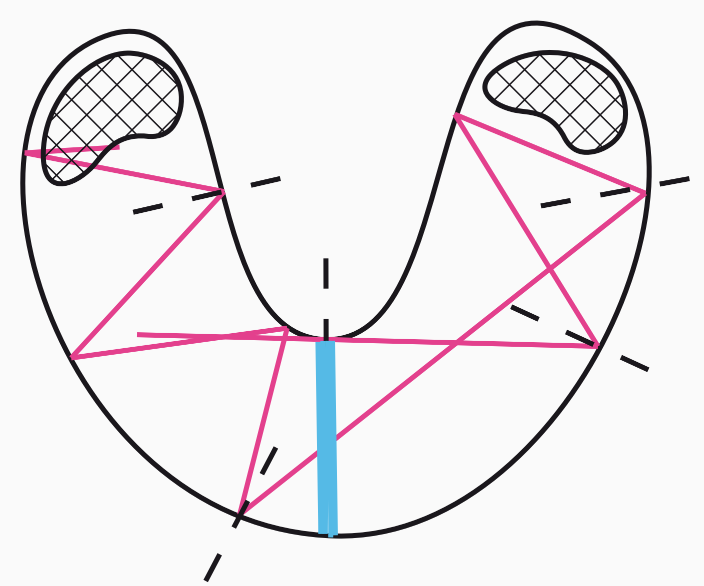
Example : The wave equation (iii)

- γ is not conditioned by δ: overdamping phenomena
- Reduced order methods do not work in the conservative
- Recover the initial condition with the *back and forth* observer (Ramdani et al., 2010)

$$\begin{cases} \dot{\hat{y}}^k = A\hat{y}^k + \gamma C^*(z^\delta - C\hat{y}^k) \\ \hat{y}(0) = \hat{y}_b^{k-1}(0) \end{cases}$$

with $\hat{y}_b^{-1}(0) = y_\diamond$ and for $k \geq 0$

$$\begin{cases} \dot{\hat{y}}_b^k = A\hat{y}_b^k - \gamma C^*(z^\delta - C\hat{y}_b^k) \\ \hat{y}_b^k(T) = \hat{y}^k(T) \end{cases}$$



- Works for unbounded domain: GCC becomes GCC-Exterior
- Works also for elastodynamics problem: GCC becomes GCC-Elasticity

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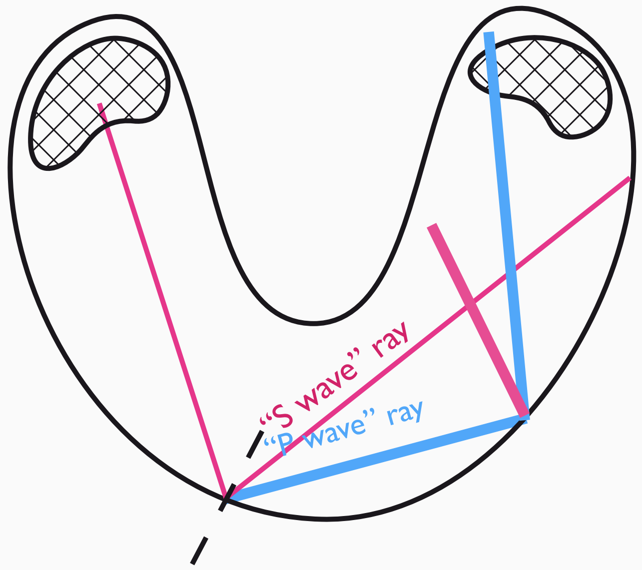

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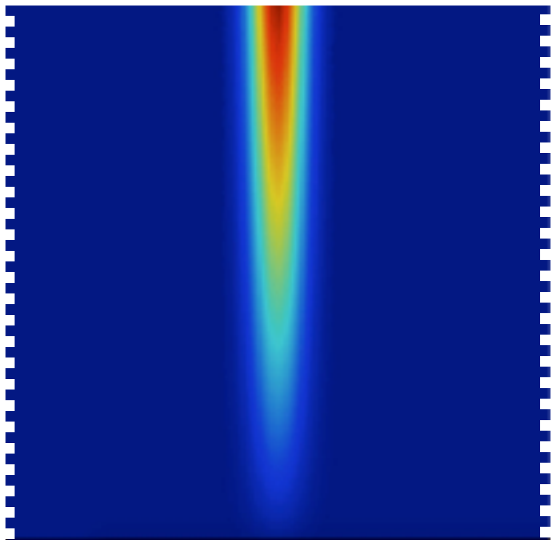
Numerical example

TBC

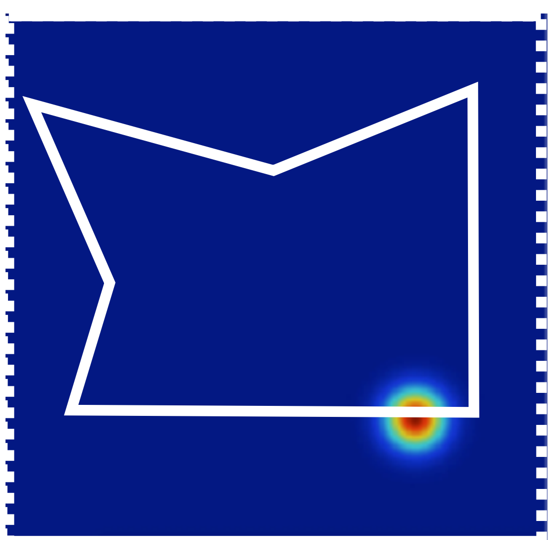


TBC

Direct problem **(D)**

Observation area ω

TBC



TBC

Forward step **(F)**

Numerical example

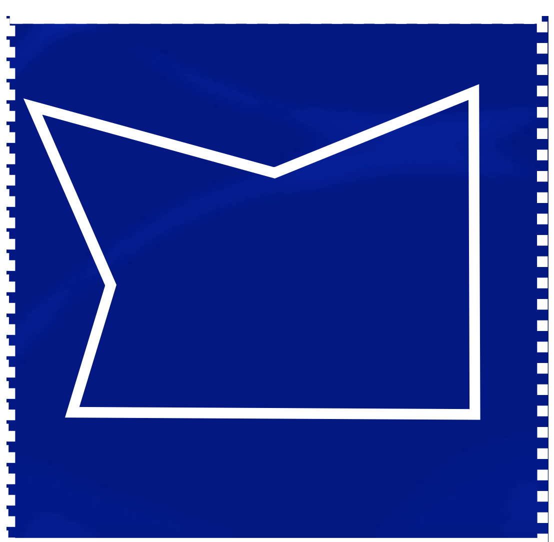
TBC

TBC

Direct problem **(D)**

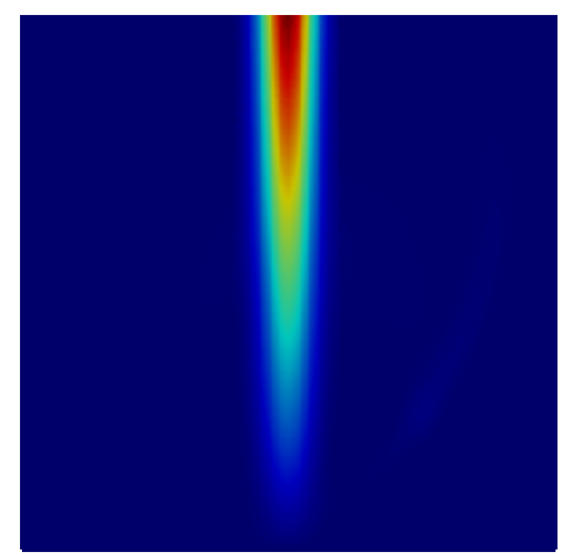
Observation area ω

TBC



TBC

Backward step **(B)**

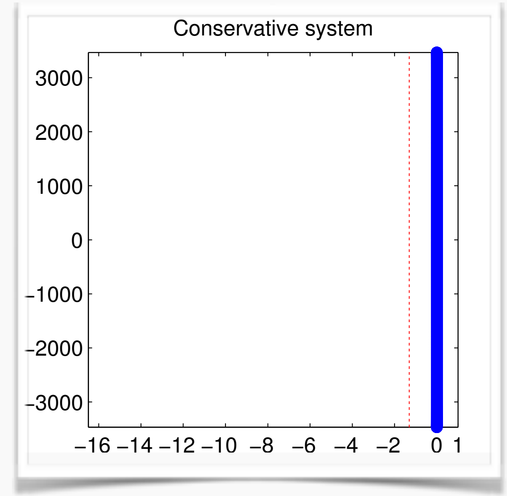


$\widehat{\mathrm{y}}^5(0)$

What about discretization

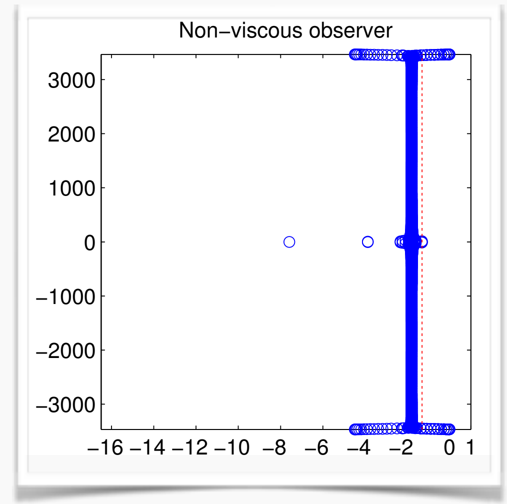
- Estimator scheme

$$\begin{cases} \mathrm{K}\dfrac{\hat{\mathrm{u}}_{n+1}-\hat{\mathrm{u}}_n}{\delta t} = \mathrm{K}\hat{\mathrm{v}}_{n+\frac12} + \gamma\,{}^\flat\mathrm{C}^\intercal\mathrm{K}_\omega\left(\mathrm{z}^\delta_{n+\frac12} - {}^\natural\mathrm{C}\hat{\mathrm{u}}_{n+\frac12}\right) \\ \mathrm{M}\dfrac{\hat{\mathrm{v}}_{n+1}-\hat{\mathrm{v}}_n}{\delta t} + \mathrm{K}\hat{\mathrm{u}}_{n+\frac12} = \mathrm{f} \end{cases}$$



- Energy balance

$$\frac{\tilde{\mathscr{E}}_{n+1}-\tilde{\mathscr{E}}_n}{\delta t} = -\gamma\tilde{\mathrm{u}}_{n+\frac12}\,{}^\flat\mathrm{C}^\intercal\mathrm{K}_\omega\,{}^\flat\mathrm{C}\tilde{\mathrm{u}}_{n+\frac12}$$

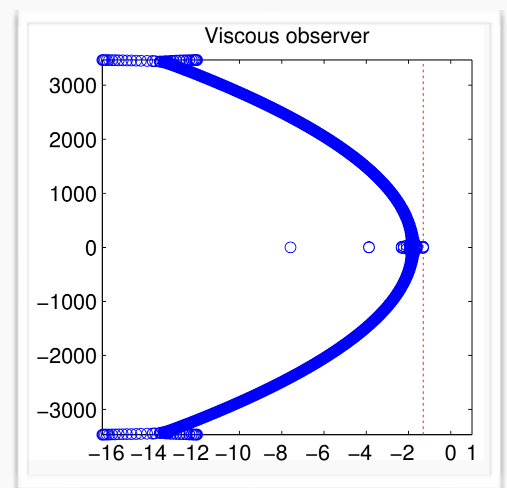


- Observability condition

$$\sum_{k=1}^{n} \tilde{y}^\intercal_{n+\frac12}\mathrm{C}^\intercal\mathrm{R}\mathrm{C}^\intercal\tilde{y}_{n+\frac12} \not\geq, \alpha\,\tilde{y}_0^\intercal\Lambda_0\tilde{y}_0$$

- Several remedies: Mixed Element Methods, Numerical viscosity, adaptative meshes...
- New estimates and paradigm

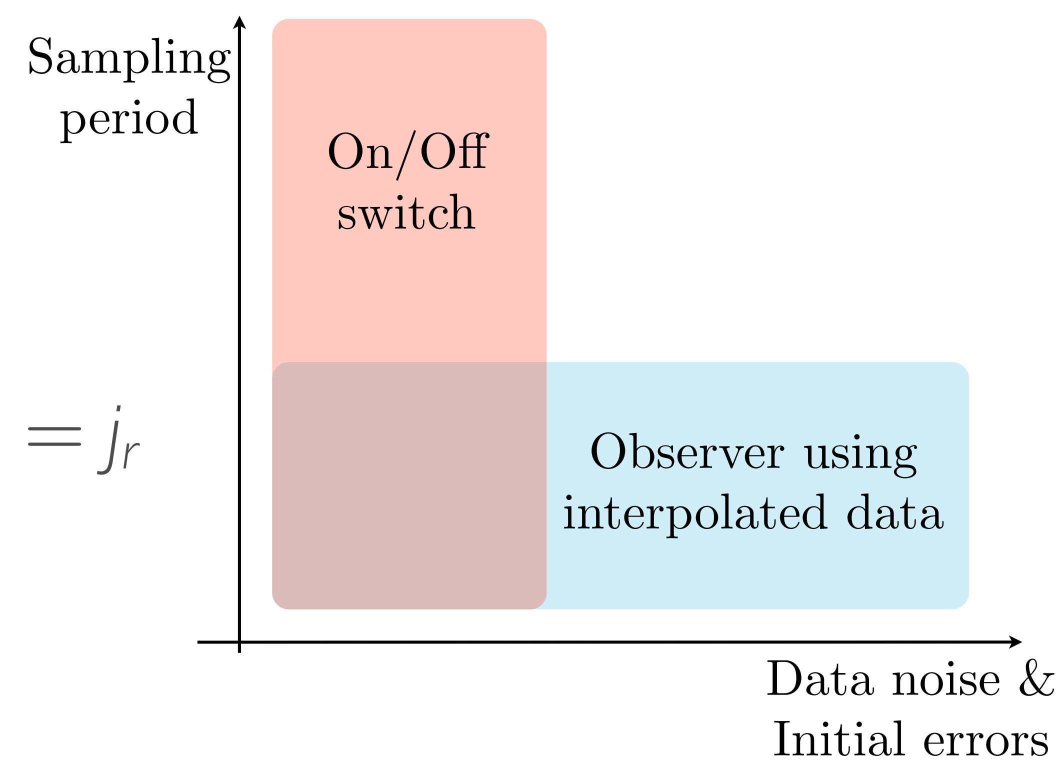
$$\|\hat{y}^n_h - \breve{y}(n\Delta t)\|_{\mathcal{Y}} \leq c(y_0)\max(\epsilon, \epsilon^2\, h^{-1}\Delta t), \qquad \forall n \in \mathbb{N}$$



Data sampling: Time sampling

- General time scheme

$$
\begin{cases}
\dfrac{\hat{y}^-_{n+1} - \hat{y}^+_n}{\delta t} = A\dfrac{\hat{y}^-_{n+1} + \hat{y}^+_n}{2}, & n > 0 \\
\dfrac{\hat{y}^+_{n+1} - \hat{y}^-_{n+1}}{\delta t} = \delta^{n+1}\gamma C^* \left(d_{n+1} - C\hat{y}^+_{n+1}\right) + \nu_{\delta t} A^2 \hat{y}^+_{n+1}, & n > 0 \\
\hat{y}^0_+ = \hat{y}_0,
\end{cases}
$$



- Use data only when they are available

$$
\delta^n = \begin{cases} 1 \\ 0 \end{cases} \qquad d_n = \begin{cases} z_r & \text{if } \exists r \in \mathbb{N} \ : \ n = j_r \\ 0 & \text{otherwise.} \end{cases}
$$

- Interpolate the data

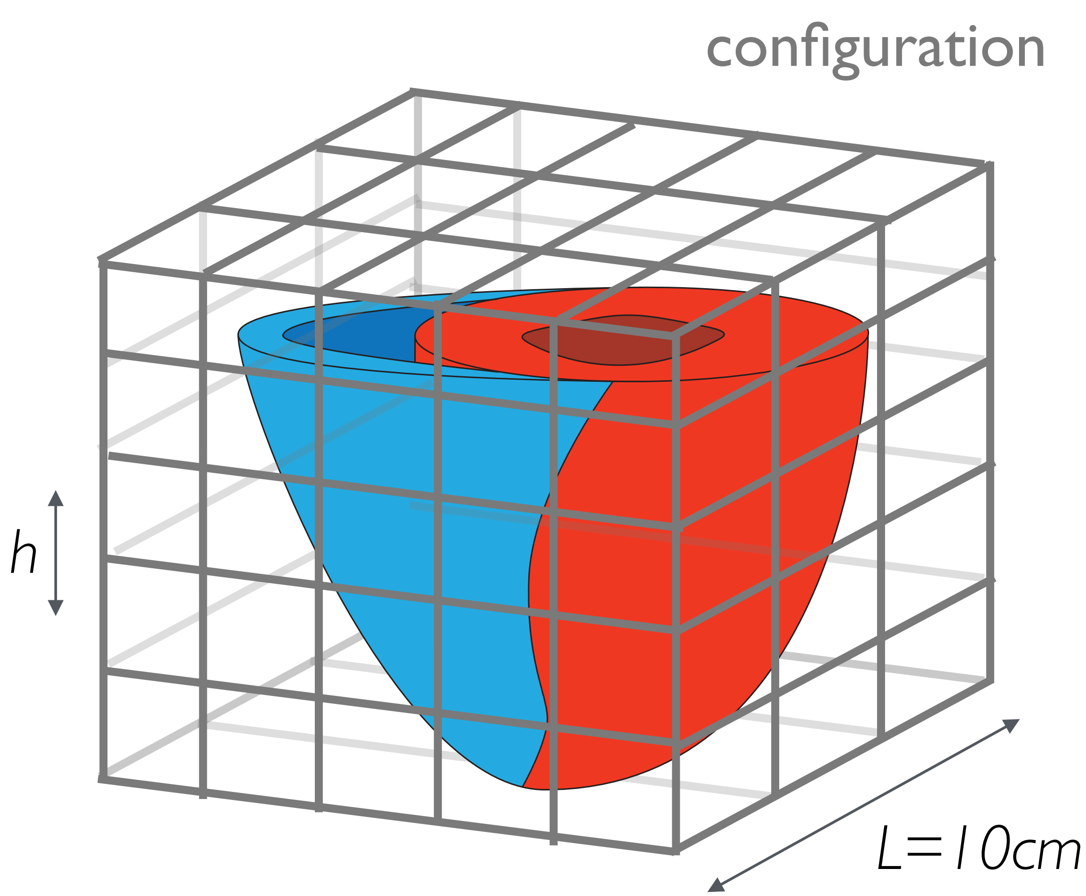
$$
\delta^n = 1 \qquad \forall n, \qquad d_n = \frac{n - j_r}{j_{r+1} - j_r} z_{r+1} + \left(1 - \frac{n - j_r}{j_{r+1} - j_r}\right) z_r \qquad j_r \leq n \leq j_{r+1}.
$$

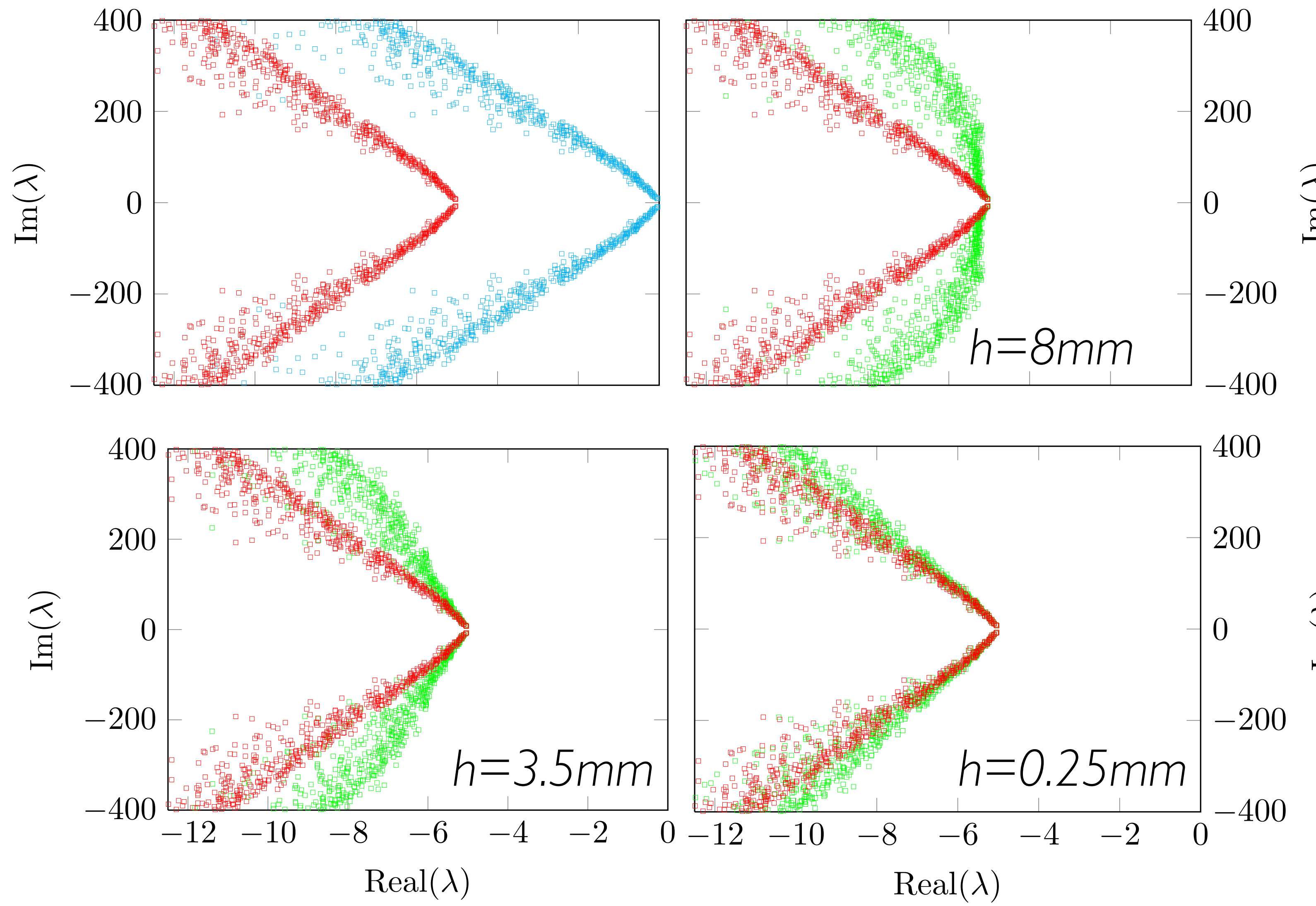
Data sampling: Space sampling

- Linear elasticity $$\int_\Omega \rho \partial_{tt} \underline{u} : \underline{w}\, d\Omega + \int_\Omega \underline{\varepsilon}(\underline{u}) : \underline{\underline{\underline{\underline{A}}}} : \underline{\varepsilon}(\underline{w})\, d\Omega = \int_\Omega \underline{f} \cdot \underline{w}\, d\Omega$$

- Eigen-problem associated with the discretization of the error system

$$\begin{bmatrix} {}^\natural\mathrm{C}^* \, {}^\natural\mathrm{C} & \mathrm{K} \\ -\mathrm{K} & -\mathrm{D}_\epsilon \end{bmatrix} \begin{bmatrix} \Phi_u \\ \Phi_v \end{bmatrix} = \lambda \begin{bmatrix} \mathrm{K} & 0 \\ 0 & \mathrm{M} \end{bmatrix} \begin{bmatrix} \Phi_u \\ \Phi_v \end{bmatrix}$$





Other example

- Transport equation
- Estimation for Saint-Venant systems (circulation flows)

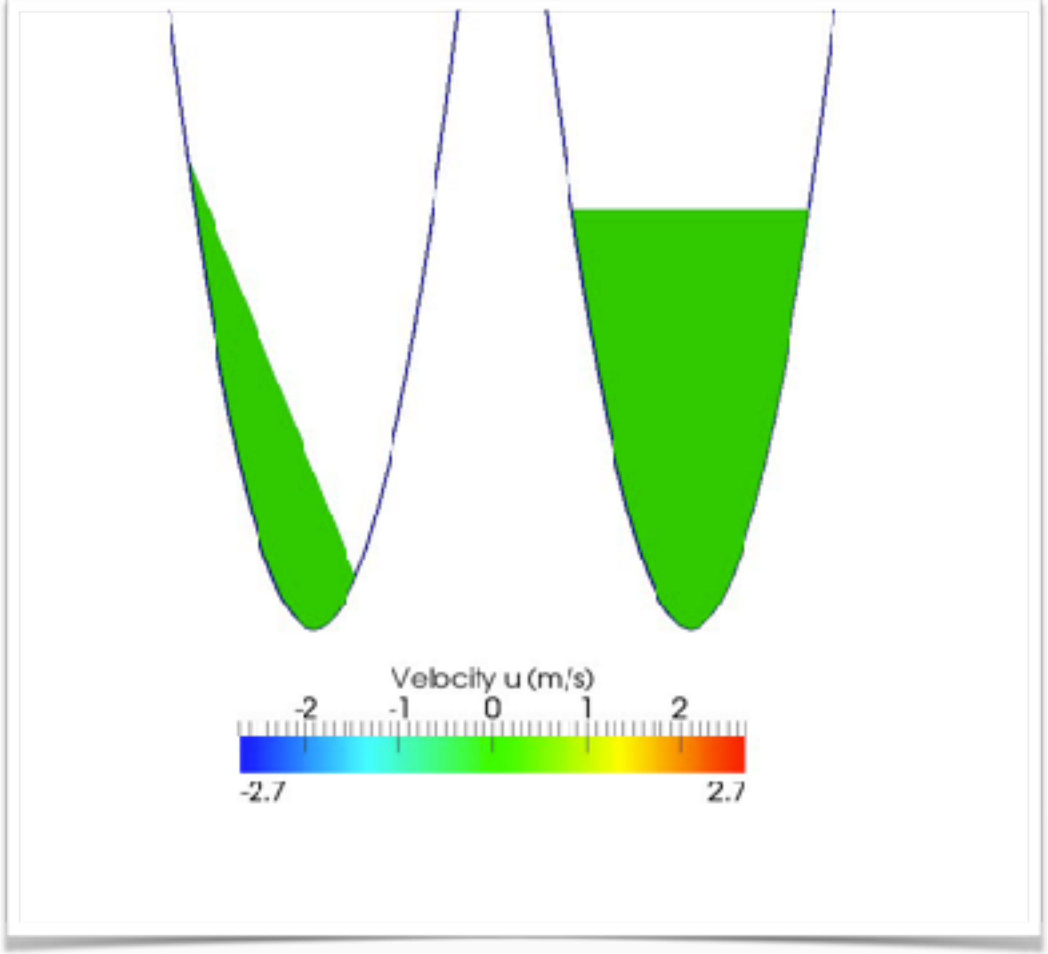
System

$$\begin{cases} \partial_t h + \partial_x \left(hu \right) = 0 \\ \partial_t \left(hu \right) + \partial_x \left(hu^2 + \frac{g}{2} h^2 \right) = 0 \end{cases}$$

Observation $z^\delta = \check{u} + \eta$

Estimator:

$$\begin{cases} \partial_t \hat{h} + \partial_x \left(\hat{h}\hat{u} \right) = \gamma (z^\delta - \hat{h}) \\ \partial_t \left(\hat{h}\hat{u} \right) + \partial_x \left(\hat{h}\hat{u}^2 + \frac{g}{2} \hat{h}^2 \right) = \gamma \hat{u} (z^\delta - \hat{h}) \end{cases}$$



Parameter estimation using asymptotic observers

Combine the strategies: the linear case

- Model: $\dot{y} = Ay + \beta + B\theta$
- Estimator (Zhang, 2002; Moireau et al., 2008)

$$\begin{cases} \dot{\hat{y}} = Ay + B\hat{\theta} + \beta + G(z^\delta - Cy) + L\dot{\theta}, \quad \hat{y}(0) = y_\diamond \\ \dot{\hat{\theta}} = U^{-1}L^*C^*R(z^\delta - C\hat{y}), \quad \hat{\theta} = \theta_\diamond \\ \dot{L} = AL + B, \quad L(0) = 0 \\ \dot{U} = L^*C^*RCL, \quad U(0) = U_\diamond = \mathbb{C}\mathrm{ov}(\theta)^{-1} \end{cases}$$

- min $\mathscr{J}(\theta) = \|\theta\|^2_{U_\diamond^{-1},\mathcal{P}} + \int_0^T \|z^\delta - C\check{y}\|^2 \,\mathrm{d}s$, where
  $\dot{\check{y}} = Ay + B\check{\theta} + \beta + G(z^\delta - C\check{y})$
- Analyzis: Triangularize the error system $(\tilde{y}, \tilde{\theta}) \rightarrow= (\mu = \tilde{y} - L\tilde{\theta}, \tilde{\theta})$

$$\begin{cases} \dot{\mu} = (A - GC)\mu \\ \dot{\tilde{\theta}} = -UL^*C^*RCL\tilde{\theta} - ULC^*RC\mu + \text{noise} \end{cases}$$

Combine the strategies: the non-linear case

- Model: $\dot{y} = A(y, \theta)$
- Estimator (Zhang and Xu, 2001; Moireau et al., 2008)

$$
\begin{cases}
\dot{\hat{y}} = A(\hat{y}, \hat{\theta}) + G(z^\delta - Cy) + L\dot{\theta}, \quad \hat{y}(0) = y_\diamond \\
\dot{\hat{\theta}} = U^{-1} L^* C^* R(z^\delta - C\hat{y}), \quad \hat{\theta} = \theta_\diamond \\
\dot{L} = \mathrm{d}_y A \, L + \mathrm{d}_\theta A, \quad L(0) = 0 \\
\dot{U} = L^* \, \mathrm{d}_y J^* R \, \mathrm{d}_y J \, L, \quad U(0) = U_\diamond = \mathbb{C}\mathrm{ov}(\theta)^{-1}
\end{cases}
$$

- No more minimization
- But analyzis: Triangularize the linearized error system $(\tilde{y}_\ell, \tilde{\theta}_\ell) \to= (\mu = \tilde{y}_\ell - L\tilde{\theta}_\ell, \tilde{\theta}_\ell)$

$$
\begin{cases}
\dot{\mu} = (\, \mathrm{d}_y A - GC)\mu \\
\dot{\tilde{\theta}}_\ell = -UL^* \, \mathrm{d}_y J^* R \, \mathrm{d}_y J \, L\tilde{\theta}_\ell - UL \, \mathrm{d}_y J^* R \, \mathrm{d}_y J \, \mu + \text{noise}
\end{cases}
$$

Combine the strategies with the reduced order UKF, (i)

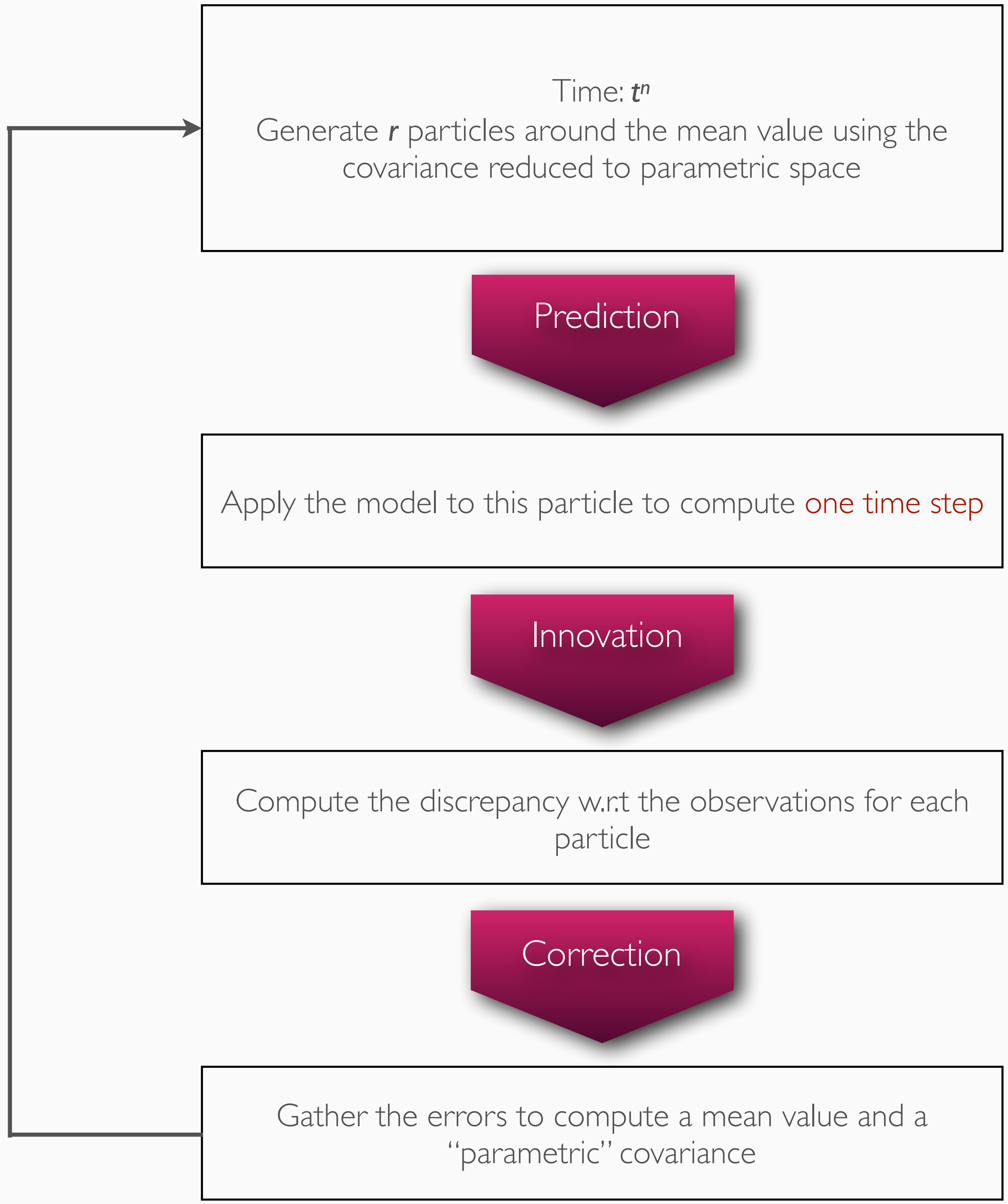
- Model: $y_{n+1} = \Phi_{n+1|n}(y_n, \theta_n)$
- Estimator (Moireau and Chapelle, 2011)

Sampling
$$\begin{cases} \hat{y}^{(i)}_{n|n} = \hat{y}_{n|n} + L^y_n \sqrt{U_n^{-1}}^T e^{(i)}, & 1 \leq i \leq p+1 \\ \hat{\theta}^{(i)}_{n|n} = \hat{\theta}^+_n + L^\theta_n \sqrt{U_n^{-1}}^T e^{(i)}, & 1 \leq i \leq p+1 \end{cases}$$

Prediction
$$\begin{cases} \hat{y}^{(i)}_{n+1|n} = \Phi_{n+1|n}(\hat{y}^{(i)}_{n|n}, \hat{\theta}_{n|n}(i)) \\ \hat{y}_{n+1|n} = E_\alpha(A_{n+1|n}(\hat{y}^*_{n|n}, \hat{\theta}^*_{n|n})), \quad \hat{\theta}_{n+1|n} = \hat{\theta}_{n|n} \end{cases}$$

Correction
$$\begin{cases} L^y_{n+1} = [\hat{y}^*_{n+1|n}] D_\alpha [e^*]^T, \quad L^\theta_{n+1} = [\theta^*_{n+1|n}] D_\alpha [e^*]^T \\ \Gamma^{(i)}_{n+1} = J(y^{(i)-}_{n+1}, t_{n+1}), \quad DJ_{n+1} = [\Gamma^{(i)}_{n+1}] D_\alpha [e^*]^T \\ U_{n+1} = \mathbb{1} + DJ^T_{n+1} W^{-1}_{n+1} DJ_{n+1} \\ \hat{y}_{n+1|n+1} = \hat{y}_{n+1|n} + L^y_{n+1} U^{-1}_{n+1} DJ^T_{n+1} R^{-1}_{n+1} (E_\alpha(\Gamma^*_{n+1})) \\ \hat{\theta}_{n+1|n+1} = \hat{\theta}_{n+1|n} + L^\theta_{n+1} U^{-1}_{n+1} DJ^T_{n+1} R^{-1}_{n+1} (E_\alpha(\Gamma^*_{n+1})) \end{cases}$$

Combine the strategies with the reduced order UKF, (ii)



Numerical illustration

- Linear elastodynamics with pre-stress loading

$$\int_\Omega \rho \partial_{tt} \underline{u} : \underline{w} \, d\Omega + \int_\Omega \underline{\underline{\varepsilon}}(\underline{u}) : \underline{\underline{\underline{\underline{A}}}} : \underline{\underline{\varepsilon}}(\underline{w}) \, d\Omega = \int_\Omega \underline{\underline{\sigma}}(t) : \underline{\underline{\varepsilon}}(\underline{w}) \, d\Omega$$

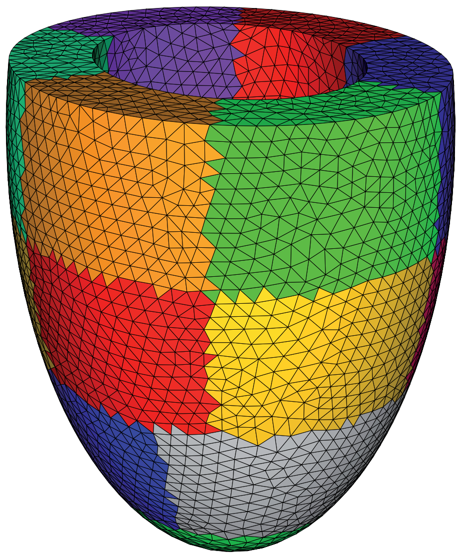
- with

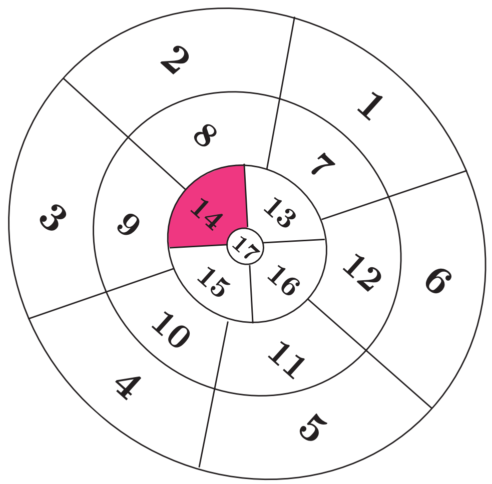
$$\underline{\underline{\sigma}}(t) = \theta \underline{\underline{\sigma_0}} \lambda(t)$$

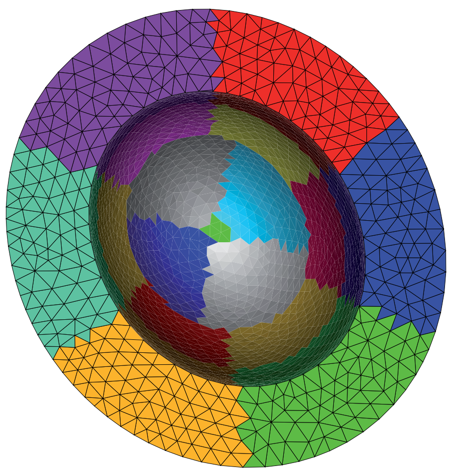
- Domain decomposition into 17 regions

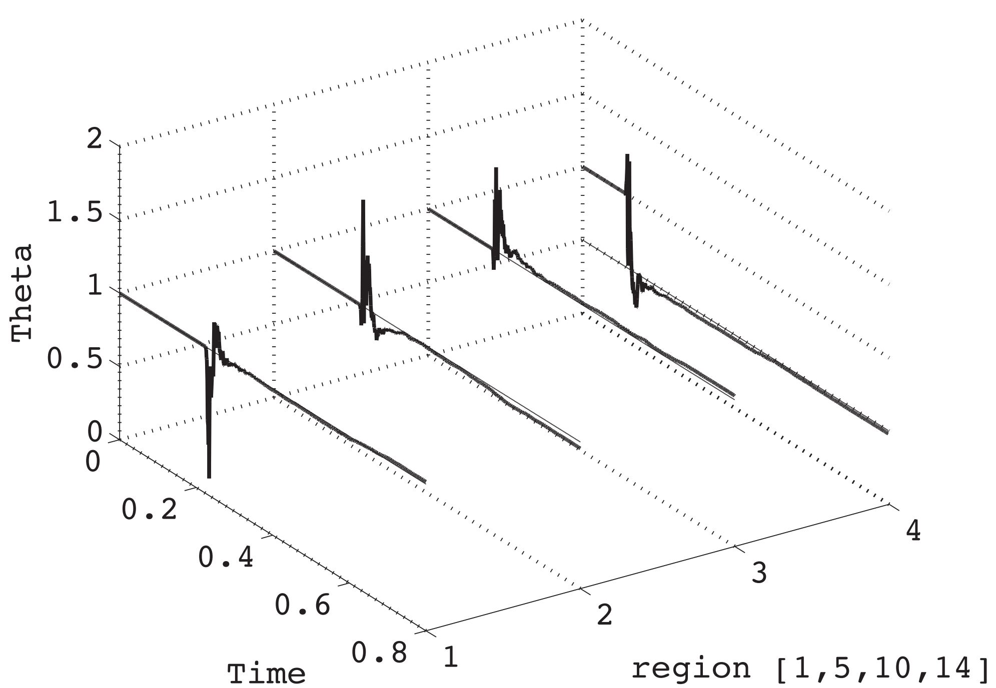
$$\check{\theta}_{14} = 0.5, \quad \check{\theta}_{[1,13]\cup[15,17]} = 1$$

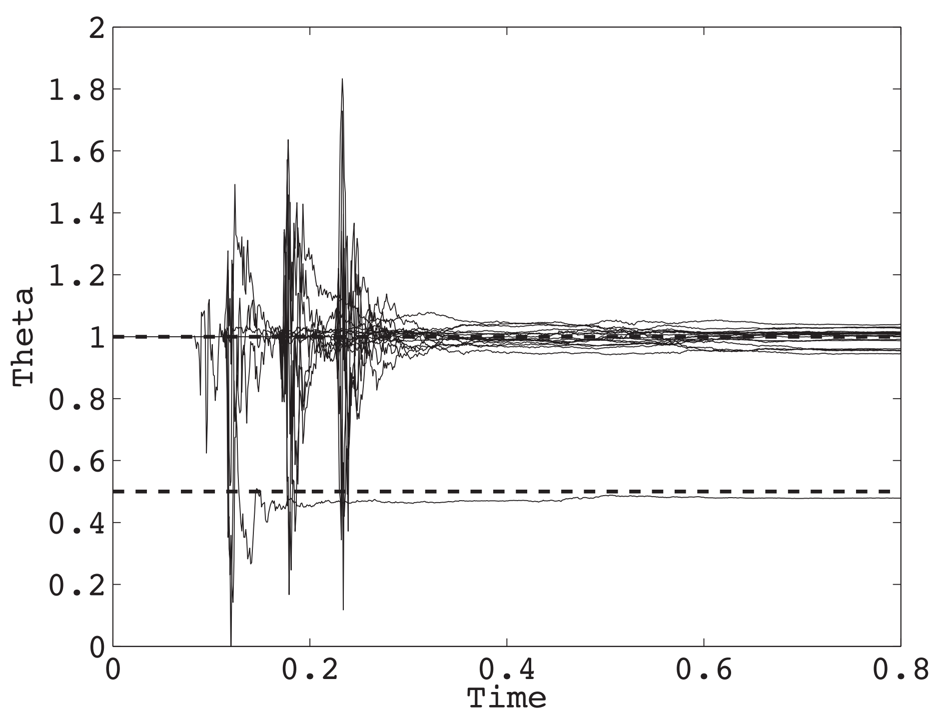
$$\hat{\theta}_{[1,17]}(0) = 1$$

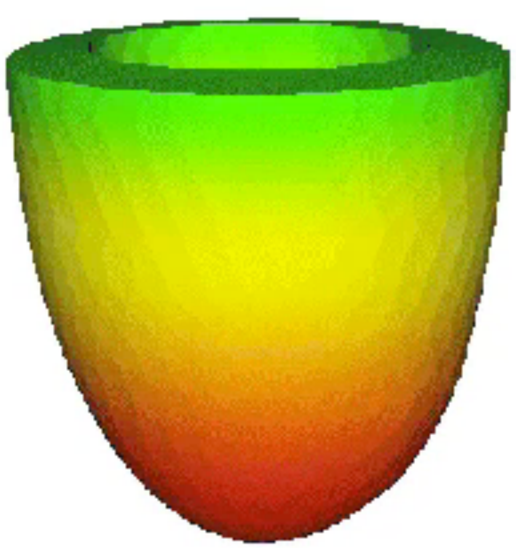












Numerical illustration

- Linear elastodynamics with pre-stress loading

$$\int_\Omega \rho \partial_{tt}\underline{u} : \underline{w}\, d\Omega + \int_\Omega \underline{\underline{\varepsilon}}(\underline{u}) : \underline{\underline{\underline{\underline{A}}}} : \underline{\underline{\varepsilon}}(\underline{w})\, d\Omega = \int_\Omega \underline{\underline{\sigma}}(t) : \underline{\underline{\varepsilon}}(\underline{w})\, d\Omega$$

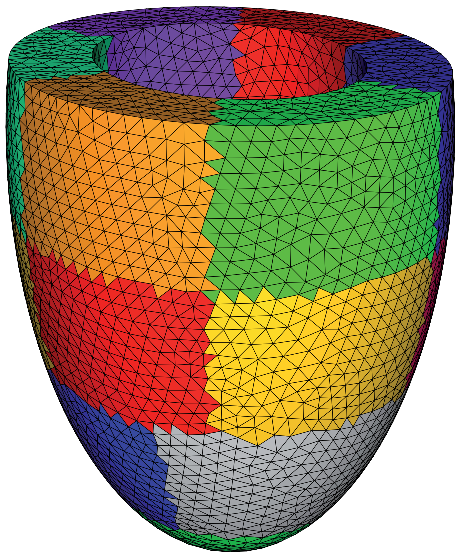
  - with

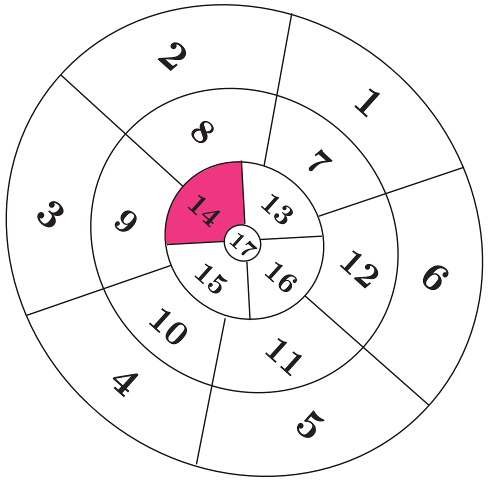
$$\underline{\underline{\underline{\underline{A}}}}(t) = 2^\theta \underline{\underline{\underline{\underline{A}}}}_0$$

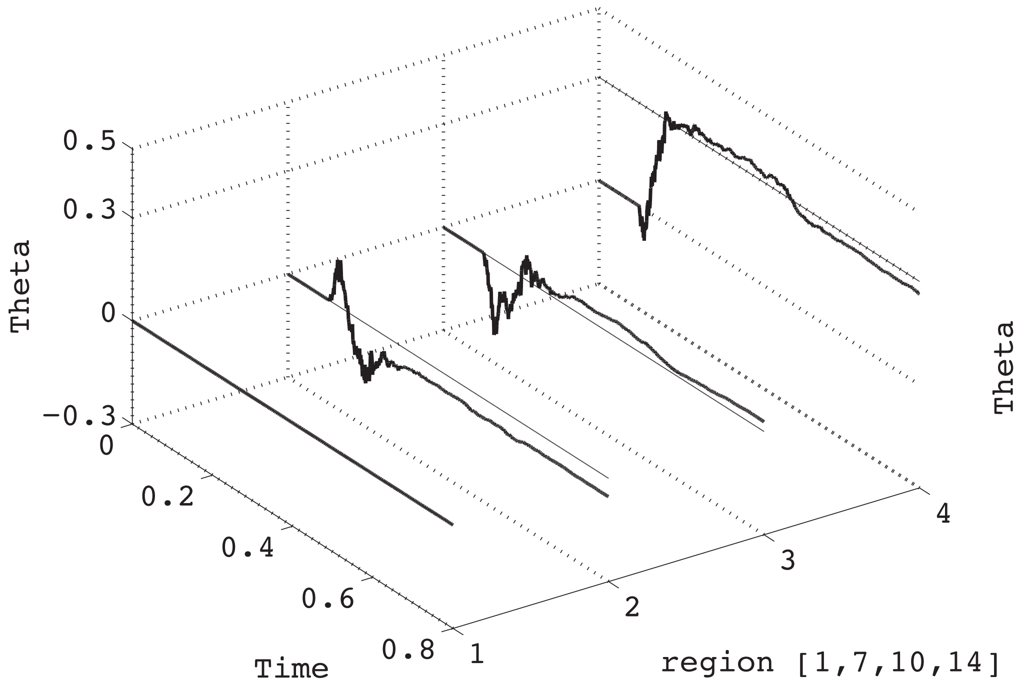
- Domain decomposition into 17 regions

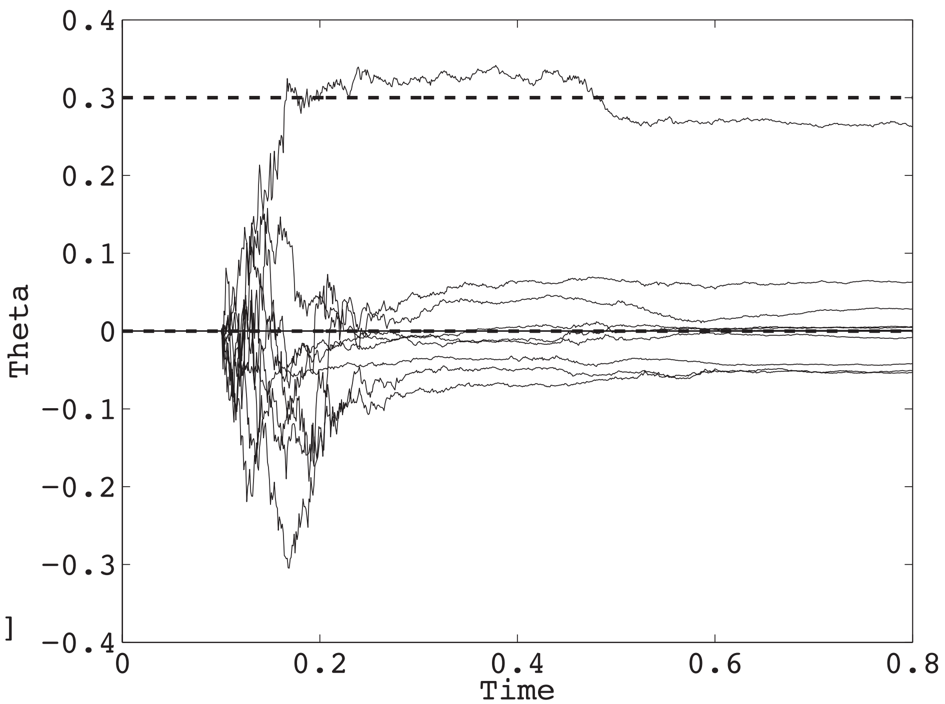
$$\check{\theta}_{14} = 0.3, \quad \check{\theta}_{[1,13]\cup[15,17]} = 0$$

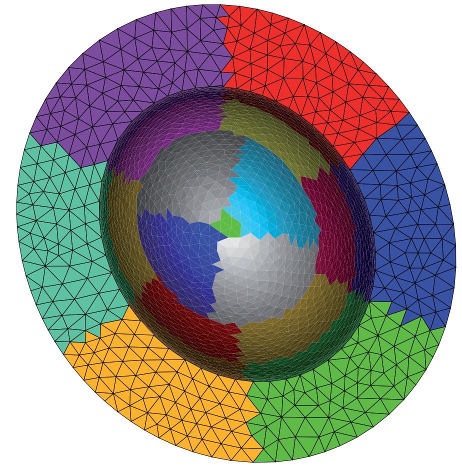
$$\hat{\theta}_{[1,17]}(0) = 0$$

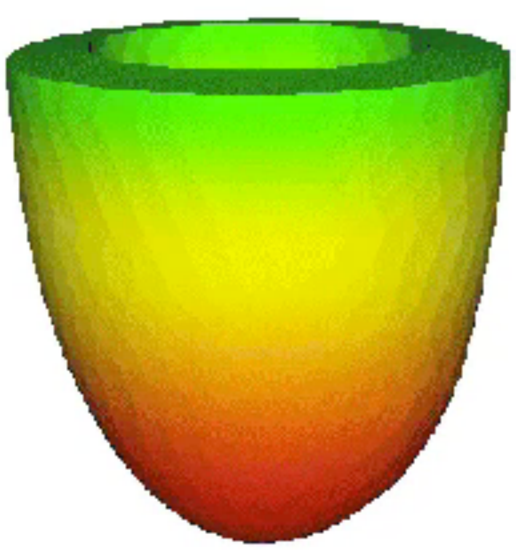




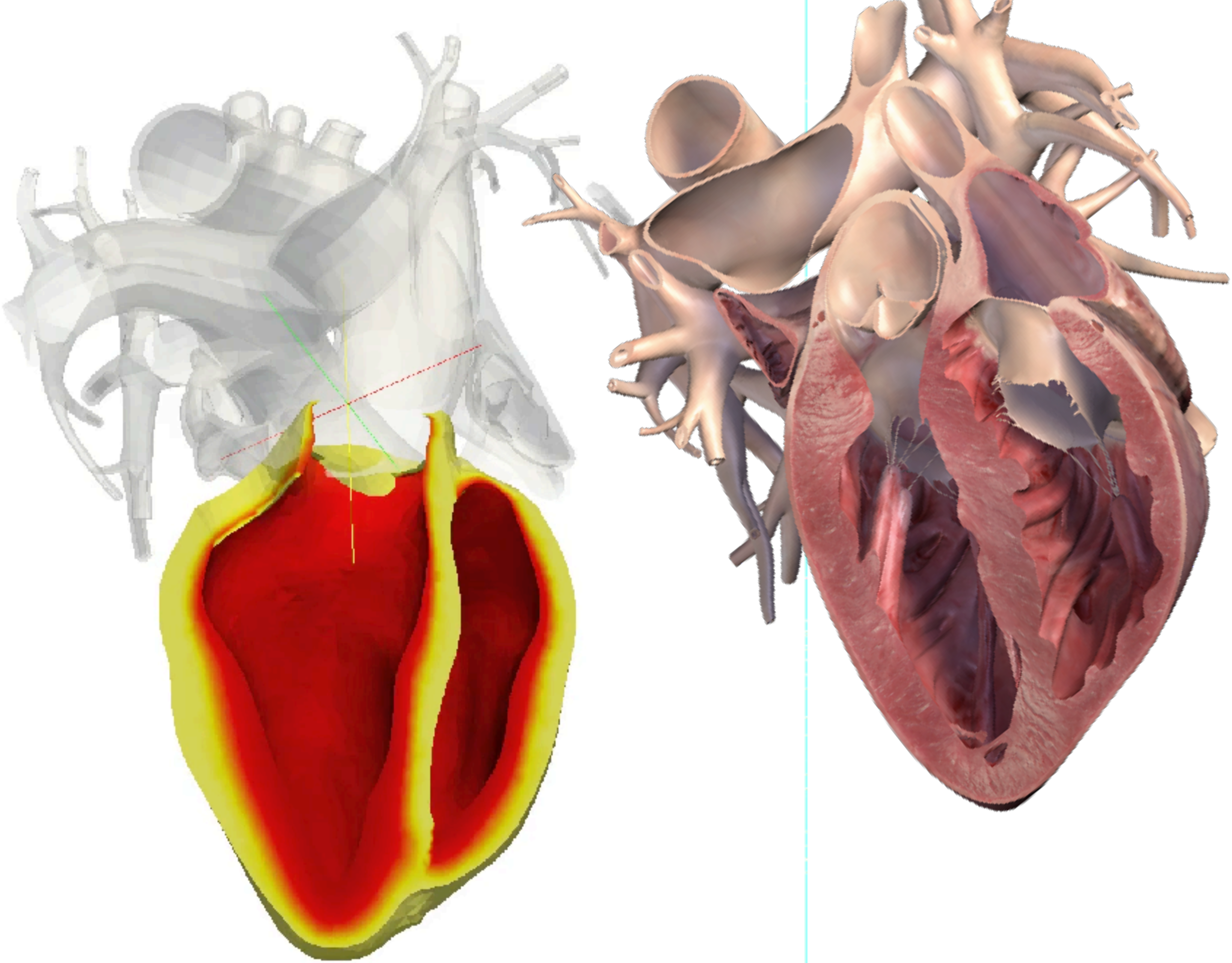






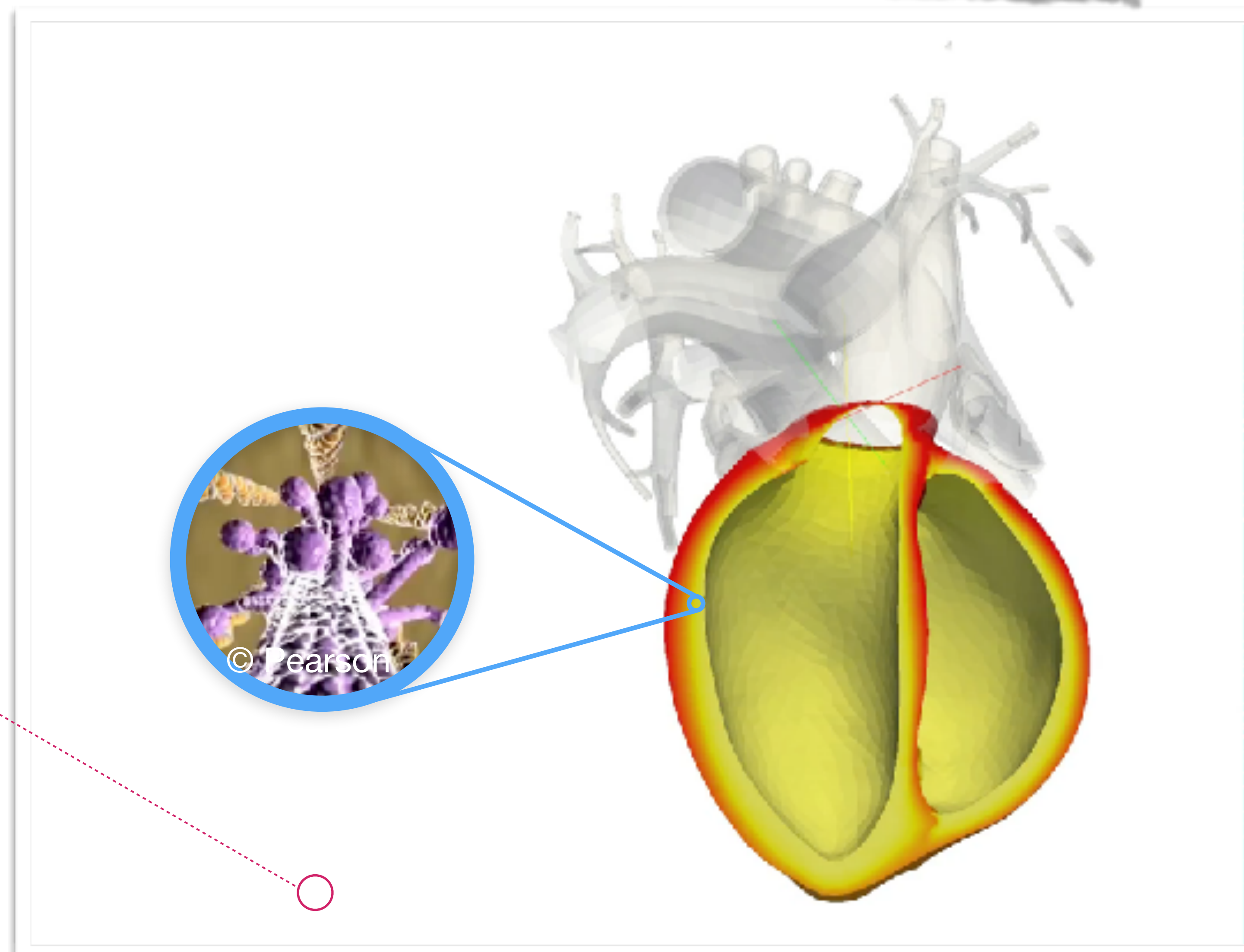


Examples



Sainte-Marie, J., Chapelle, D., Cimrman, R., & Sorine, M. - **Modeling and estimation of the cardiac electromechanical activity. Computers & Structures**, *84(28), 1743–1759 - 2006*
Chapelle, D., Le Tallec, P., Moireau, P., & Sorine, M - **An energy-preserving muscle tissue model: formulation and compatible discretizations**. *International Journal of Multiscale Computational Engineering, 2011*

Finite strains solid mechanics with multi-scale constitutive law and coupling with reaction diffusion system modeling the electrophysiology and a fluid (simplified or not)



$$\begin{cases} \partial_t \underline{u} = \underline{v}, & \text{in } \Omega_0\\ \rho\partial_t \underline{v} - \mathrm{div}(\underline{\underline{T}}) = 0, & \text{in } \Omega_0 \\ \underline{\underline{T}}\cdot\underline{n} = k_s\underline{u} + c_s\underline{v}, & \text{on } \Gamma_n\\ \underline{\underline{T}}\cdot \underline{n} = -Jp_{v,i}\underline{\underline{F}}^{-\intercal}\cdot\underline{n}, & \text{on } \Gamma_{c,i}\\ \underline{\underline{T}}\cdot\underline{n}=0, & \text{on } \partial\Omega_0\backslash((\cup_i\Gamma_{c,i})\cap\Gamma_n)\end{cases}$$

simplified fluid entry

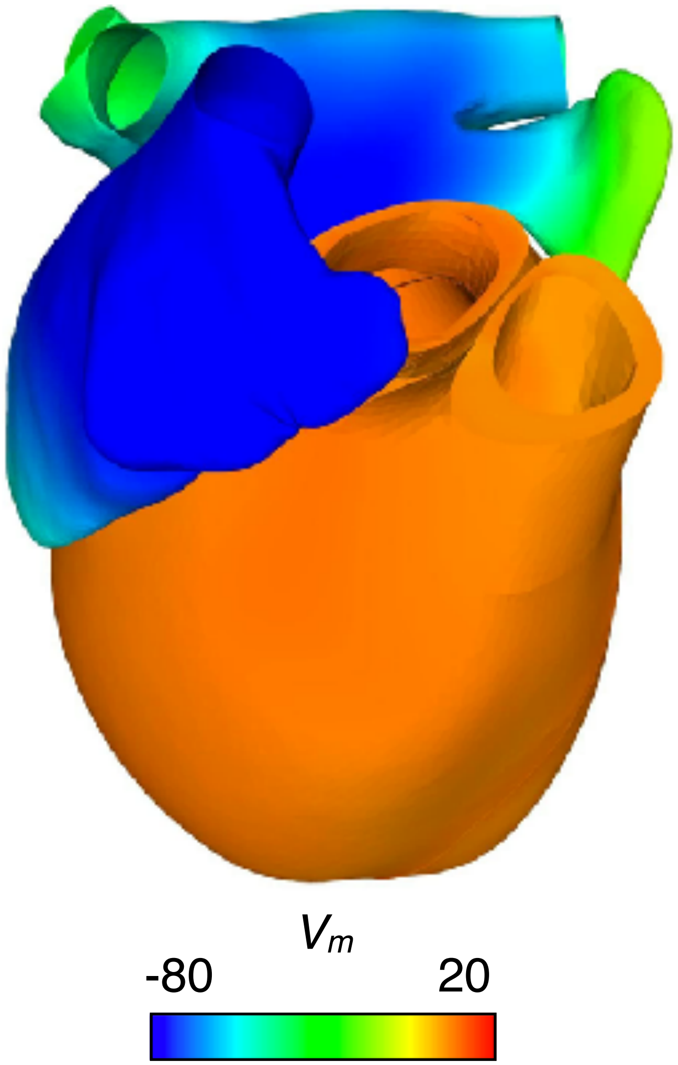
$$\underline{\underline{\Sigma}} = \underbrace{\underline{\underline{F}}}_{\underline{\underline{1}}+\underline{\underline{\nabla}}_{\xi}\underline{u}}{}^{-1}\cdot\underline{\underline{T}} = \underline{\underline{\Sigma}}_p + \sigma_{1D}(e_{1D},t)\,\underline{\tau}\otimes\underline{\tau}$$

fiber direction

internal variables

passive

active

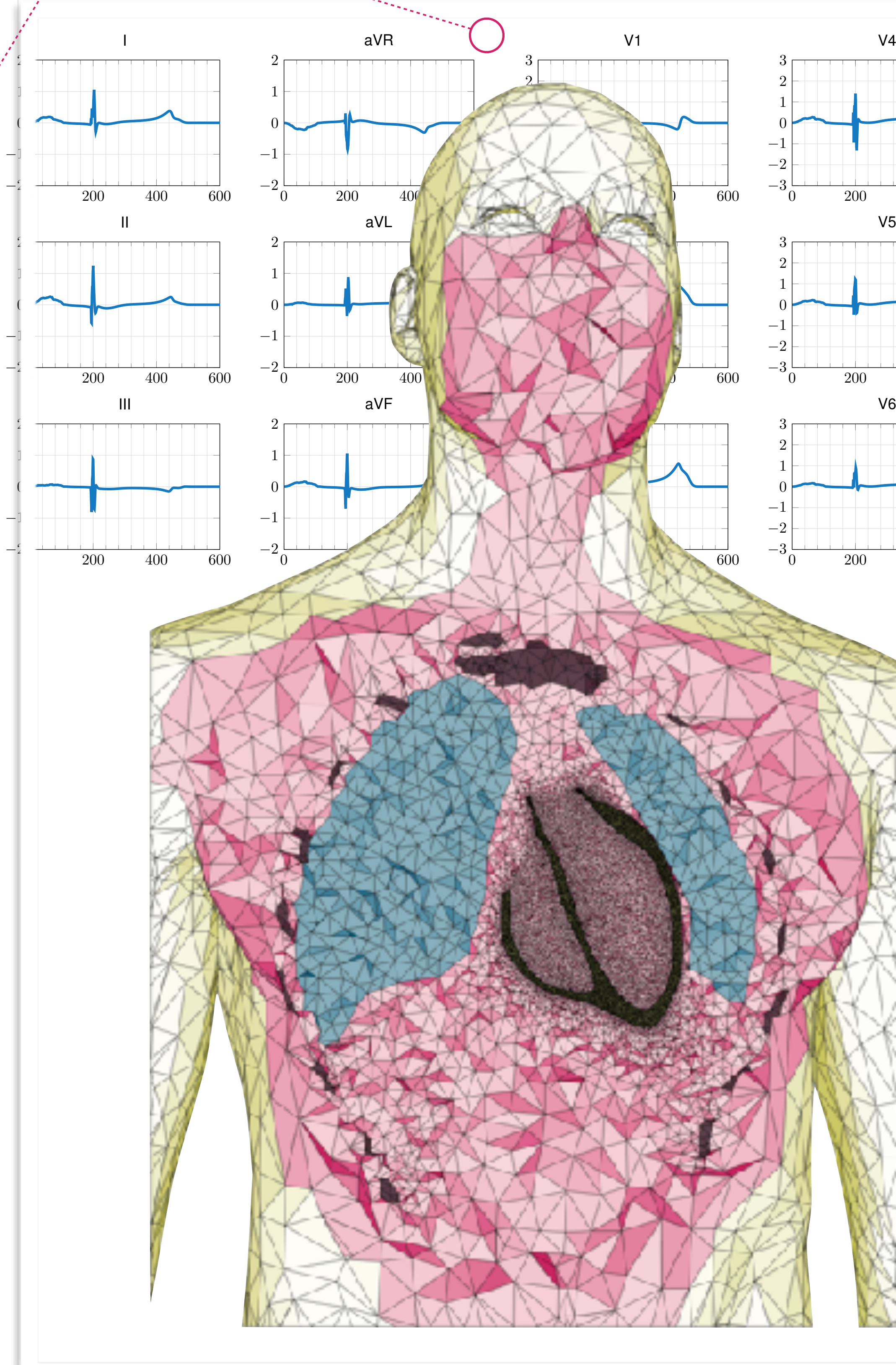


© Annabelle Collin

Reaction diffusion (bidomain) model in the heart + Poisson diffusion over the thorax

D Chapelle, M A Fernández, J-F Gerbeau, P Moireau, J Sainte-Marie, N Zemzemi- Numerical simulation of the electromechanical activity of the heart — FIMH 2009

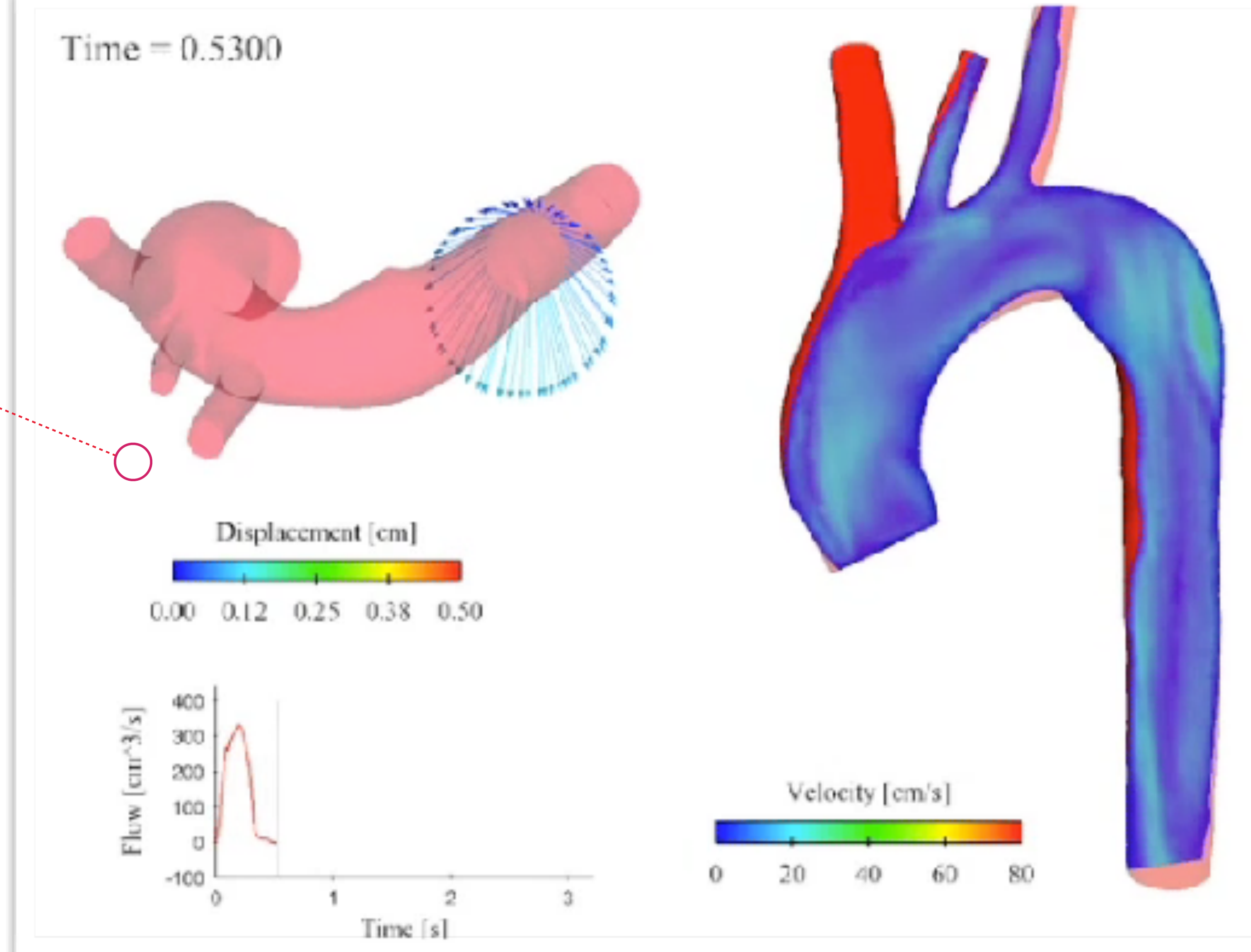
E Schenone, A Collin, J-F Gerbeau. - **Numerical simulation of electrocardiograms for full cardiac cycles in healthy and pathological conditions —** *International Journal for Numerical Methods in Biomedical Engineering 2015*

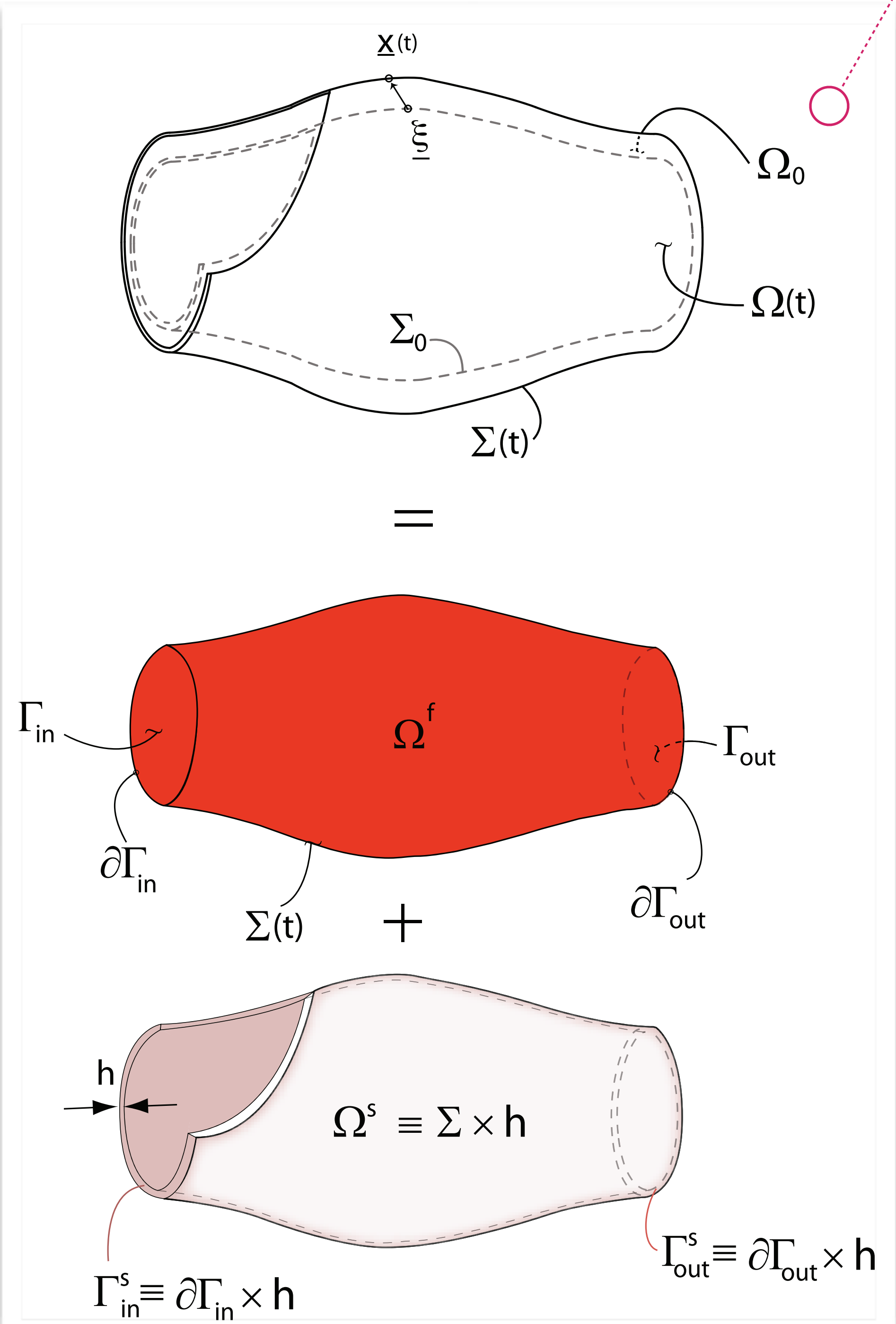


$$
\begin{aligned}
A_m\left(C_m\frac{\partial V_m}{\partial t}+I_{ion}(V_m,\cdots)\right)-\operatorname{div}\left(\underline{\underline{\sigma}}_i\cdot\underline{\nabla}V_m\right) &= \operatorname{div}\left(\underline{\underline{\sigma}}_i\cdot\vec{\nabla}u_e\right)+I_{app}(w), && \text{in } \Omega^{heart}\\
\operatorname{div}\left(\left(\underline{\underline{\sigma}}_i+\underline{\underline{\sigma}}_e\right)\cdot\underline{\nabla}u_e\right) &= -\operatorname{div}\left(\underline{\underline{\sigma}}_i\cdot\vec{\nabla}V_m\right), && \text{in } \Omega^{heart}\\
\partial_t w+g(V_m,w) &= 0, && \text{in } \Omega^{heart},
\end{aligned}
$$

$$
\begin{aligned}
-\operatorname{div}(\underline{\underline{\sigma_T}}\underline{\nabla}u_T)=0, &\quad \text{in } \Omega_B\\
R_p(\underline{\underline{\sigma_T}}\underline{\nabla}u_T)\cdot\underline{n}+u_T=u_e, &\quad \text{on } \partial\Omega_B^{heart}\\
\underline{\underline{\sigma_T}}\underline{\nabla}u_T\cdot\underline{n}=0, &\quad \text{on } \partial\Omega_B^{ext}
\end{aligned}
$$

ALE - FSI with Windkessel boundary conditions for the fluid and viscoelastic BC for the solid





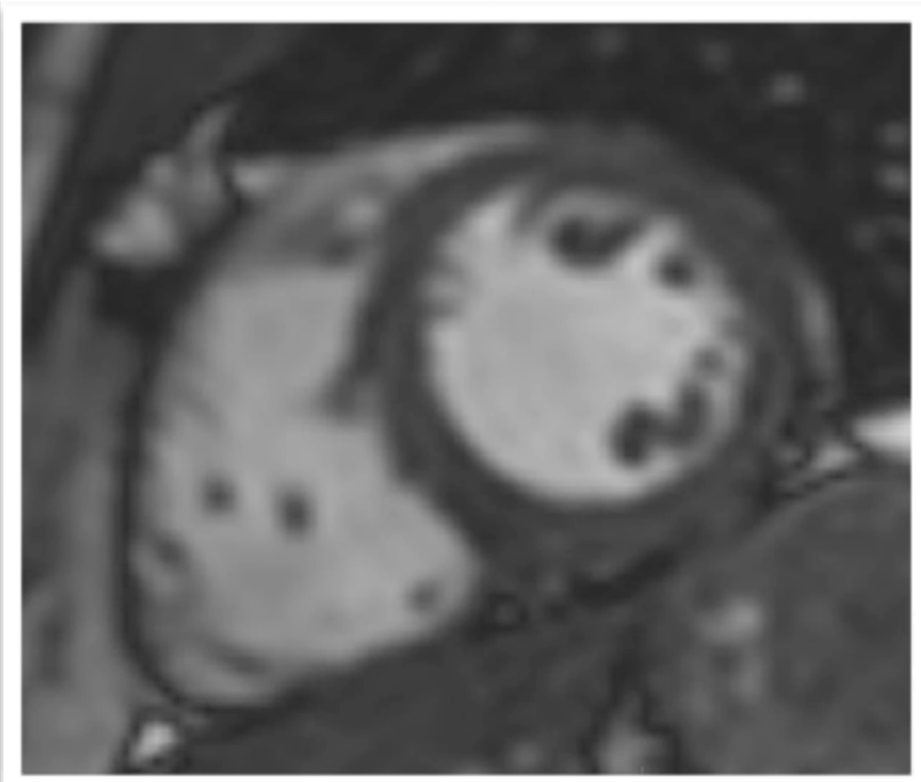
$$
\begin{cases}
\underline{w} = \mathrm{Ext}(\underline{v}_s|_{\Sigma_0}), & \text{in} \quad \Omega^f_0, \\
\dfrac{\rho_f}{J^f}\dfrac{\partial (J^f \underline{v}_f)}{\partial t}\Big|_{\xi} + \rho_f(\underline{v}_f - \underline{w})\cdot \underline{\underline{\nabla}}_x \underline{v}_f - 2\mu \underline{\underline{\nabla}}_x \cdot (\underline{\underline{\varepsilon}}_f(\underline{v}_f)) + \underline{\underline{\nabla}}_x p = \rho_f \underline{f}_f, & \text{in} \quad \Omega^f(t), \\
\underline{\underline{\nabla}}_x \cdot \underline{v}_f = 0, & \text{in} \quad \Omega^f(t),
\end{cases}
$$

$$
\begin{cases}
\underline{v}_f = \underline{v}_s, & \text{on} \quad \Sigma(t), \\
\underline{\underline{\Pi}}_s \cdot \underline{n}_s = J_f \underline{\underline{\sigma}}_f(\underline{v}_f, p_f)\cdot (\underline{\underline{F}}^f)^{-T} \cdot \underline{n}_s, & \text{on} \quad \Sigma_0
\end{cases}
$$

$$
\begin{cases}
\rho_s \dfrac{\partial \underline{v}_s}{\partial t} - \underline{\underline{\nabla}}_{\xi} \cdot (\underline{\underline{\Pi}}_s) = 0, & \text{in} \quad \Omega^s_0, \\
\underline{\underline{\sigma}}_s \cdot \underline{n}_s = -k_s(\underline{y}_s - \underline{y}_b), & \text{on} \quad \Gamma^s_{in}, \\
\underline{\underline{\sigma}}_s \cdot \underline{n}_s = -k_s \underline{y}_s - c_s \underline{v}_s, & \text{on} \quad \Gamma_s
\end{cases}
$$

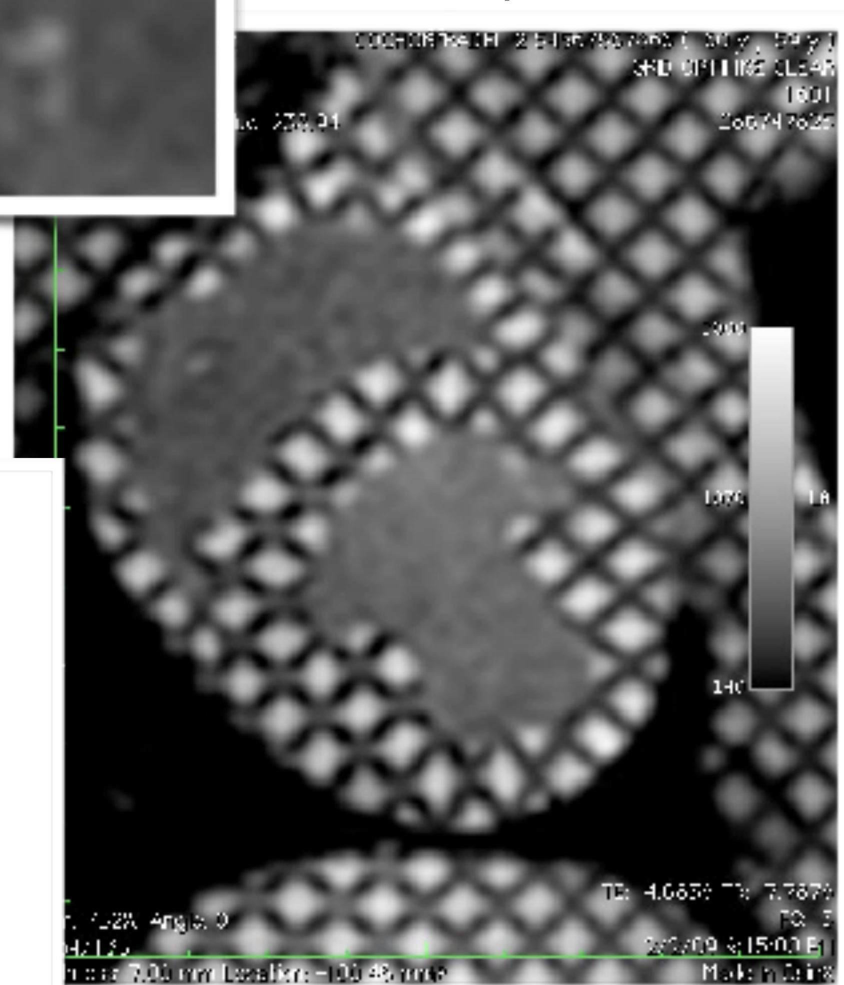
P. Moireau, N. Xiao, M. Astorino, C.A. Figueroa, D. Chapelle, C.A. Taylor & J.-F. Gerbeau — **External tissue support and fluid–structure simulation in blood flows**. *BMMB, 2012*

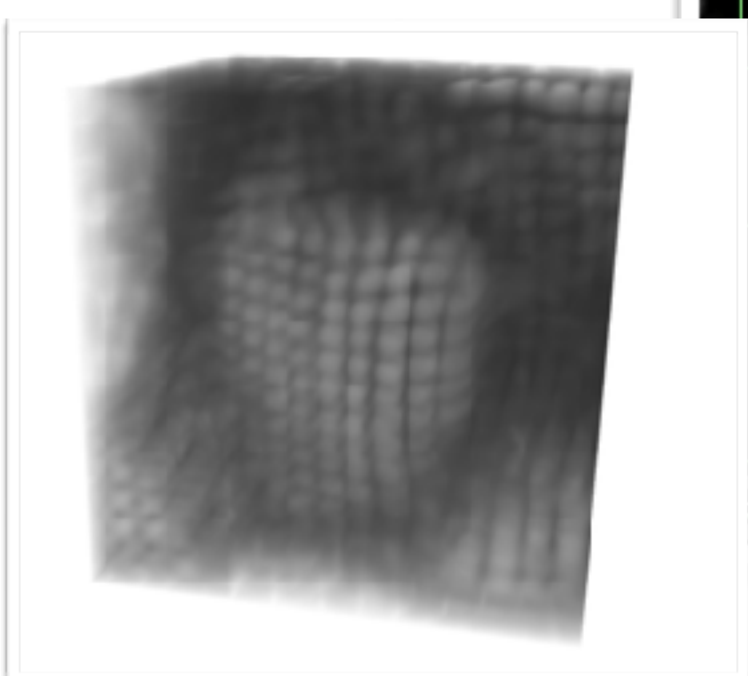
Measurements



Courtesy: King's College London

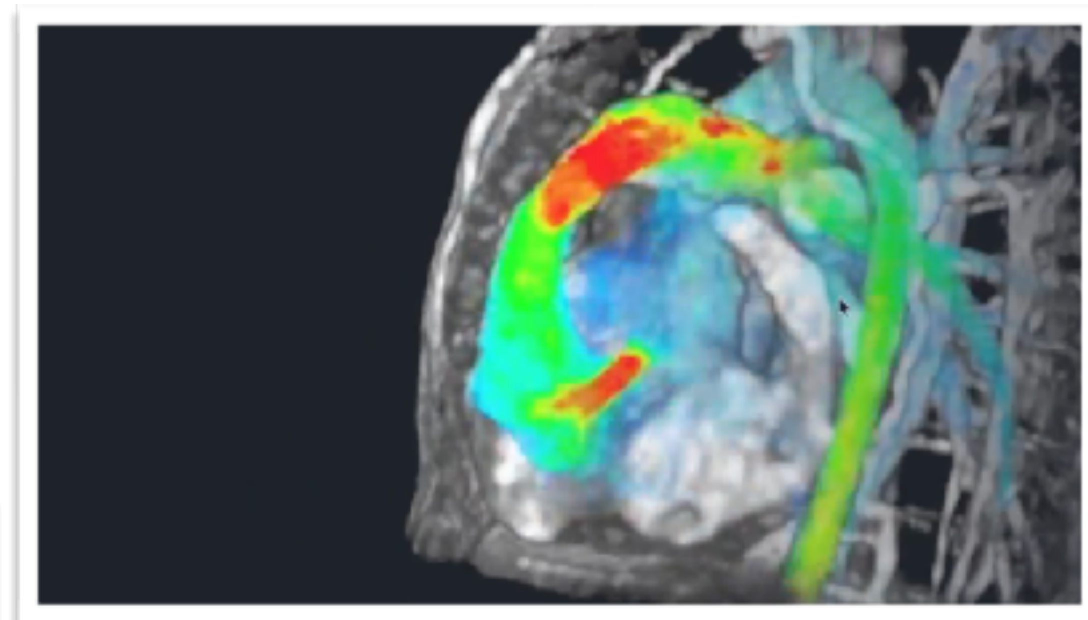
Courtesy: Mondor Créteil



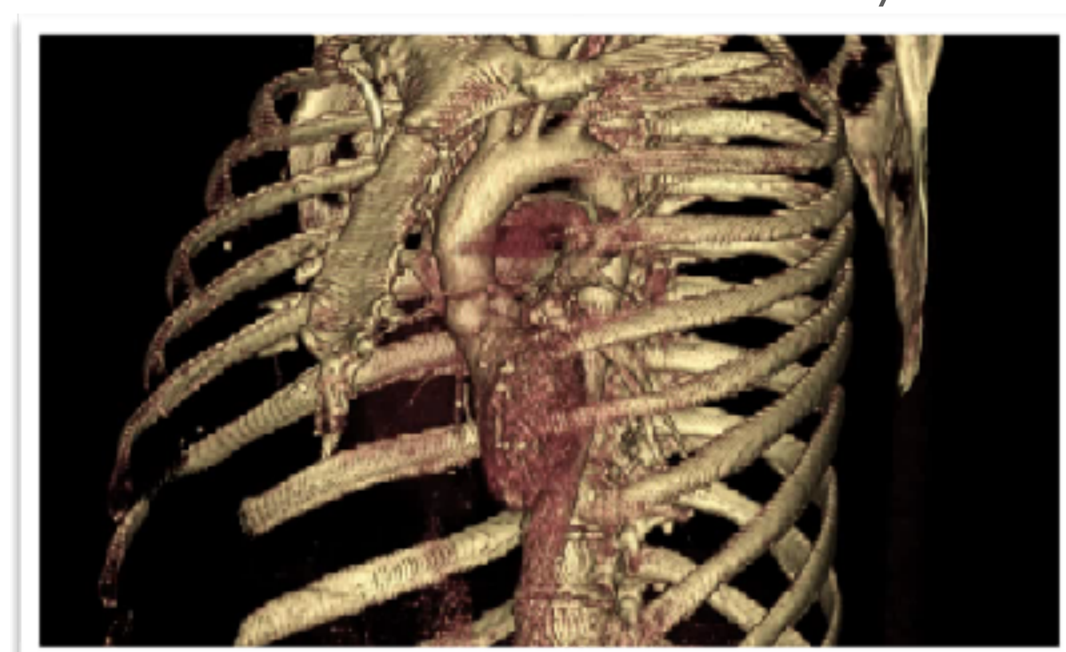


Courtesy: ETH

Courtesy: Phase Contrast MRI from Arterys



Courtesy Stanford



New challenges: shape observations

Observations of surfaces

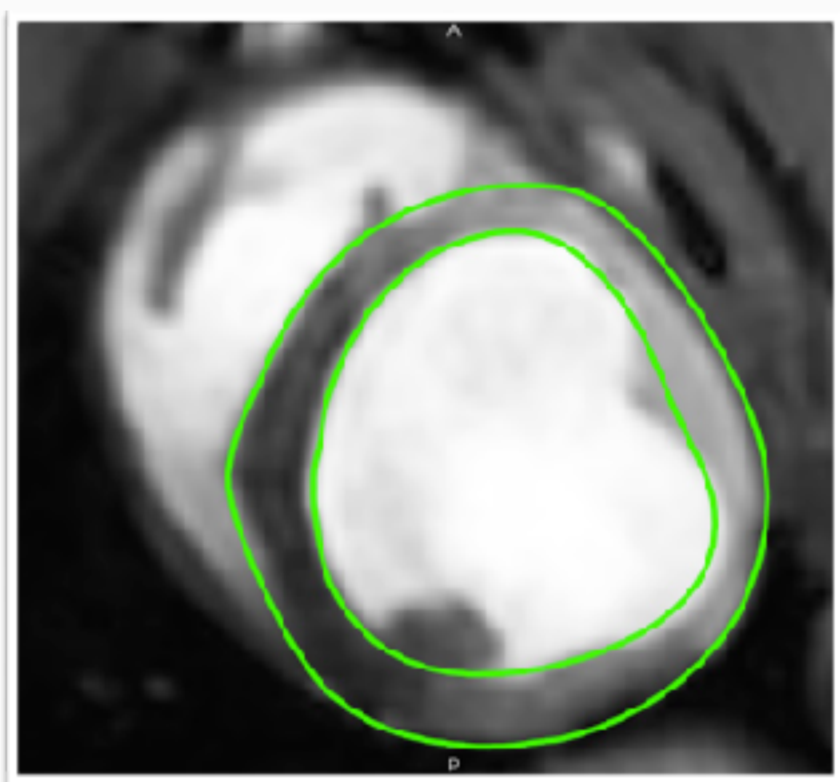
- Remember we can consider complex data termes

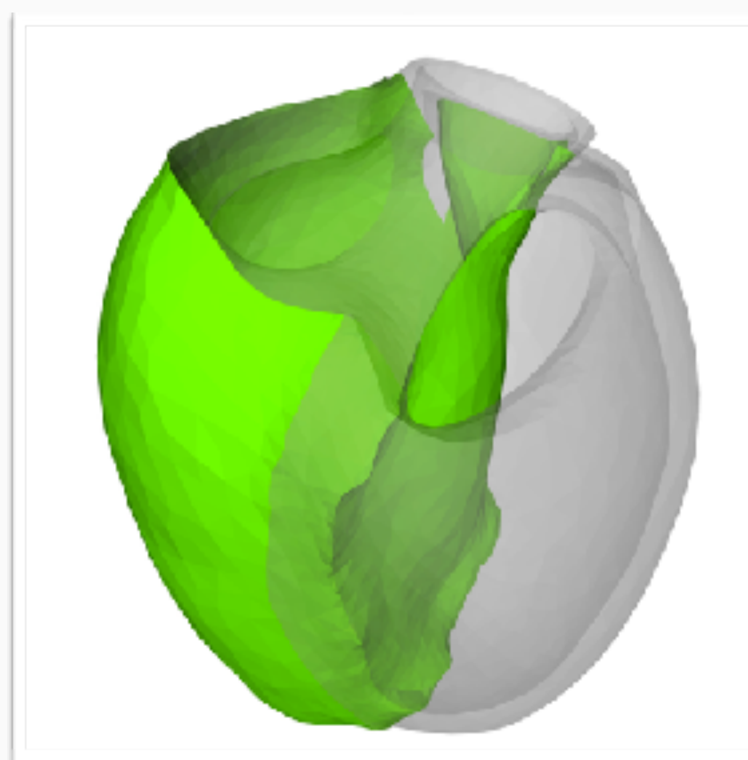
$$\mathscr{J}_{data}(z^{\delta}, y) = \int_{\Sigma} \left|\mathrm{dist}_{\mathfrak{S}}(\underline{x} + \underline{u}(\underline{x}, t), t)\right|^{2} \, \mathrm{d}\Gamma$$

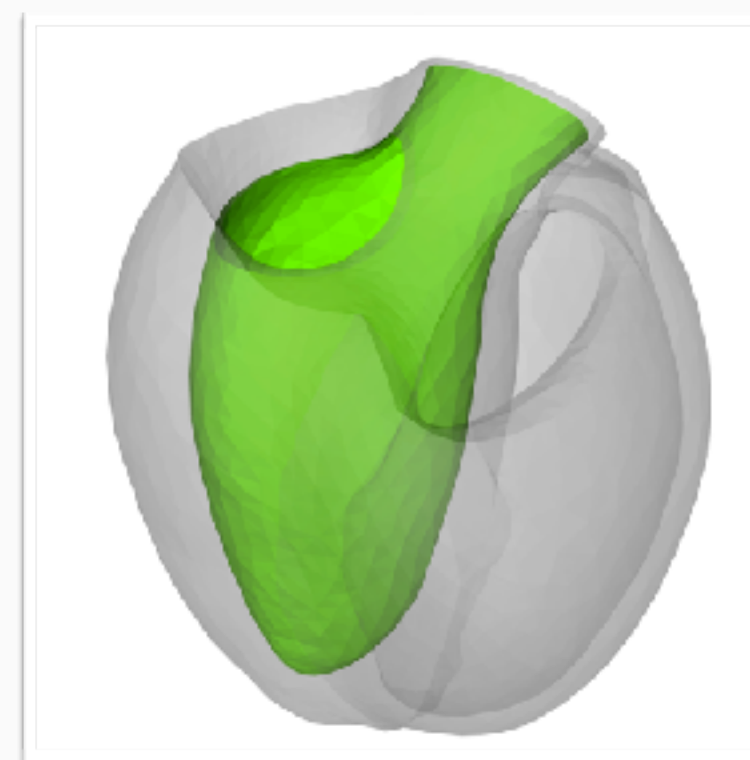
- in a Non-linear elasticity case

$$\forall \underline{w} \in \mathcal{V}, \left(\partial_t \hat{\underline{u}}, \underline{w}_s\right)_{\mathcal{E}_l} = \left(\hat{\underline{v}}, \underline{w}\right)_{\mathcal{E}_l} + \gamma\left(L_{\underline{n}_{\Sigma}}(\mathrm{dist}_{\mathfrak{S}}(\underline{x} + \hat{\underline{u}}, t)\underline{n}_{\mathfrak{S}}), \underline{w}\right)_{\mathcal{E}_l},$$

- Stability: after linearisation equivalent to observation of normal displacement to the surface







Observations of surfaces

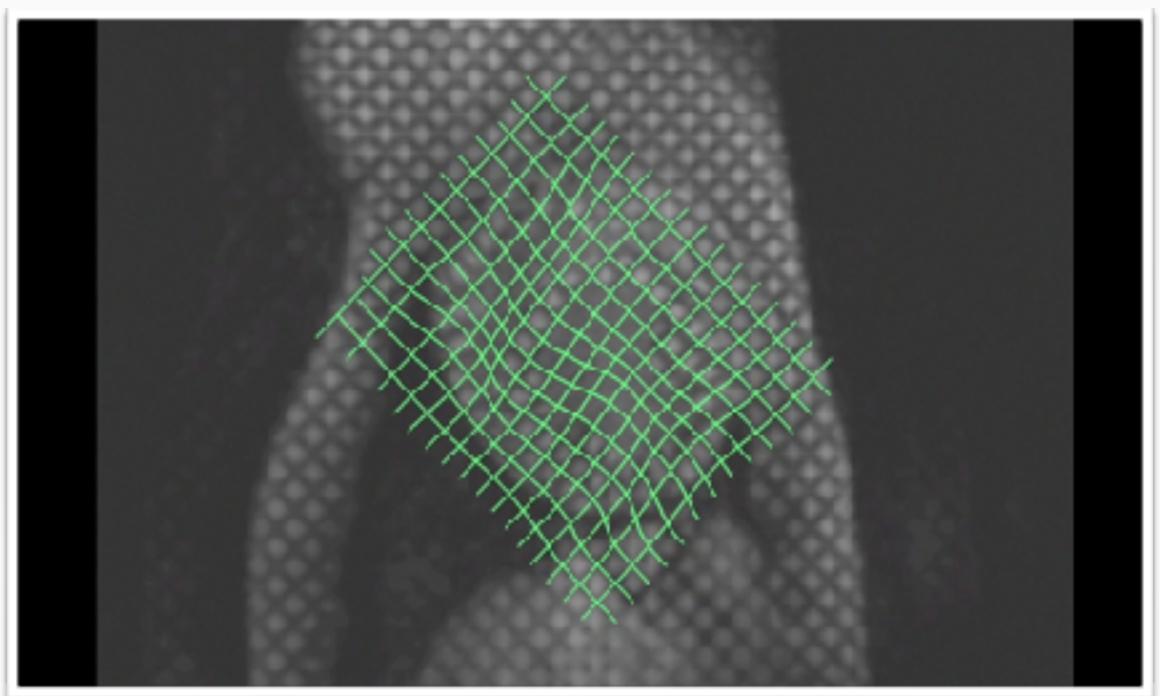
- Remember we can consider complex data termes

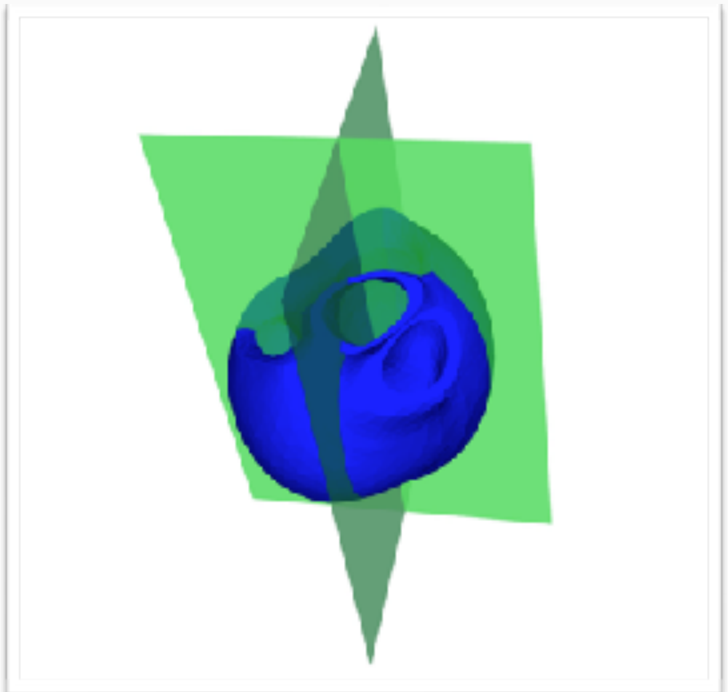
$$\mathscr{J}_{data}(z^{\delta}, y) = \int_{\Sigma} \left|\mathrm{dist}_{\mathfrak{S}}(\underline{x} + \underline{u}(\underline{x}, t), t)\right|^2 \mathrm{d}\Gamma$$

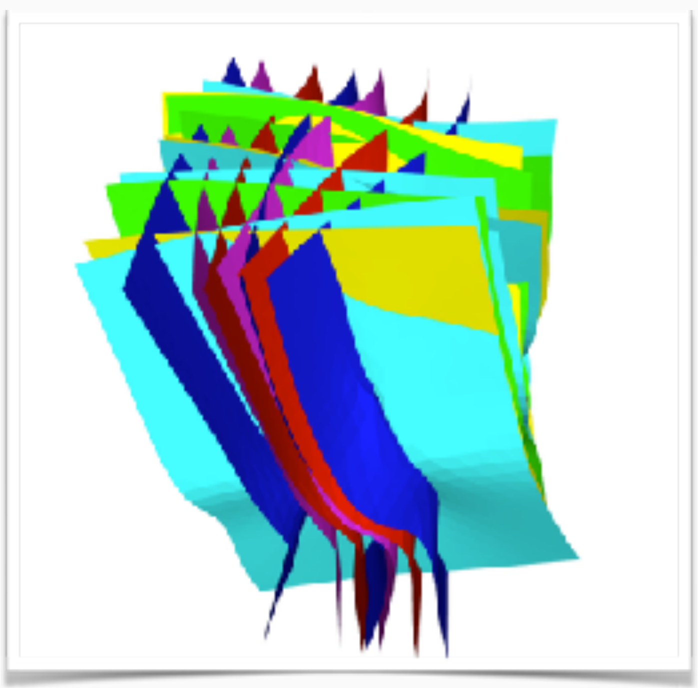
- in a Non-linear elasticity case

$$\forall \underline{w} \in \mathcal{V}, \left(\partial_t \underline{\hat{u}}, \underline{w}_s\right)_{\mathcal{E}_l} = \left(\underline{\hat{v}}, \underline{w}\right)_{\mathcal{E}_l} + \gamma\left(L_{\underline{n}_{\Sigma}}(\mathrm{dist}_{\mathfrak{S}}(\underline{x} + \underline{\hat{u}}, t)\underline{n}_{\mathfrak{S}}), \underline{w}\right)_{\mathcal{E}_l},$$

- Stability: after linearisation equivalent to observation of normal displacement to the surface

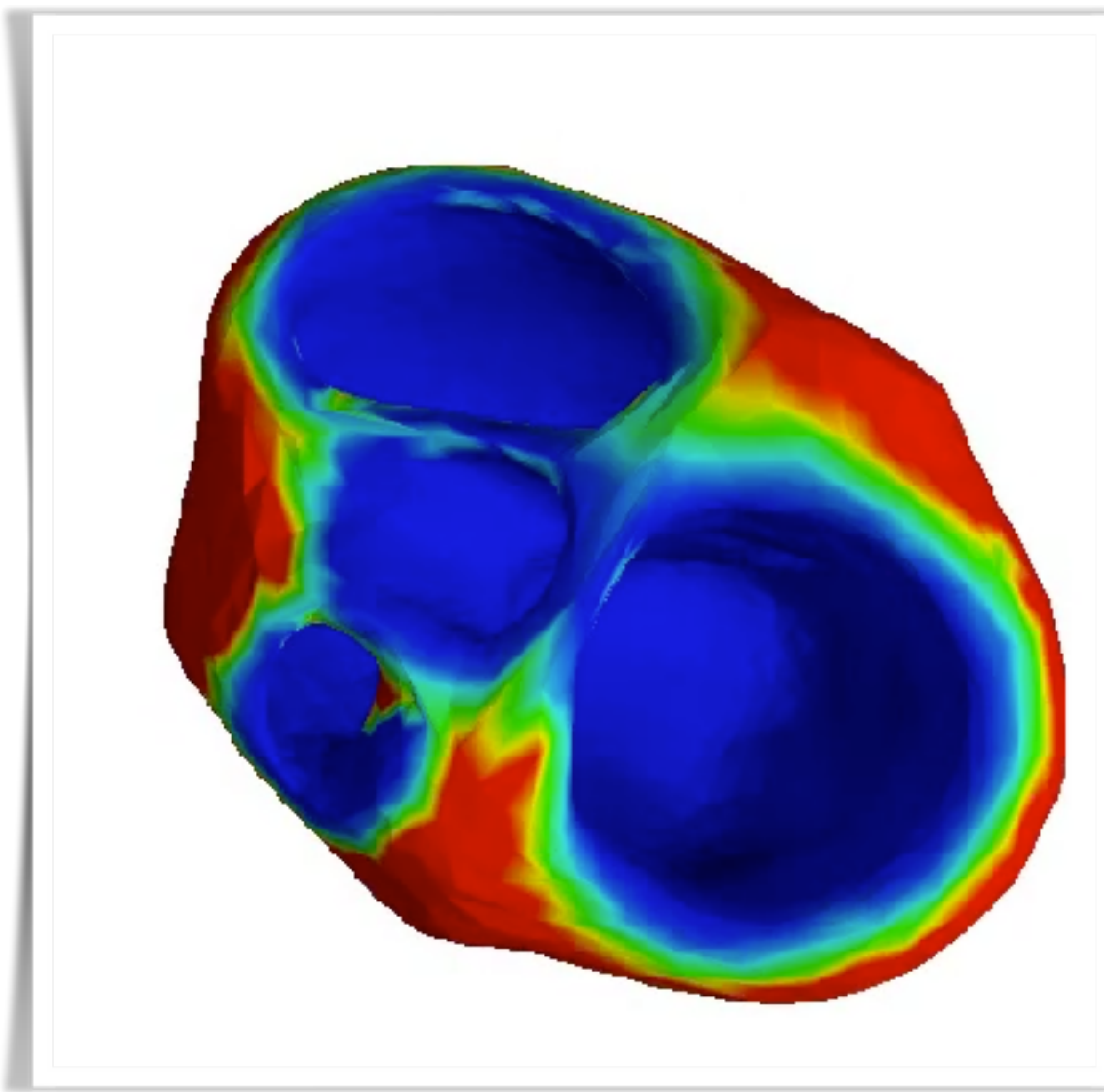


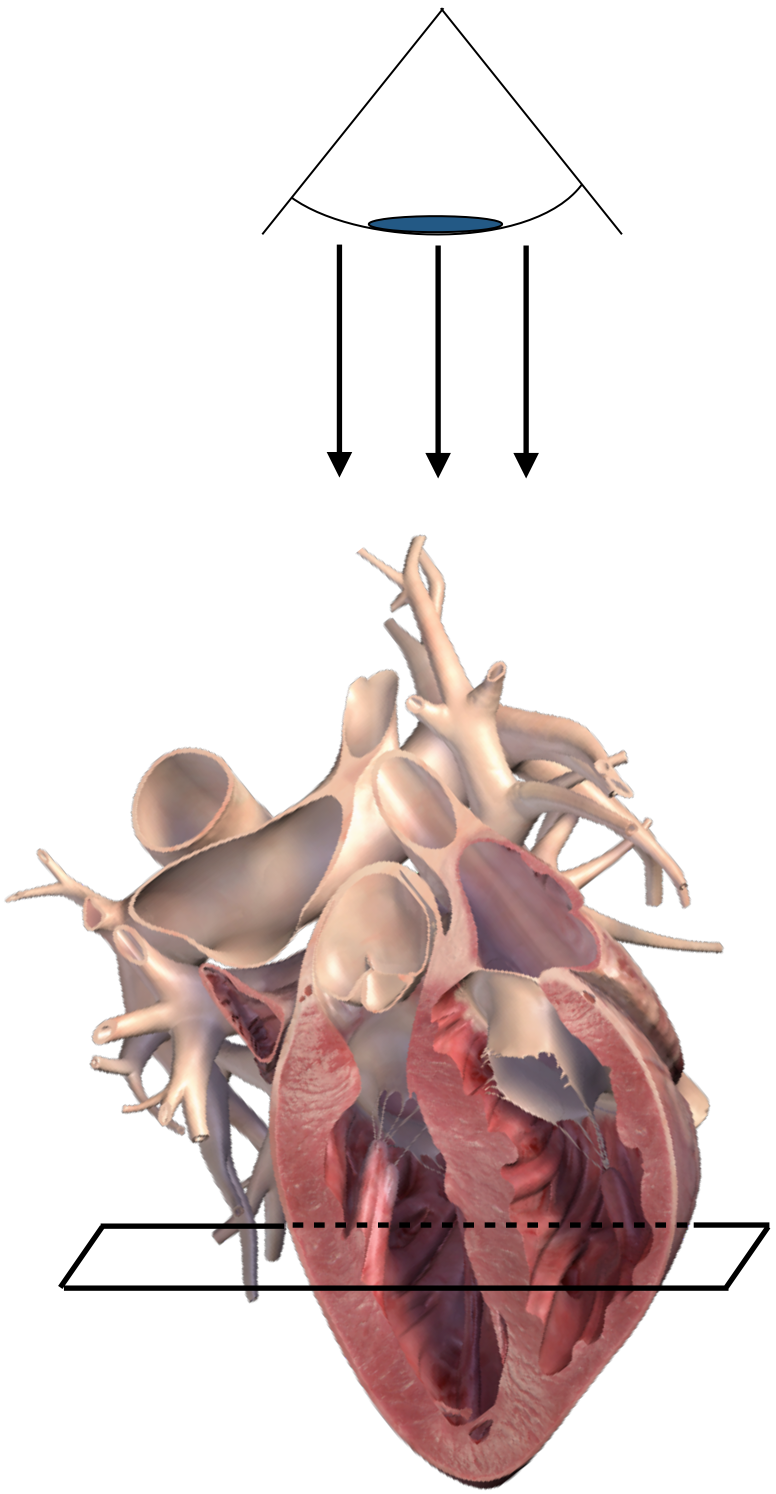




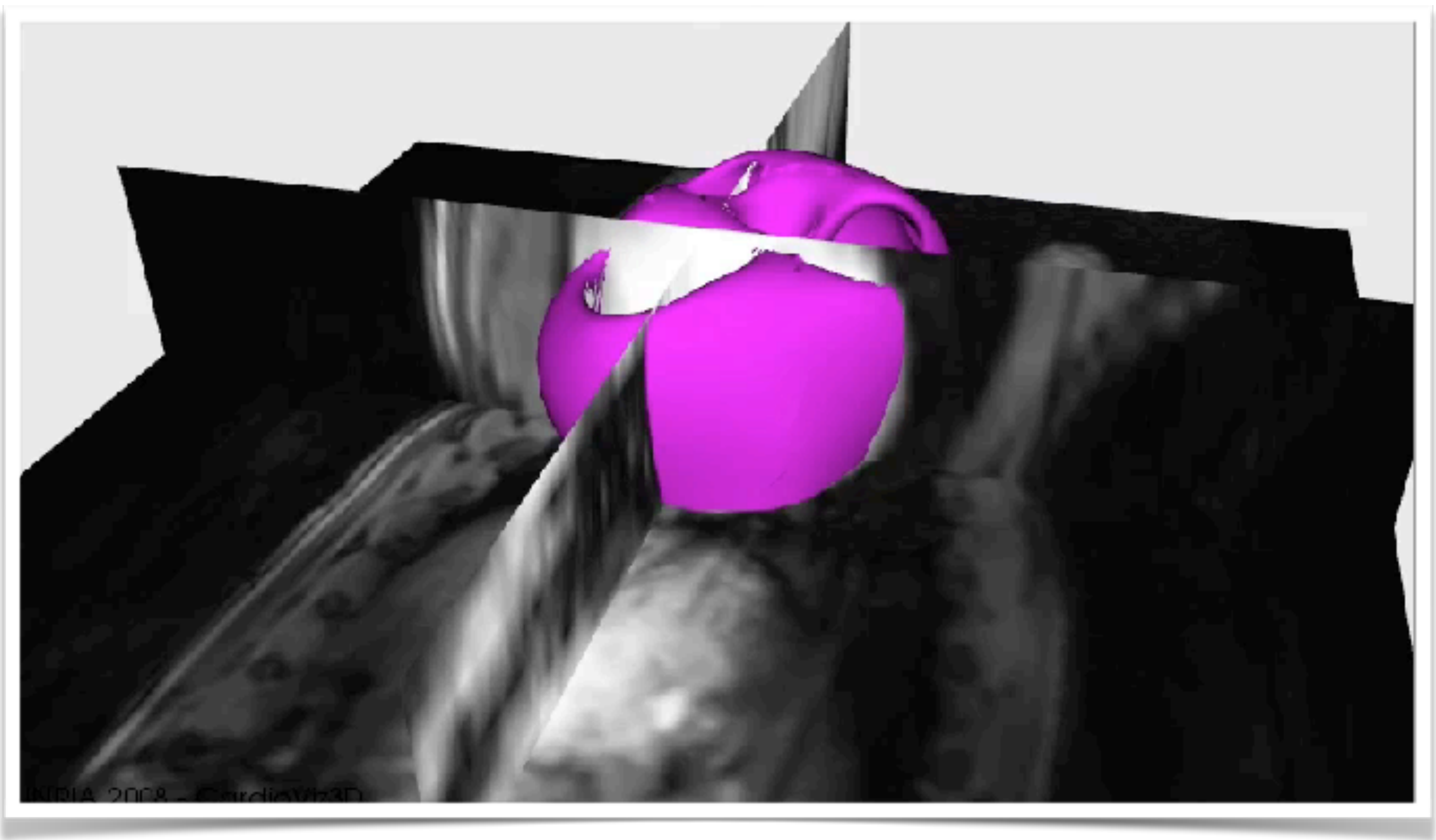
A first real data case

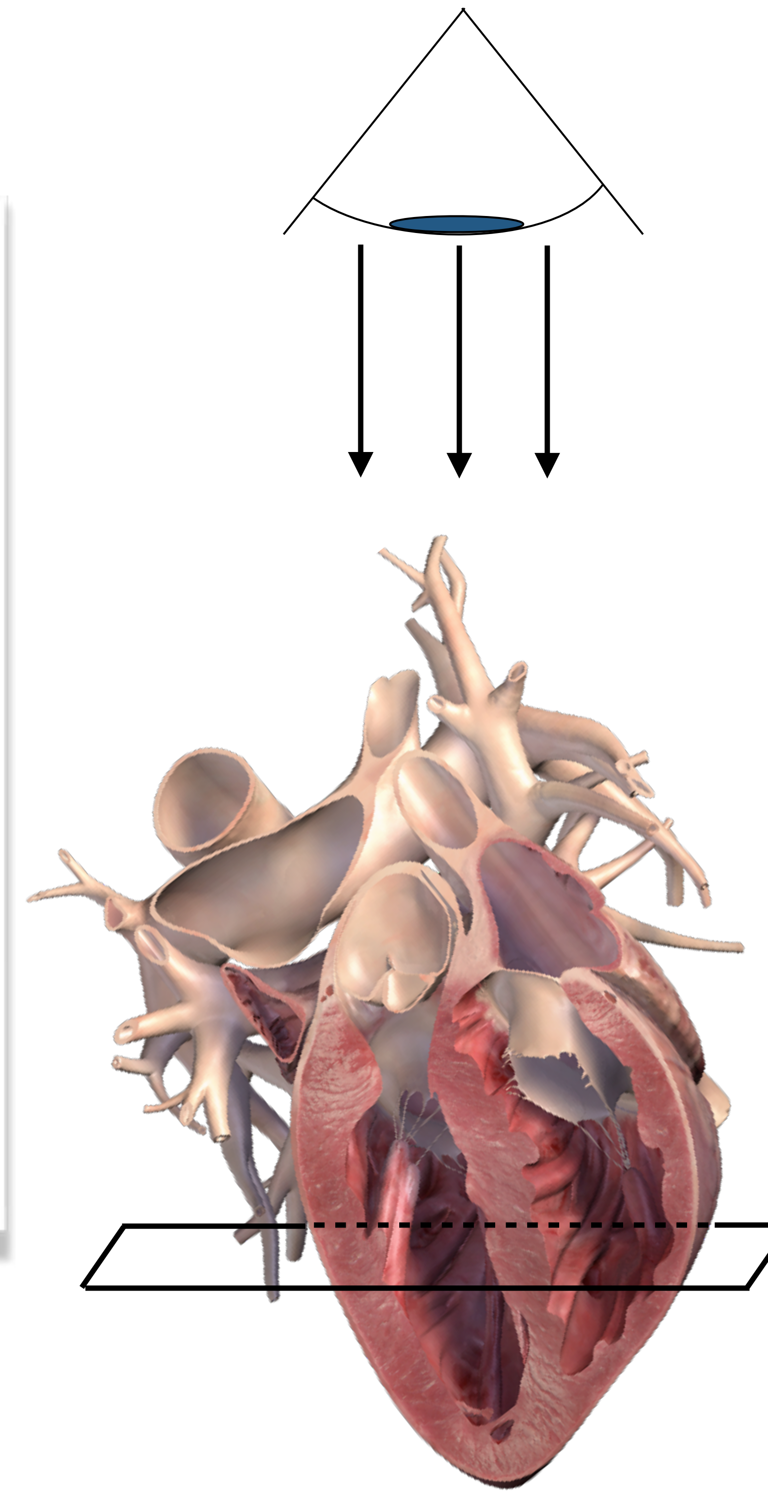
Model Calibration on Baseline (T0)



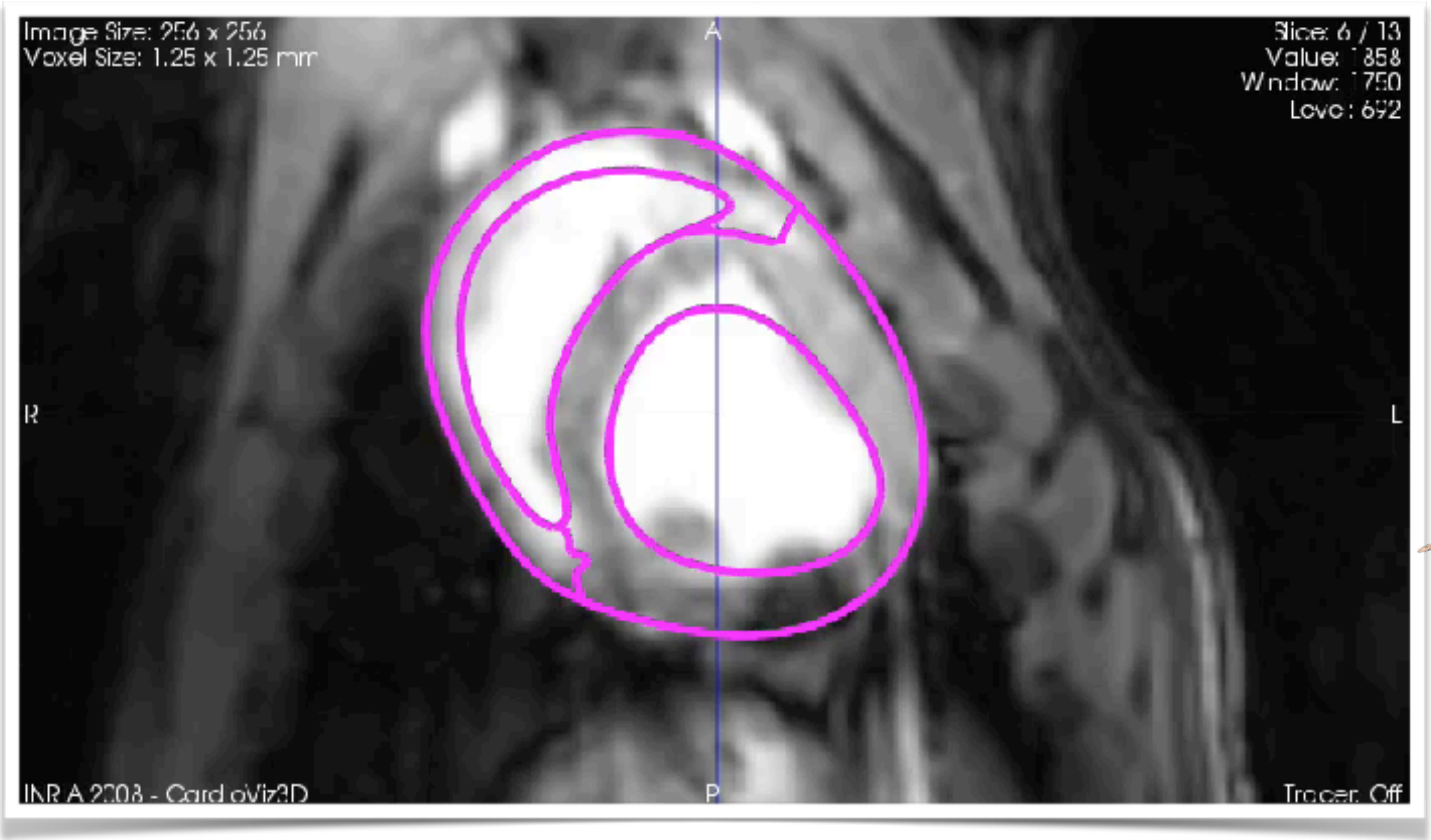


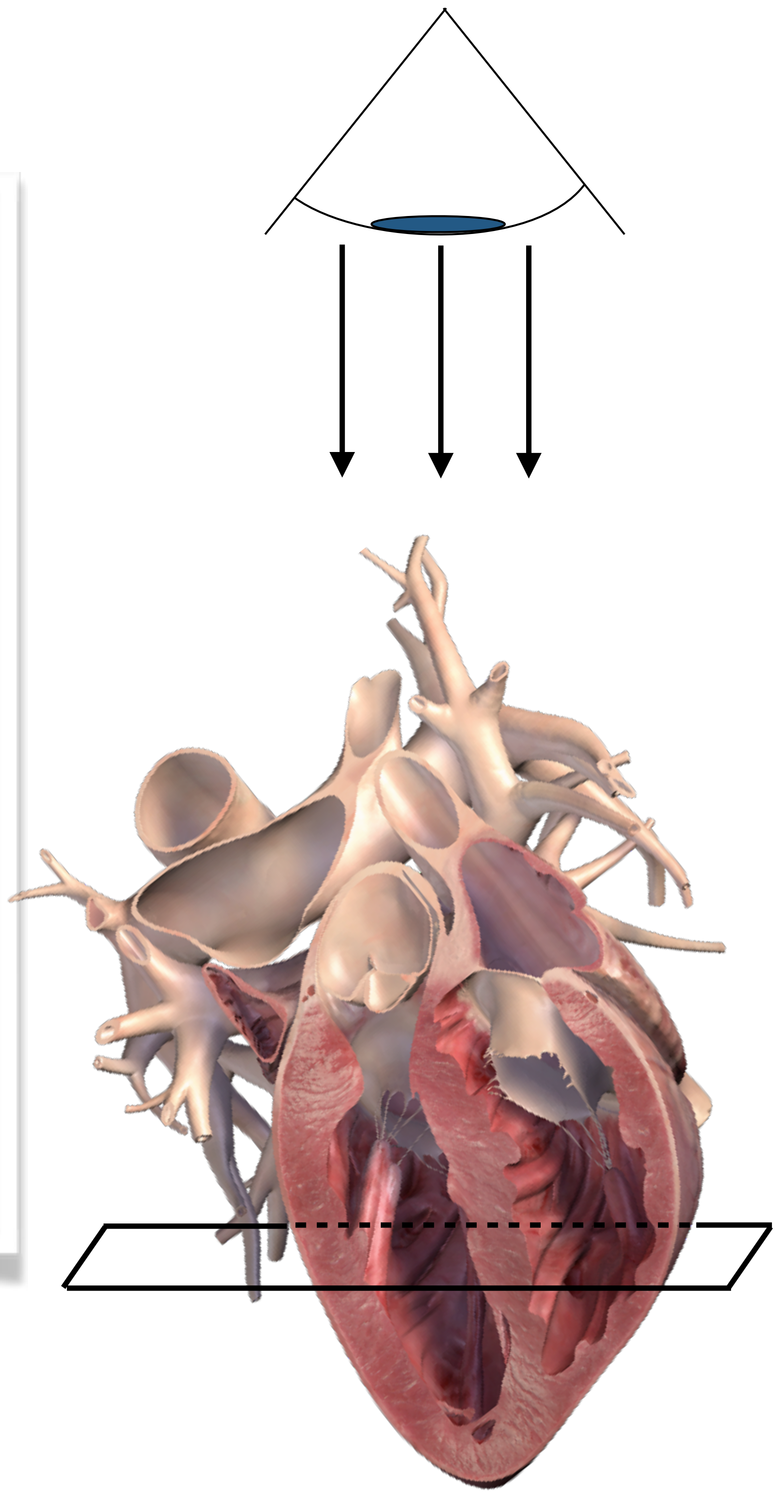
Model Calibration on Baseline (T0)



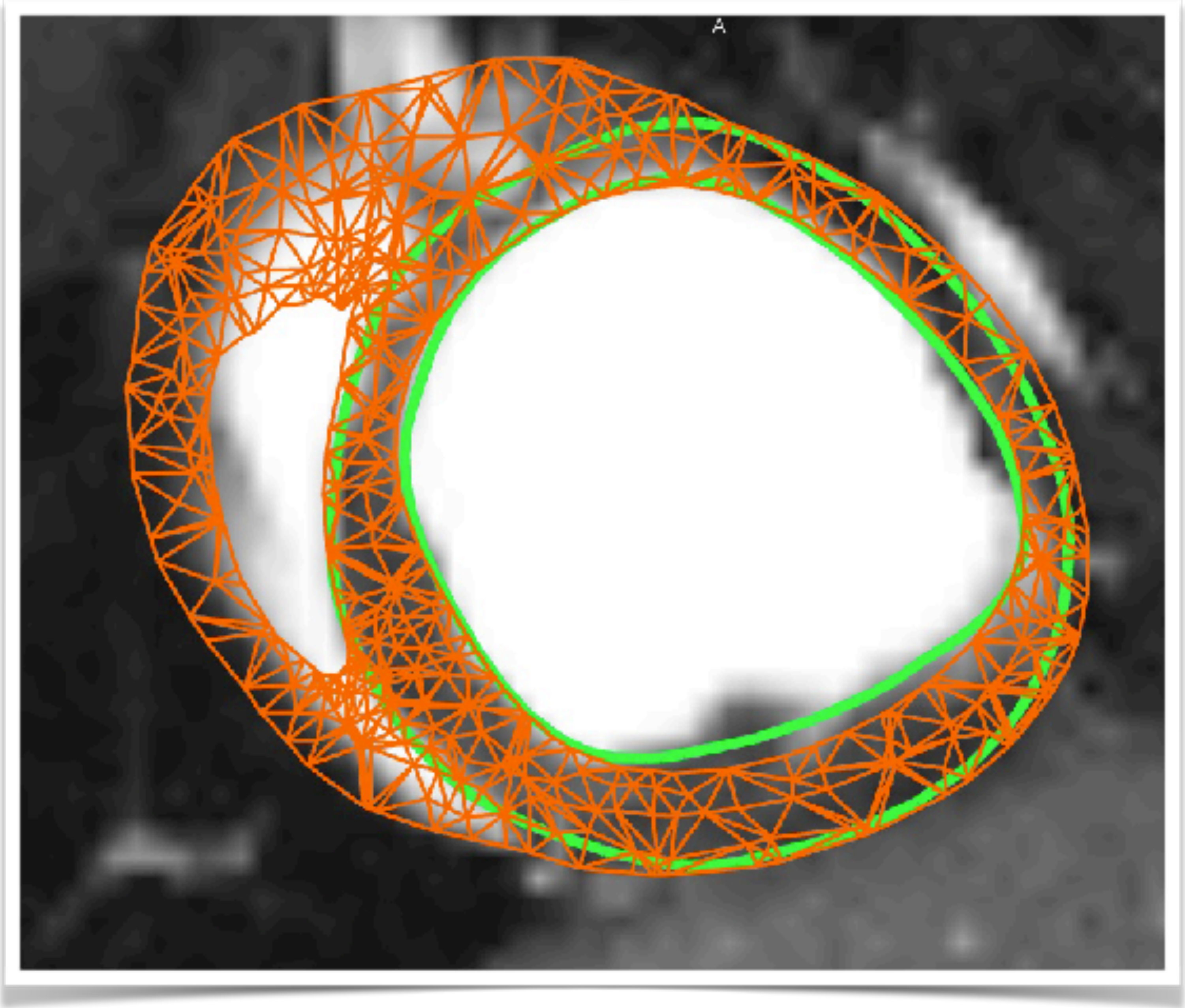


Model Calibration on Baseline (T0)





Baseline (T0) simulations compared to T38 images

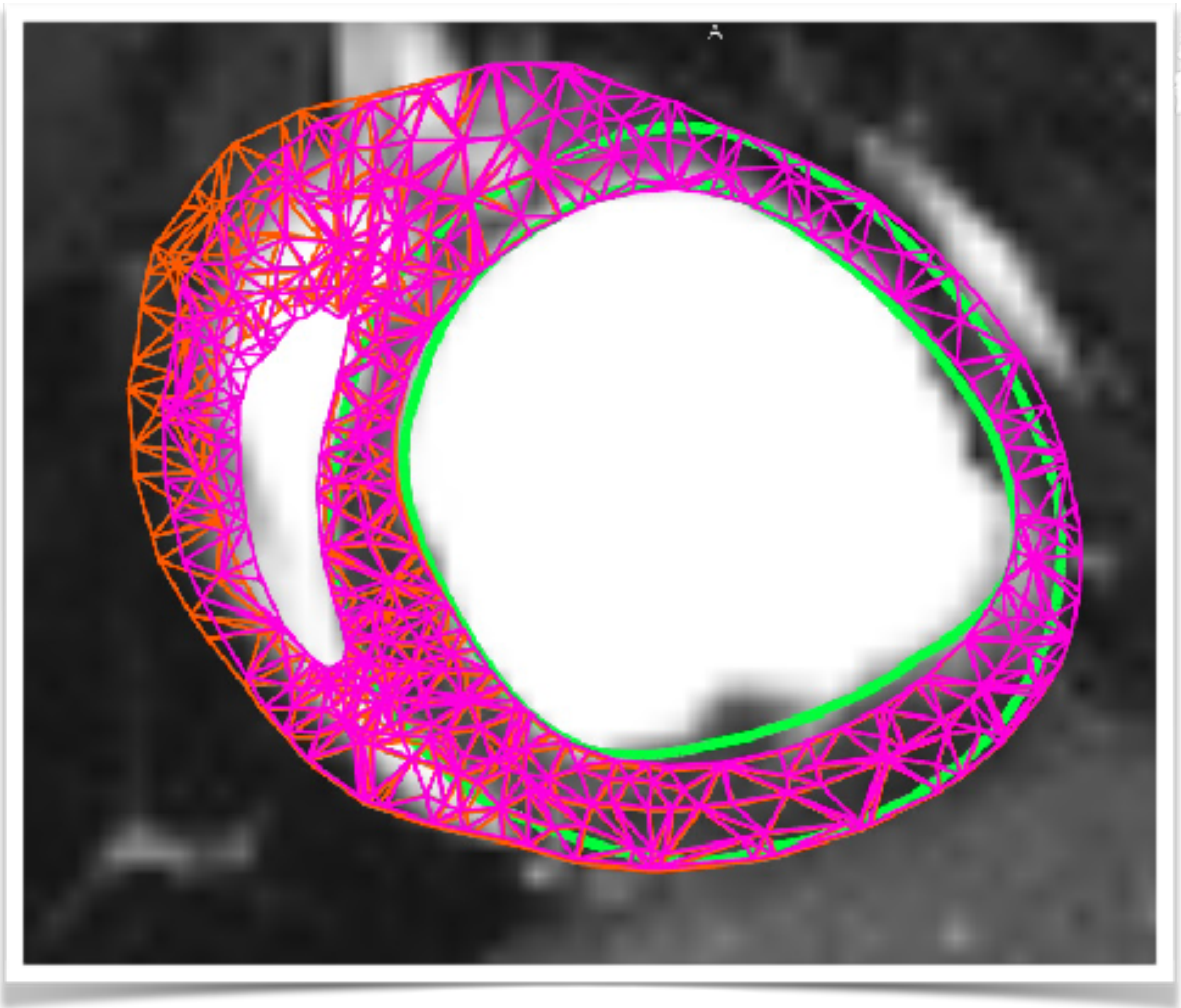


Model *without* infarct

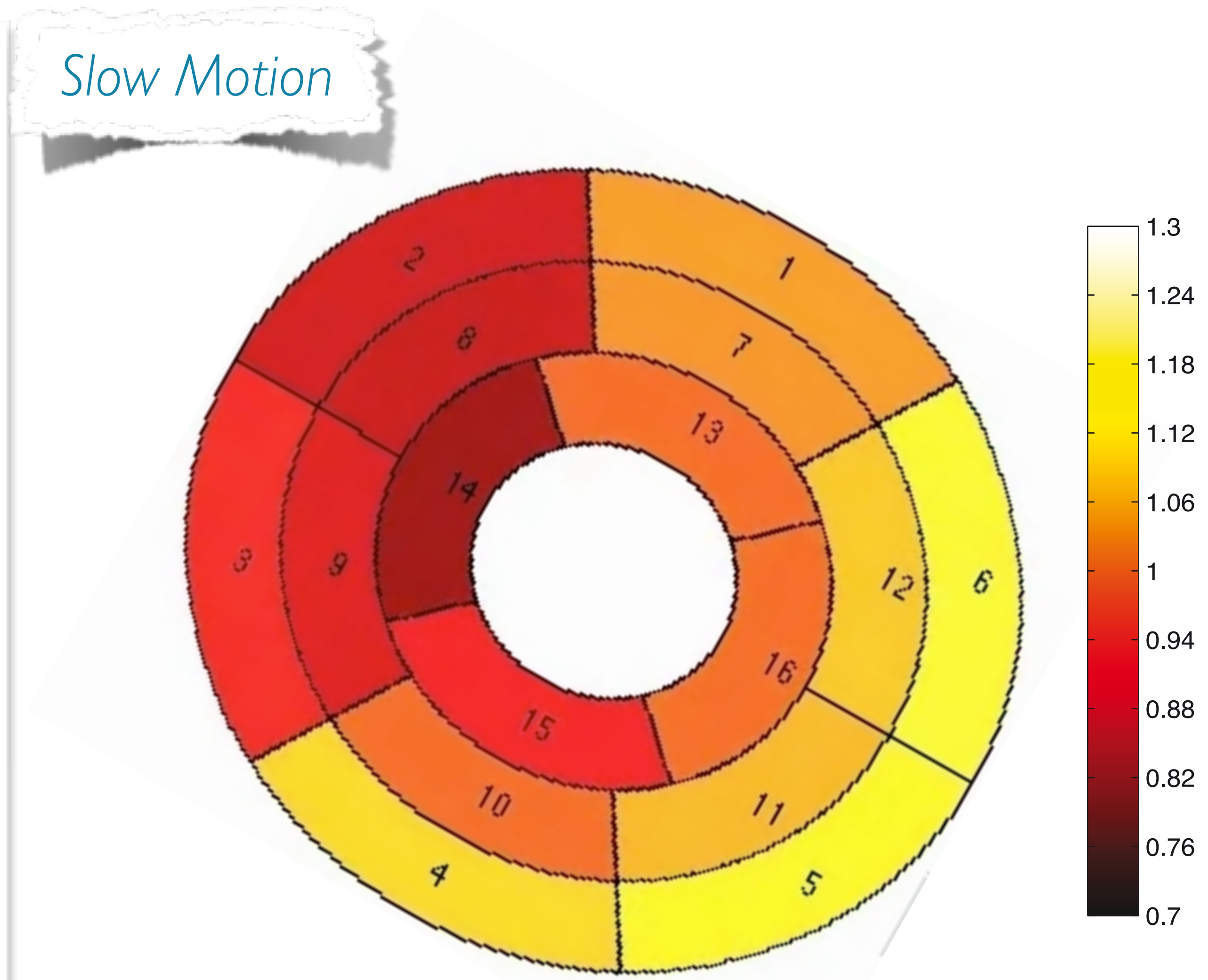
Segmentation of the *infarcted* left ventricle

Slow Motion

Adapt model contractility



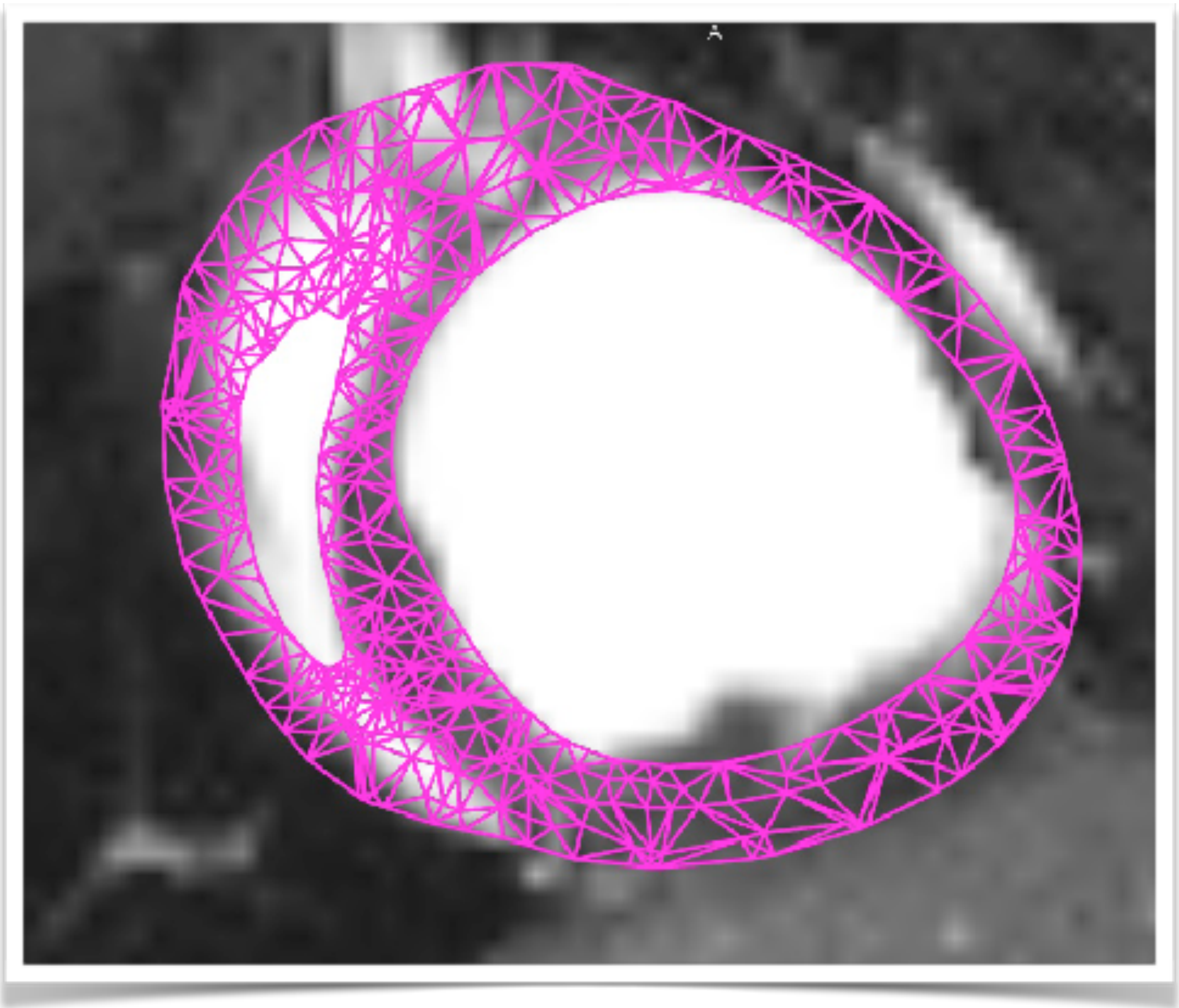
Model without infarct
Estimator
Segmentation

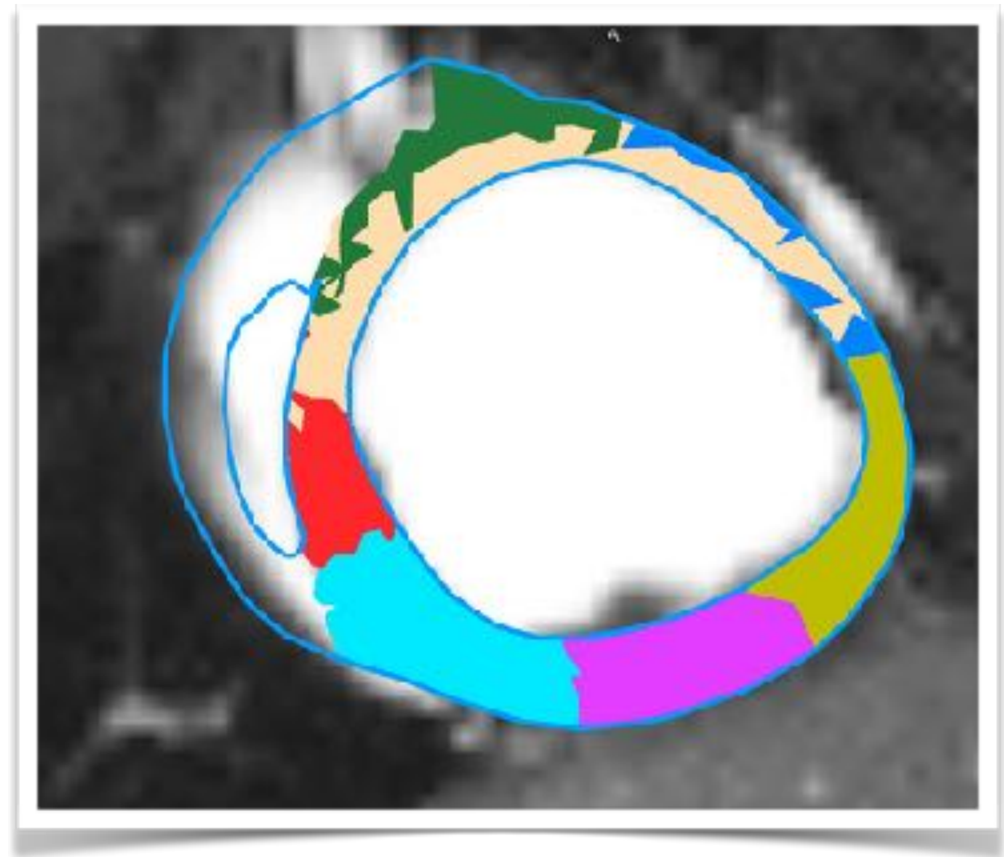


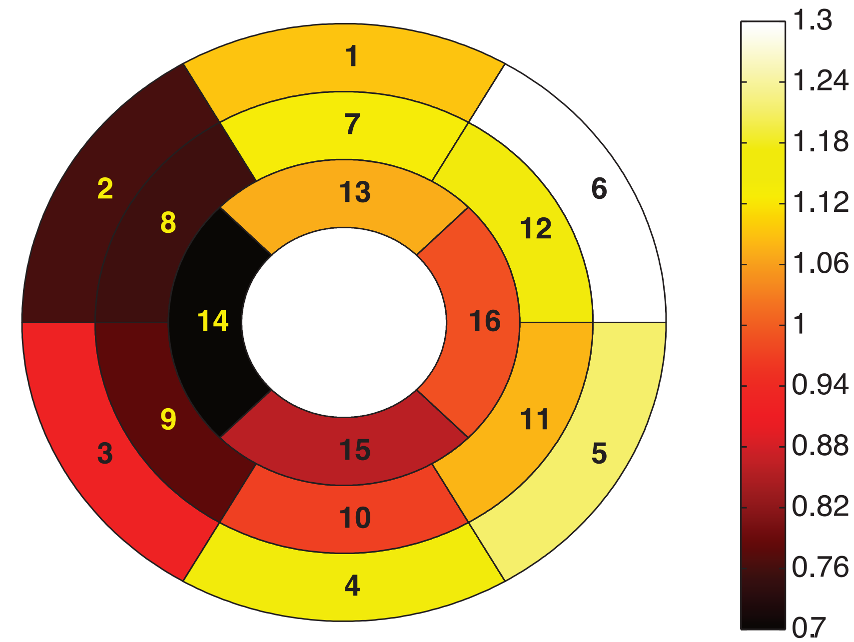
Cardiologist representation of the various regions of the heart

1 parameter estimated by region

Infarct quantification

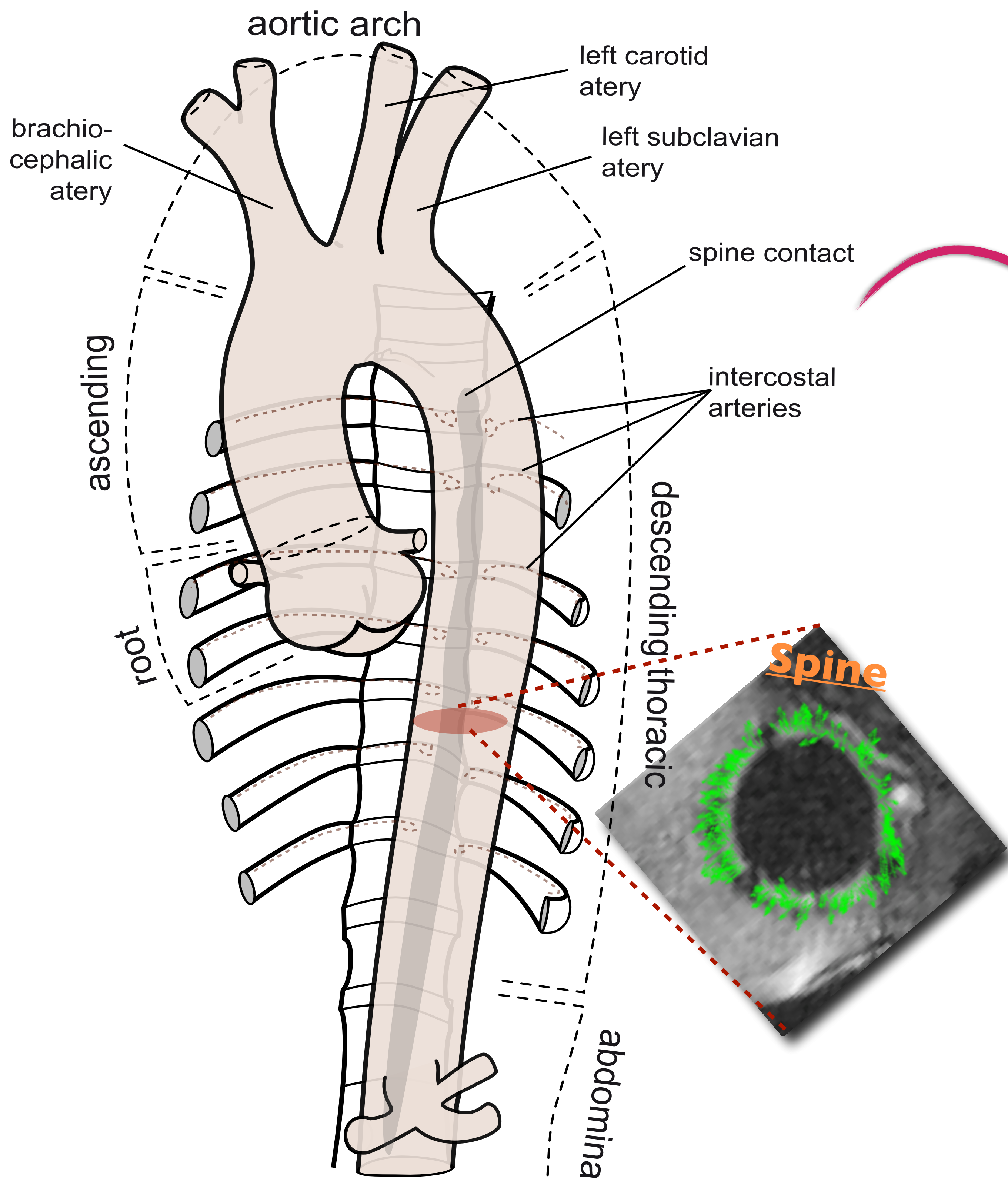


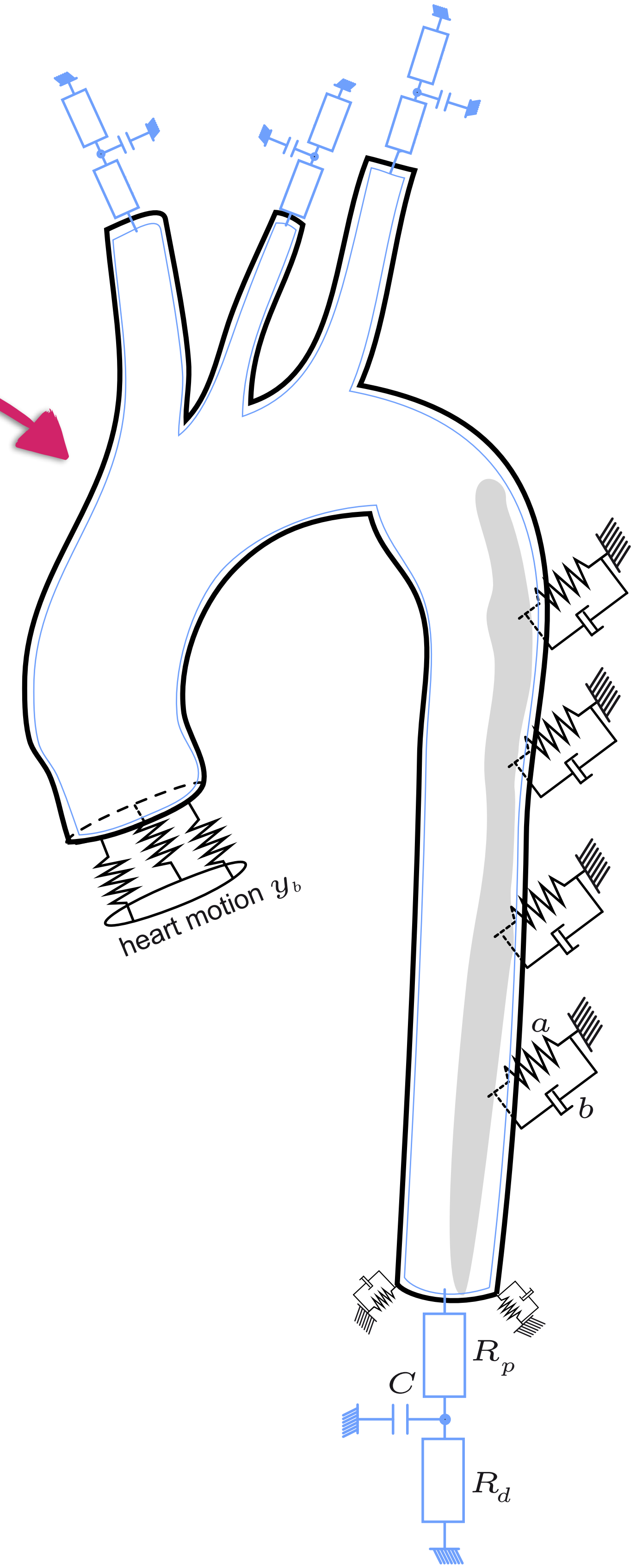




A second real data case

Estimation of elastic boundary support

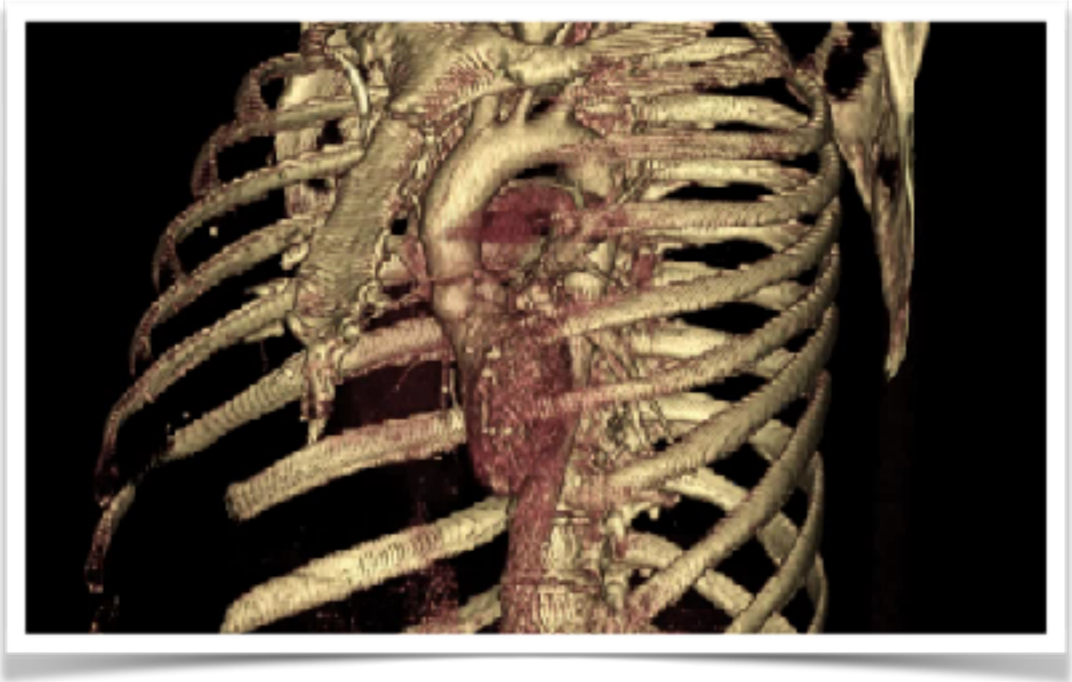




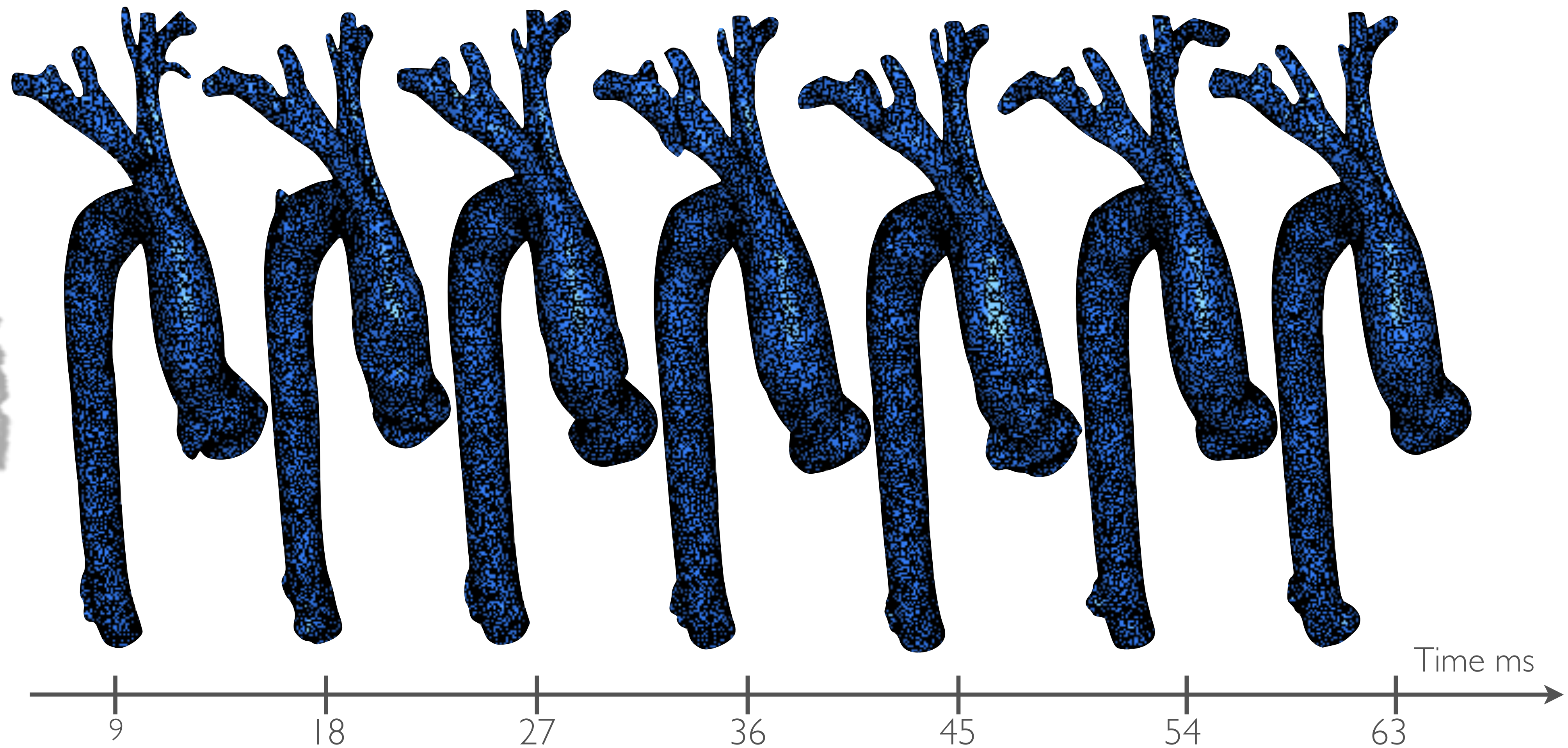
P. Moireau, N. Xiao, M. Astorino, C.A. Figueroa, D. Chapelle, C.A. Taylor & J.-F. Gerbeau — **External tissue support and fluid–structure simulation in blood flows**. *BMMB, 2012*

P. Moireau, C. Bertoglio, N. Xiao, C.A. Figueroa, C.A. Taylor, D. Chapelle & J.-F. Gerbeau — **Sequential identification of boundary support parameters in a fluid-structure vascular model using patient image data — BMMB 2013**

Real Data



CT sequence from Stanford

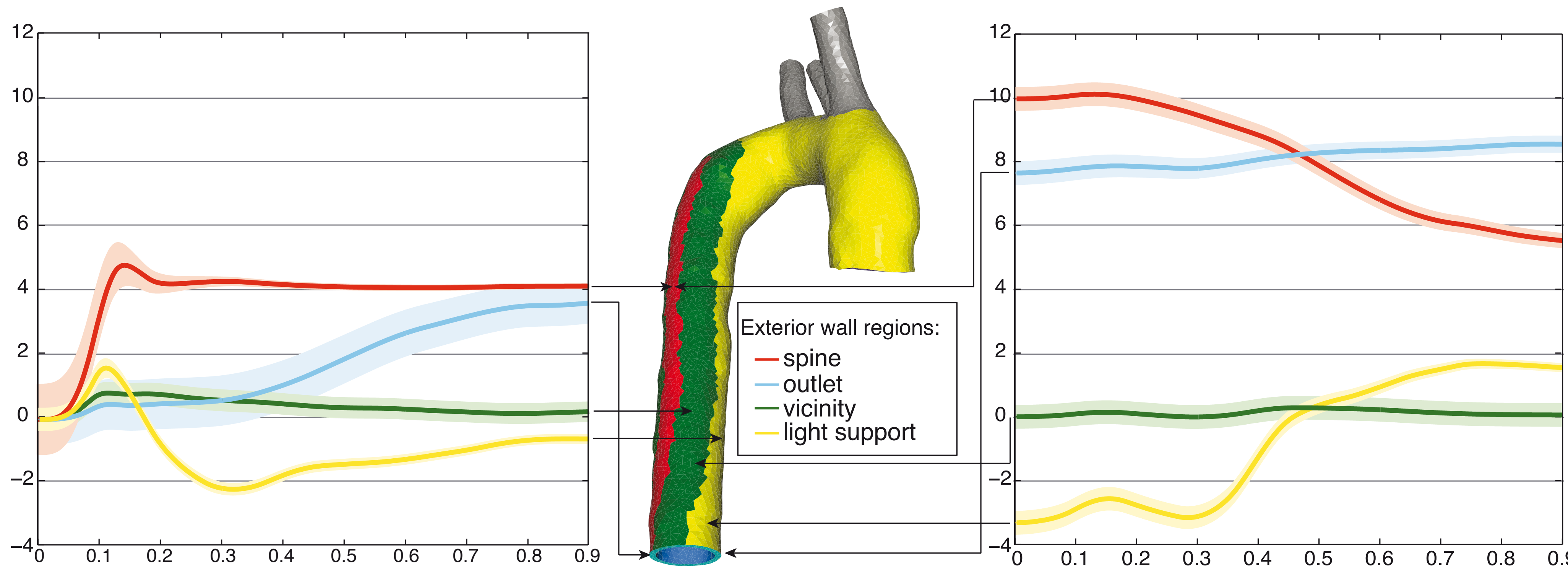


Heart rate: 0.9s

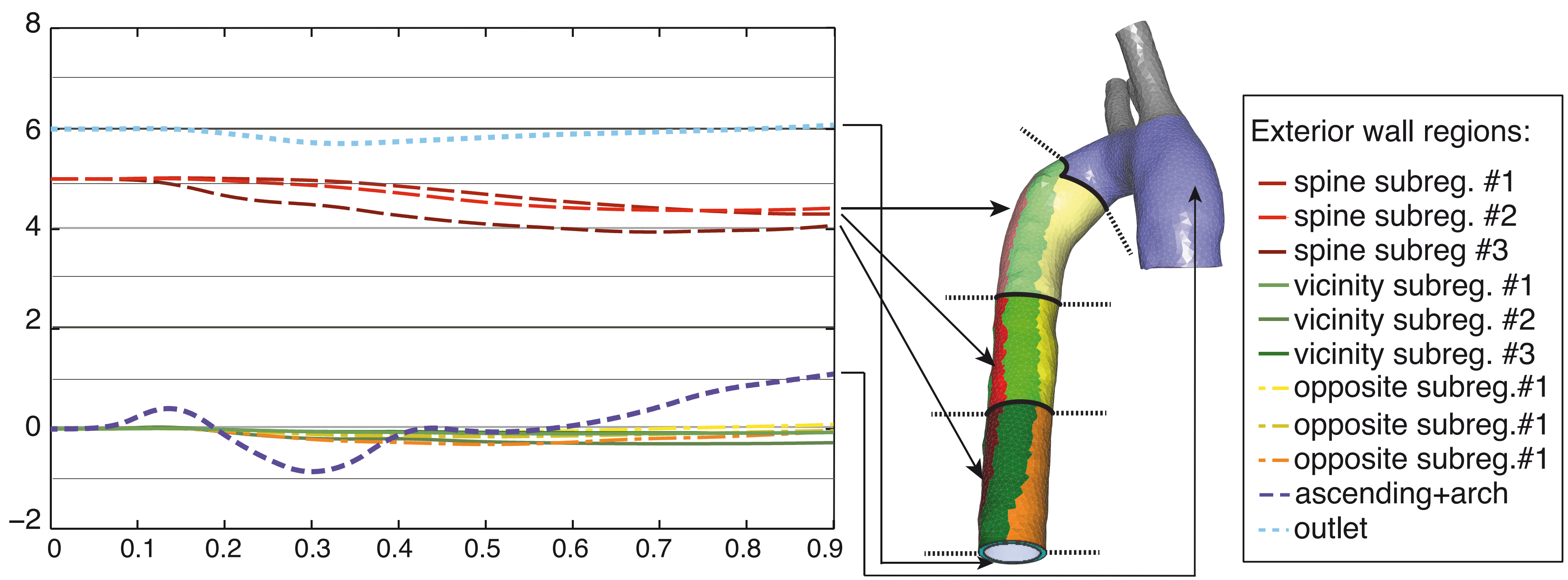
Data from Stanford: N. Xiao, A. Figueroa, C. Taylor

Sequential Estimation Results

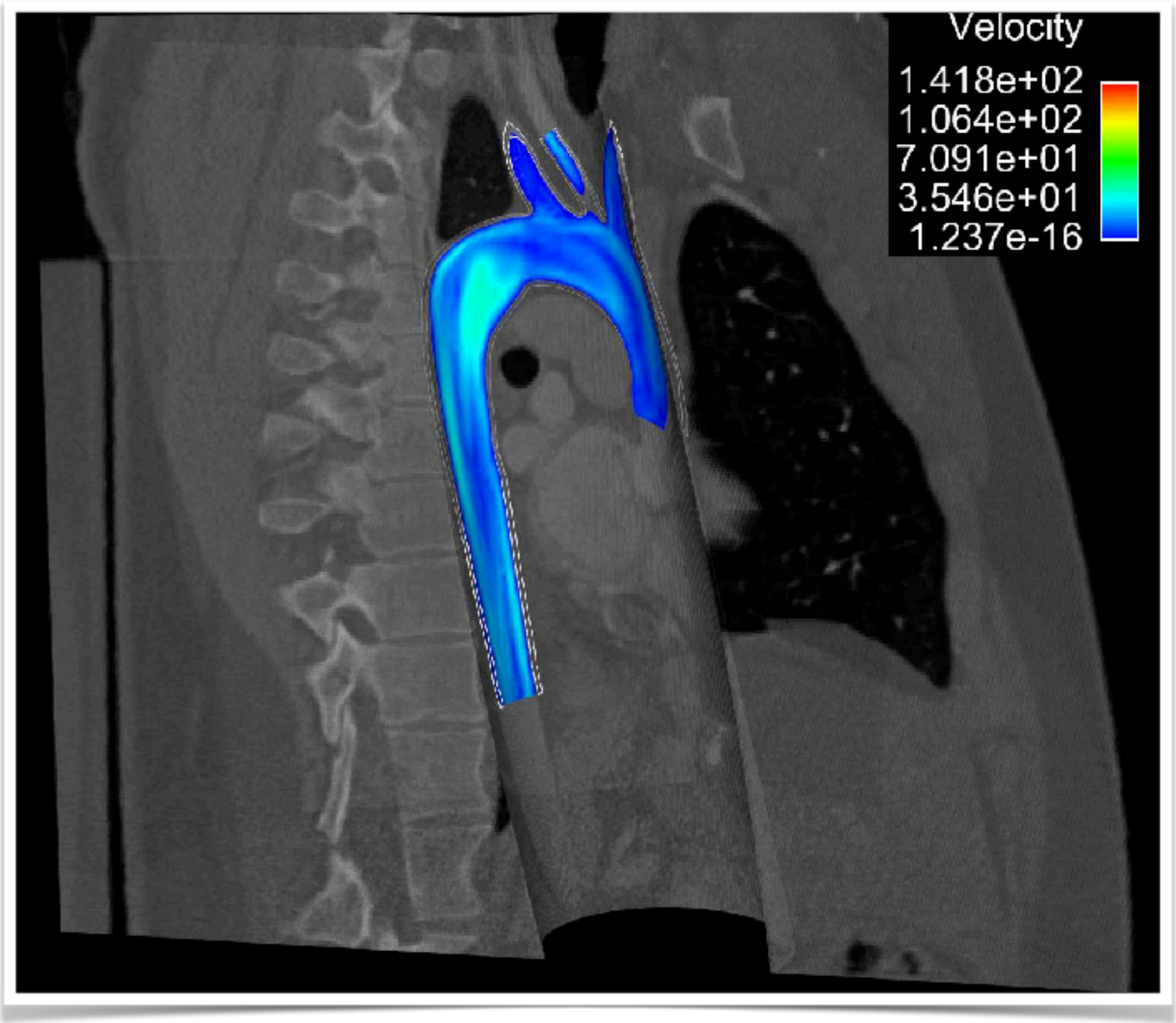
- Case 1a) and 1b)
  5 Regions,
  2 A priori.



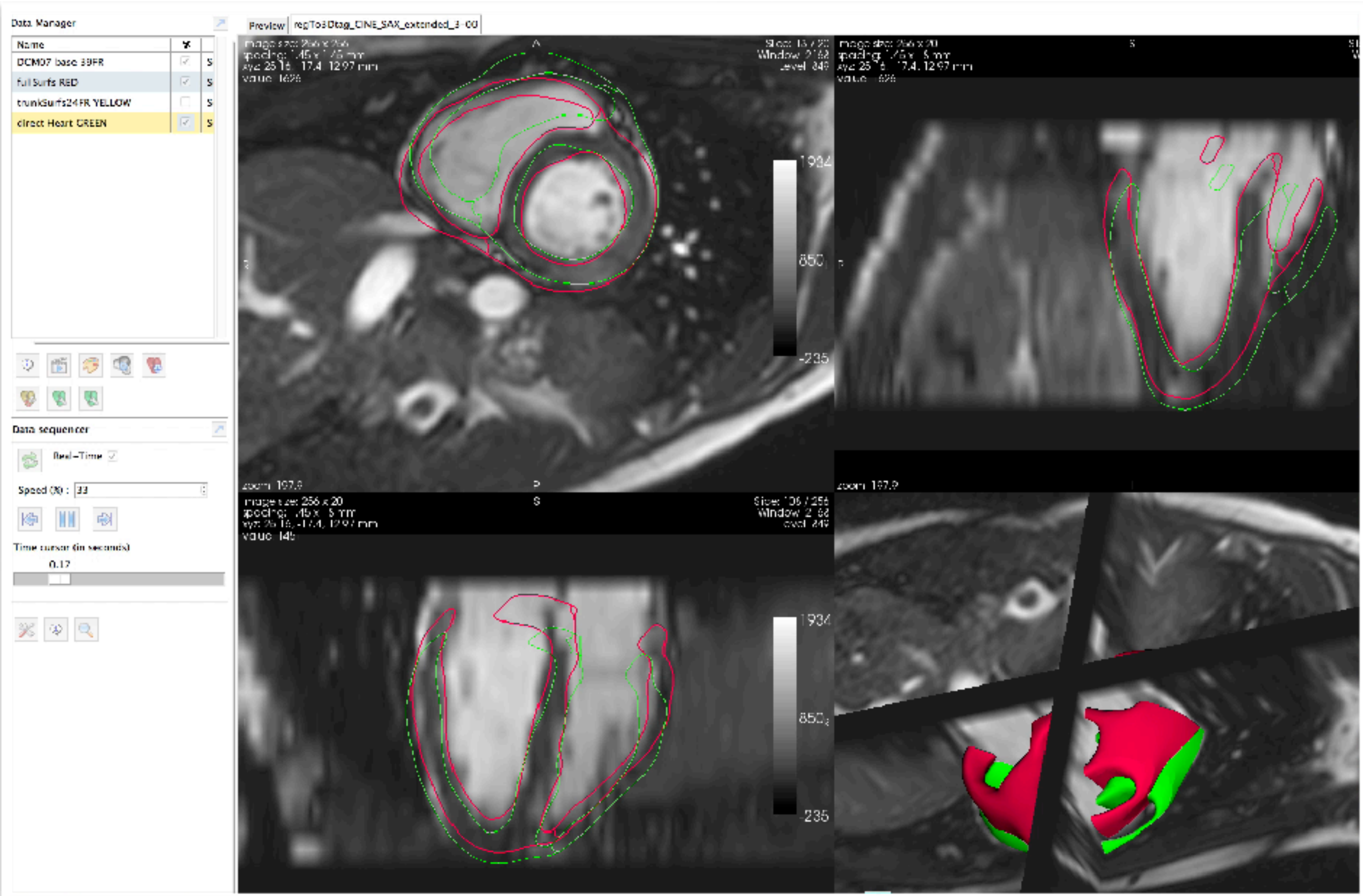
- Case 2:
  11 Regions



Patient specific simulation

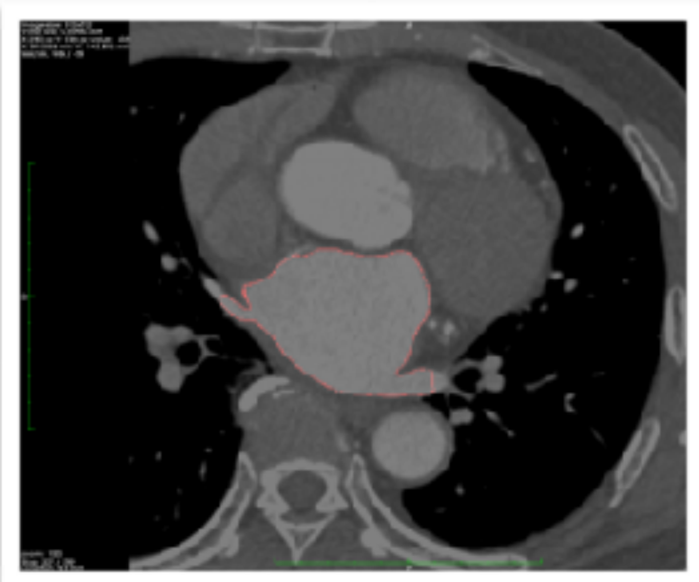


Use directly the image

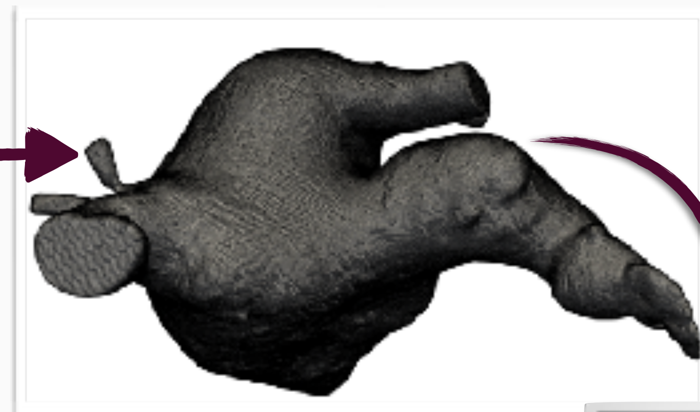


Electrophysiology and shapes

Observations of fronts

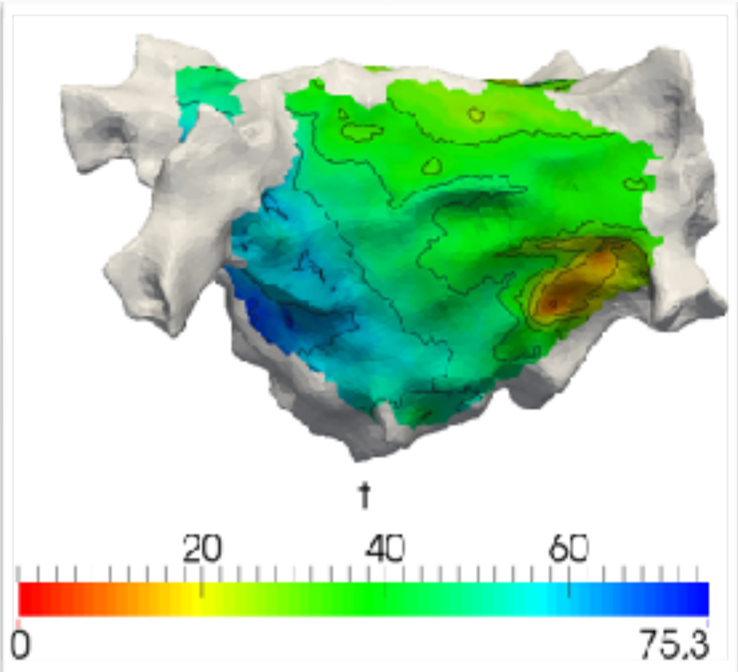


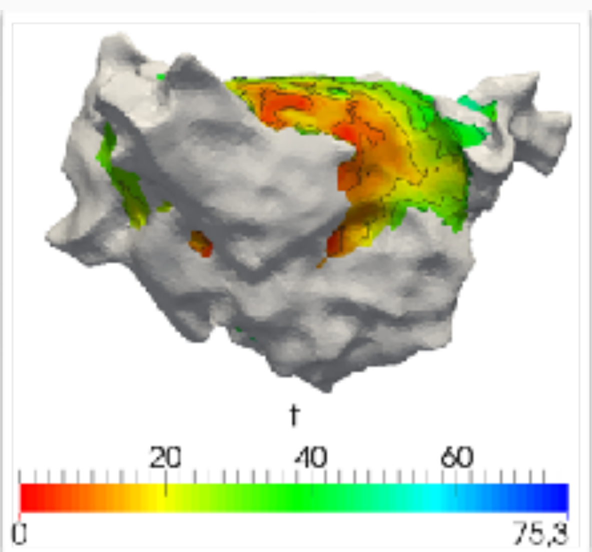
Images (CT, IRM)



Anatomy

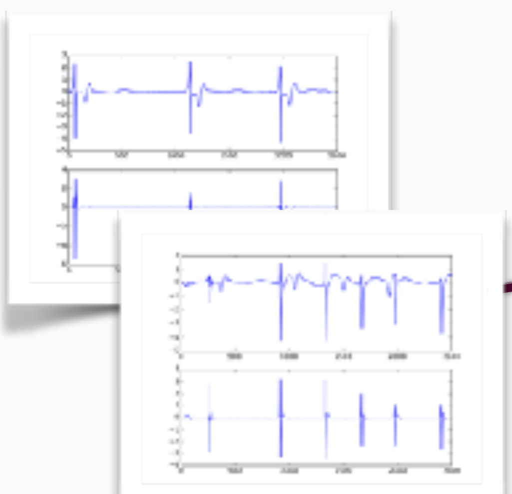
Signal Processing





Maps of electrical activation
(isochronous)

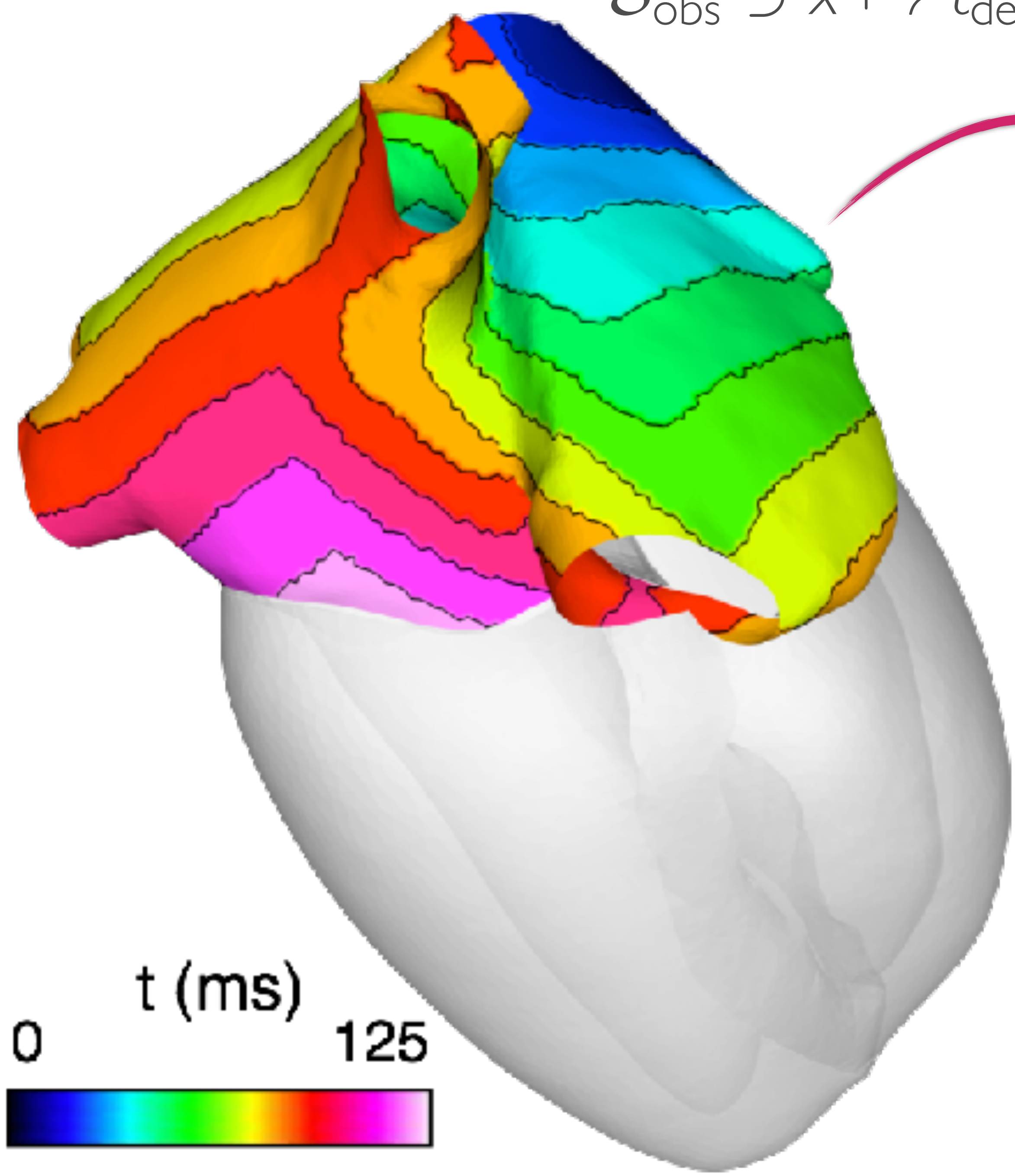
Cloud points

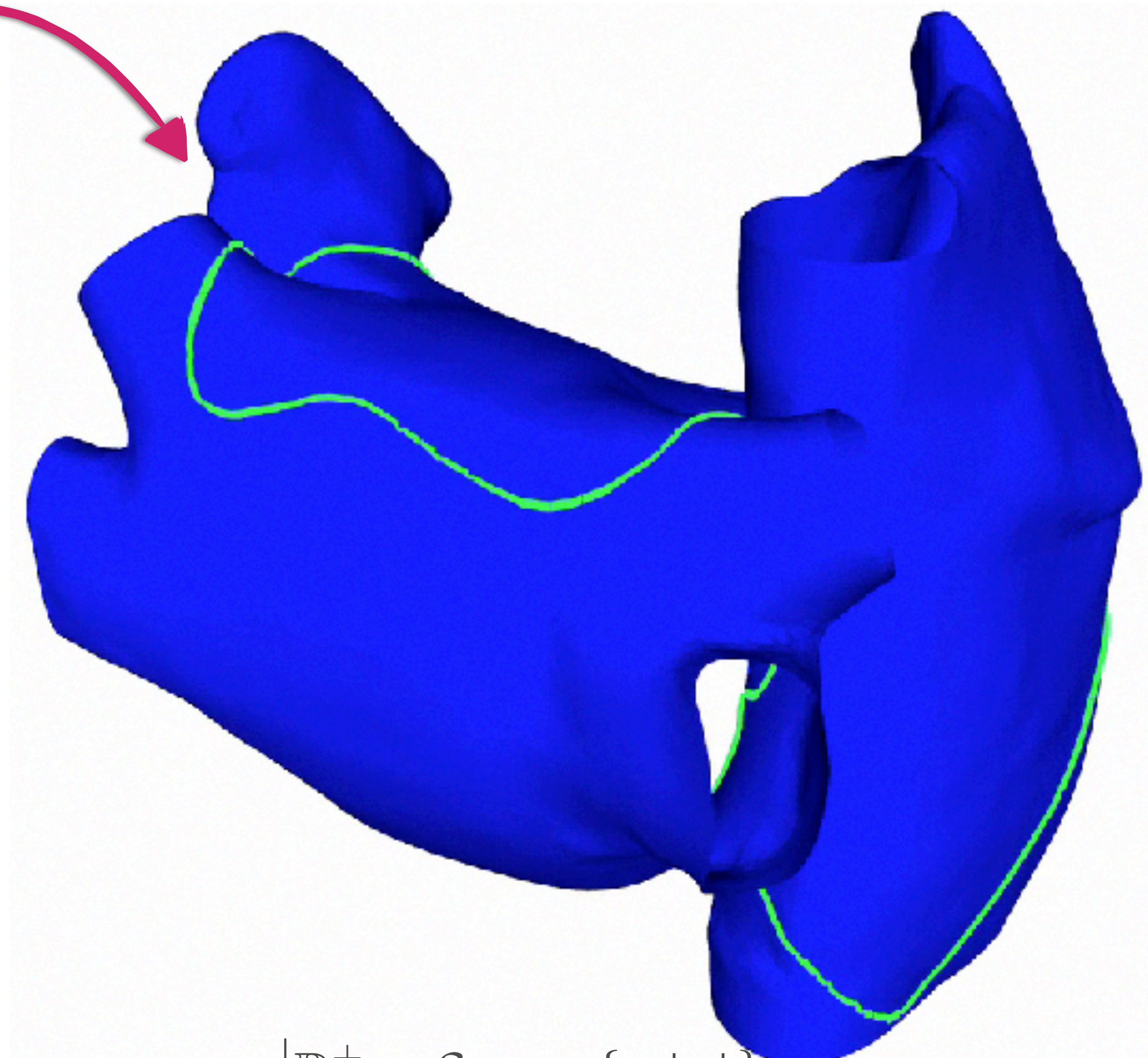


Signals

Data problem statement

$$\mathcal{S}_{\mathrm{obs}} \ni x \mapsto \breve{t}_{\mathrm{dep}}(x) = \min_{t}\{\breve{w}(x,t) > c_{th}\} \in \mathbb{R}^+$$



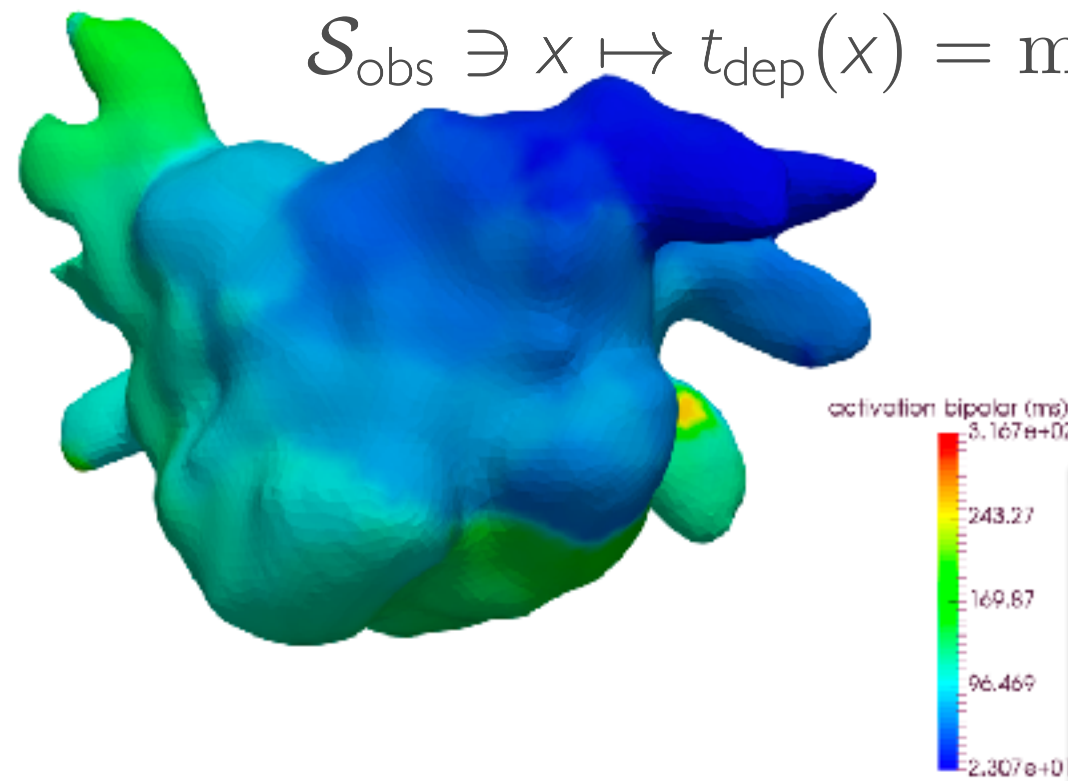


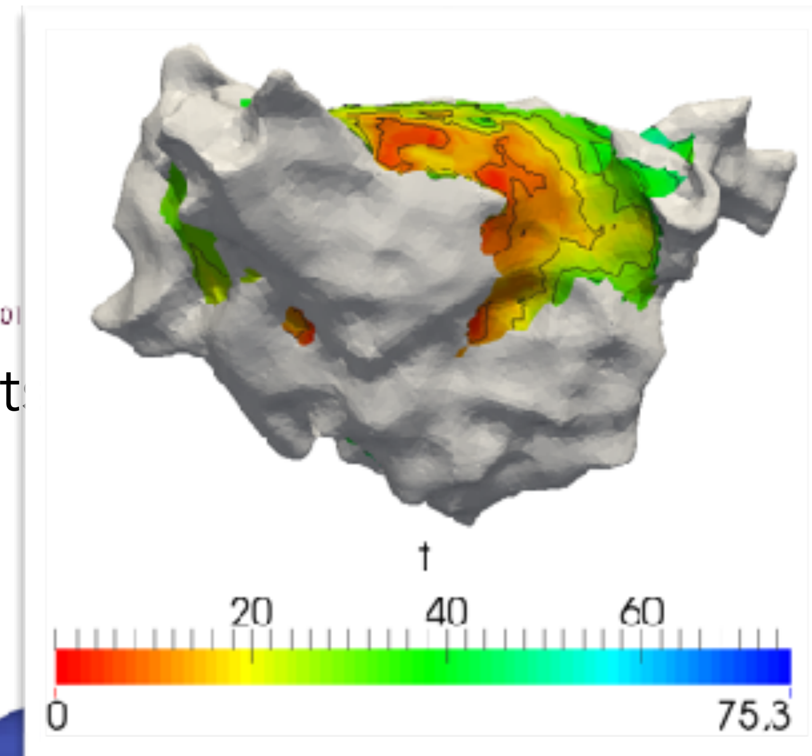
$$z \in L^2([0,T]; L^2(\mathcal{S}_{\mathrm{obs}})) : \left| \begin{array}{l} \mathbb{R}^+ \times \mathcal{S}_{\mathrm{obs}} \to \{-1, 1\} \\ \quad (t,x) \mapsto 1_{t > \breve{t}_{\mathrm{dep}}}(x) - 1_{t < \breve{t}_{\mathrm{dep}}}(x) \end{array} \right.$$

Real data

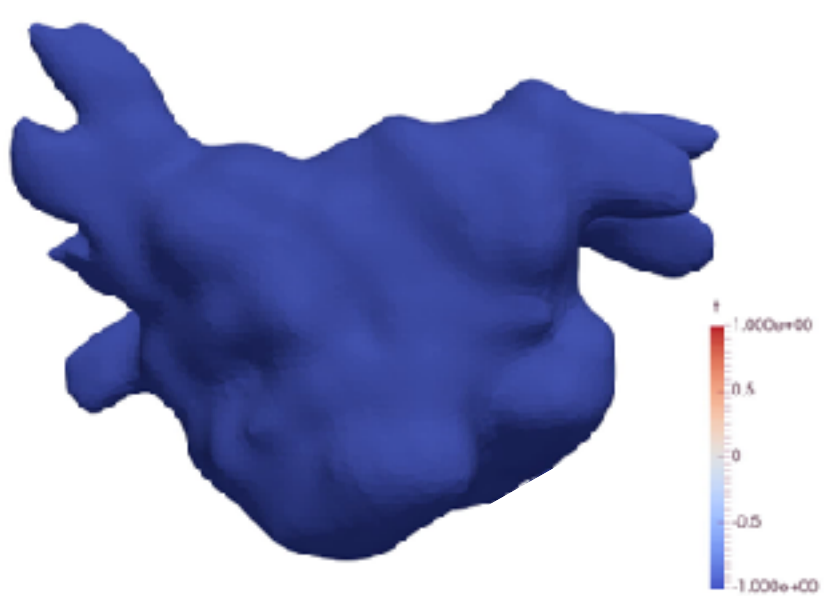
- Isochrone maps

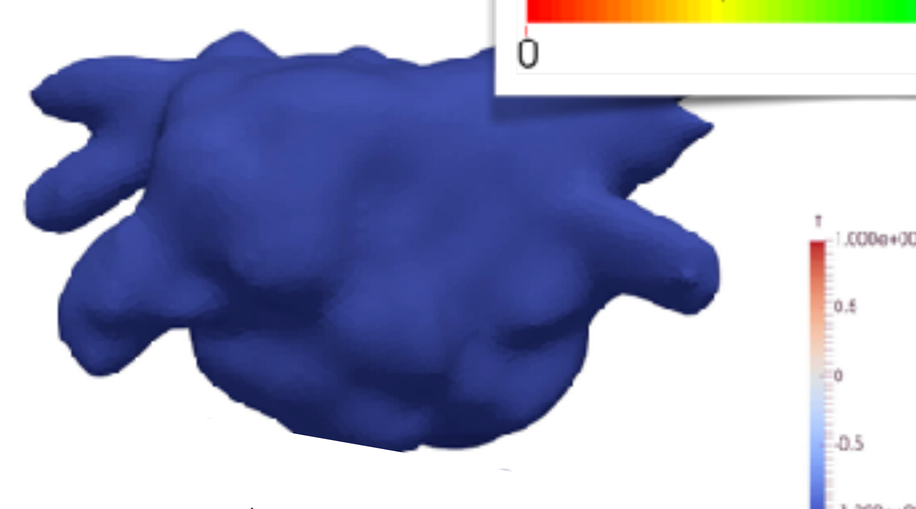
$$\mathcal{S}_{\text{obs}} \ni x \mapsto t_{\text{dep}}(x) = \min_{t}\{w(x,t) > c_{th}\} \in \mathbb{R}^+$$





- Sequence of activation maps: $z = 1$ inside the front and 0 out
  - Some data are only partial measurements





$$z \in L^2([0,T]; L^2(\mathcal{S}_{\text{obs}})) : \left| \begin{array}{l} \mathbb{R}^+ \times \mathcal{S}_{\text{obs}} \to \{-1, 1\} \\ (t,x) \mapsto 1_{t>t_{\text{dep}}}(x) - 1_{t<t_{\text{dep}}}(x) \end{array} \right.$$

Similarity measure

- Inspired from a segmentation problem in image processing from level set

  TF Chan, L Vese, Active contours without edges — IEEE Trans Image Proc. 2001.

- To be compared to: $\phi(w) = w - c_{th}$

- We define c_{th} as the depolarisation constant, namely the depolarisation area is given by

$$\Omega_+(t) = \{\underline{x} \in \mathcal{S} \,|\, w(\underline{x}, t) > c_{th}\}$$
$$\Gamma_w(t) = \{\underline{x} \in \mathcal{S} \,|\, w = c_{th}\} = \Gamma_{\phi=0}(t)$$

- Our comparator is based on the Mumford-Shah functional (Chan-Vese)

$$\boxed{\begin{aligned}\mathcal{J}(z,w) &= \int_{\mathcal{S}_{\mathrm{obs}}} H(w - c_{th})\big(z - C_1(z,w)\big)^2 \mathrm{d}x + \big(1 - H(w - c_{th})\big)\big(z - C_2(z,w)\big)^2 \mathrm{d}x,\\ C_1(z,w) &= \frac{\int_{\mathcal{S}_{\mathrm{obs}}} H(\phi(w)) z \,\mathrm{d}x}{\int_{\mathcal{S}_{\mathrm{obs}}} H(\phi(w))\,\mathrm{d}x}, \quad C_2(z,w) = \frac{\int_{\mathcal{S}_{\mathrm{obs}}} \big(1 - H(\phi(w))\big) z \,\mathrm{d}x}{\int_{\mathcal{S}_{\mathrm{obs}}} \big(1 - H(\phi(w))\big)\,\mathrm{d}x}.\end{aligned}}$$

Front-based state observer

- Target $\begin{cases} c\partial_t \breve{w} - \nabla\cdot(\mathbf{D}\cdot\nabla\breve{w}) &= kf(\breve{w},\kappa), & \text{in } \mathcal{S}\times(0,T), \\ \dot{\breve{\kappa}} &= \eta(\kappa,\breve{w}) & \text{in } \mathcal{S}\times(0,T) \\ (\mathbf{D}\cdot\nabla\breve{w})\cdot n &= 0, & \text{on } \partial\mathcal{S}\times(0,T), \\ \breve{w}(x,0) &= \breve{w}_0(x), & \text{in } \mathcal{S}. \end{cases}$

- Observer $\begin{cases} c\partial_t \hat{w} - \nabla\cdot(\mathbf{D}\cdot\nabla\hat{w}) &= kf(\hat{w},\kappa) + g(z,\hat{w}), & \text{in } \mathcal{S}\times(0,T), \\ \dot{\hat{\kappa}} &= \eta(\hat{\kappa},\hat{w}) & \text{in } \mathcal{S}\times(0,T) \\ (\mathbf{D}\cdot\nabla\hat{w})\cdot n &= 0, & \text{on } \partial\mathcal{S}\times(0,T), \\ w(x,0) &= \hat{w}_0(x), & \text{in } \mathcal{S}. \end{cases}$

  with $g(z,\hat{w}) = \gamma_{sh}(x)\,\delta(\hat{w}-c_{th})\Big(-\big(z(t)-C_1(z,\hat{w})\big)^2 + \big(z(t)-C_2(z,\hat{w})\big)^2\Big)$

  $$+\gamma_{topo}(x) H\Big(\big((z-C_1(z,\hat{w}))^2 - (z-C_2(z,\hat{w}))^2\big)(\hat{w}-c_{th})\Big)\Big((z-C_1(z,\hat{w}))^2 - (z-C_2(z,\hat{w}))^2\Big)$$

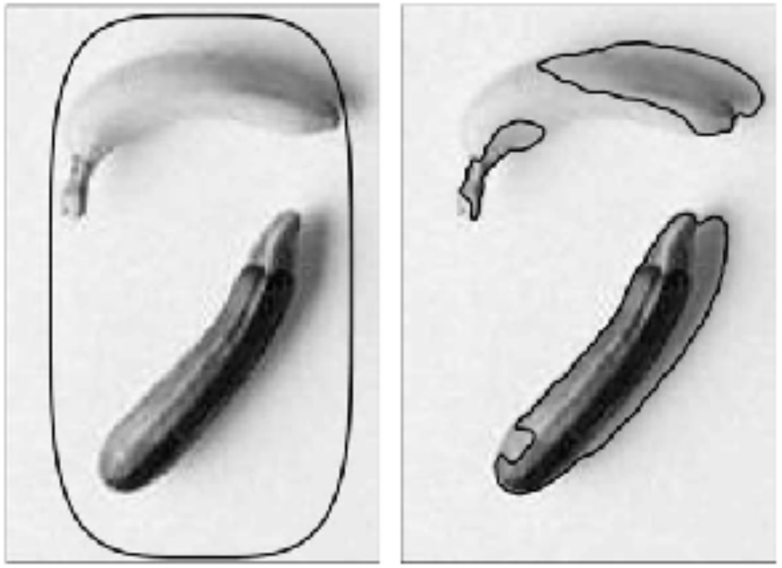
- Analysis

Proposition.

The data correction stabilises the observer on the target trajectory for sufficiently small errors.

Mathematical justification

- Compute the error norm $\tilde{u} = \breve{u} - \hat{u}$



$$\int_{\mathcal{S}} \partial_t \hat{w}^2 \, dx = -\int_{\mathcal{S}} D \, \underline{\nabla} \tilde{w} \cdot \underline{\nabla} \tilde{w} \, dx + \int_{\mathcal{S}} k(f(\breve{w}) - f(\hat{w})) \, \tilde{w} \, dx$$
$$\boxed{+\gamma \int_{\Gamma_{\hat{w}}} \frac{1}{|\underline{\nabla} \hat{w}|} \left((z - C_1)^2 - (z - C_2)^2 \right) \tilde{w} \, d\Gamma}$$

Dissipative ?

- Stabilization property of

$$\mathcal{Q}_w(\tilde{w}) = \int_{\Gamma_{\hat{w}}} \frac{\gamma}{|\underline{\nabla} w|} \left((z - C_1)^2 - (z - C_2)^2 \right) \tilde{w} \, d\Gamma.$$

- Around the target trajectory

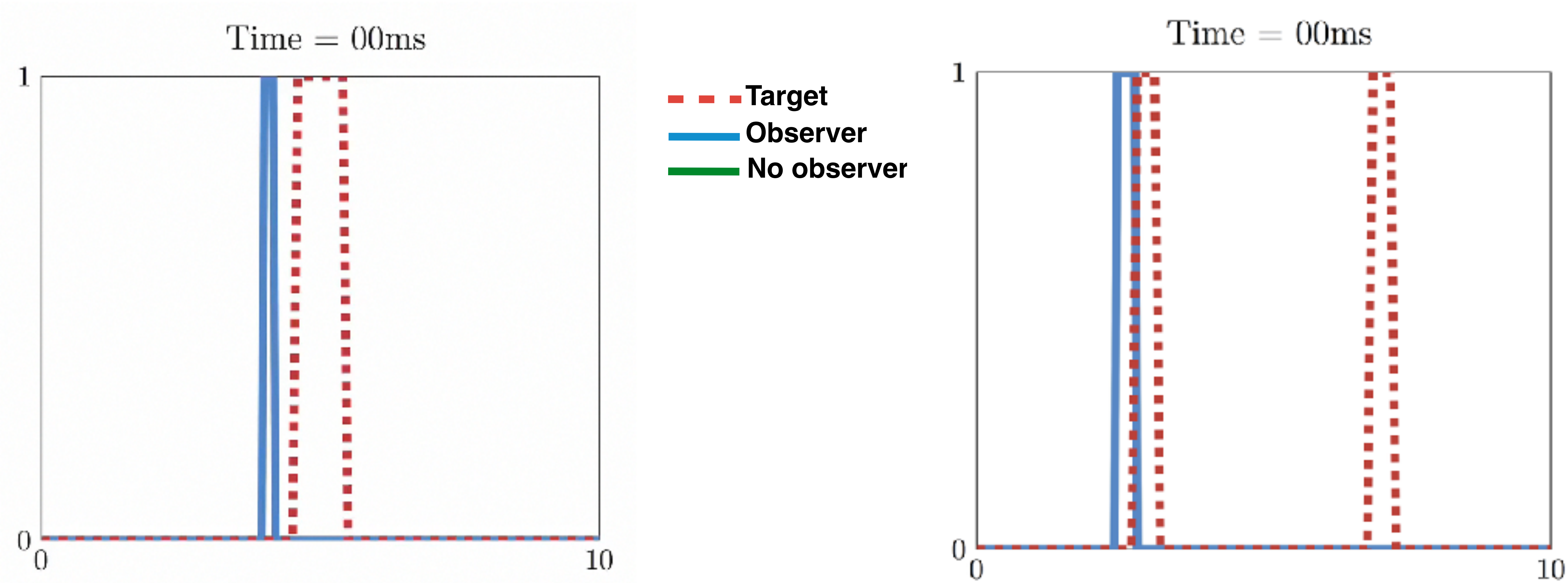
$$\left(d_{\hat{w}} \mathcal{Q}_{\breve{w}}(\tilde{w}) \, ; \, \tilde{w} \right) = 2\left(C_2(z, \breve{w}) - C_1(z, \breve{w}) \right) \int_{\Gamma_{\breve{w}}} \gamma \, \partial_n z_{\breve{w}} \frac{\tilde{w}}{|\underline{\nabla} \breve{w}|} \frac{\tilde{w}}{|\underline{\nabla} \breve{w}|} \, d\Gamma$$
$$- 2 \left(\frac{C_2(z, \breve{w}) - C_1(z, \breve{w})}{2} \right)^2 \left(\frac{1}{|\Omega_{\breve{w}}|} + \frac{1}{|\mathcal{S} \setminus \overline{\Omega_{\breve{w}}}|} \right) \int_{\Gamma_{\breve{w}}} \frac{\tilde{w}}{|\underline{\nabla} \breve{w}|} \, d\Gamma \times \int_{\Gamma_{\breve{w}}} \gamma \frac{\tilde{w}}{|\underline{\nabla} \breve{w}|} \, d\Gamma$$

A. Collin, D. Chapelle, P. Moireau, A Luenberger observer for reaction–diffusion models with front position data — Journal Of Computational Physics, 2015

1D illustration: state estimation

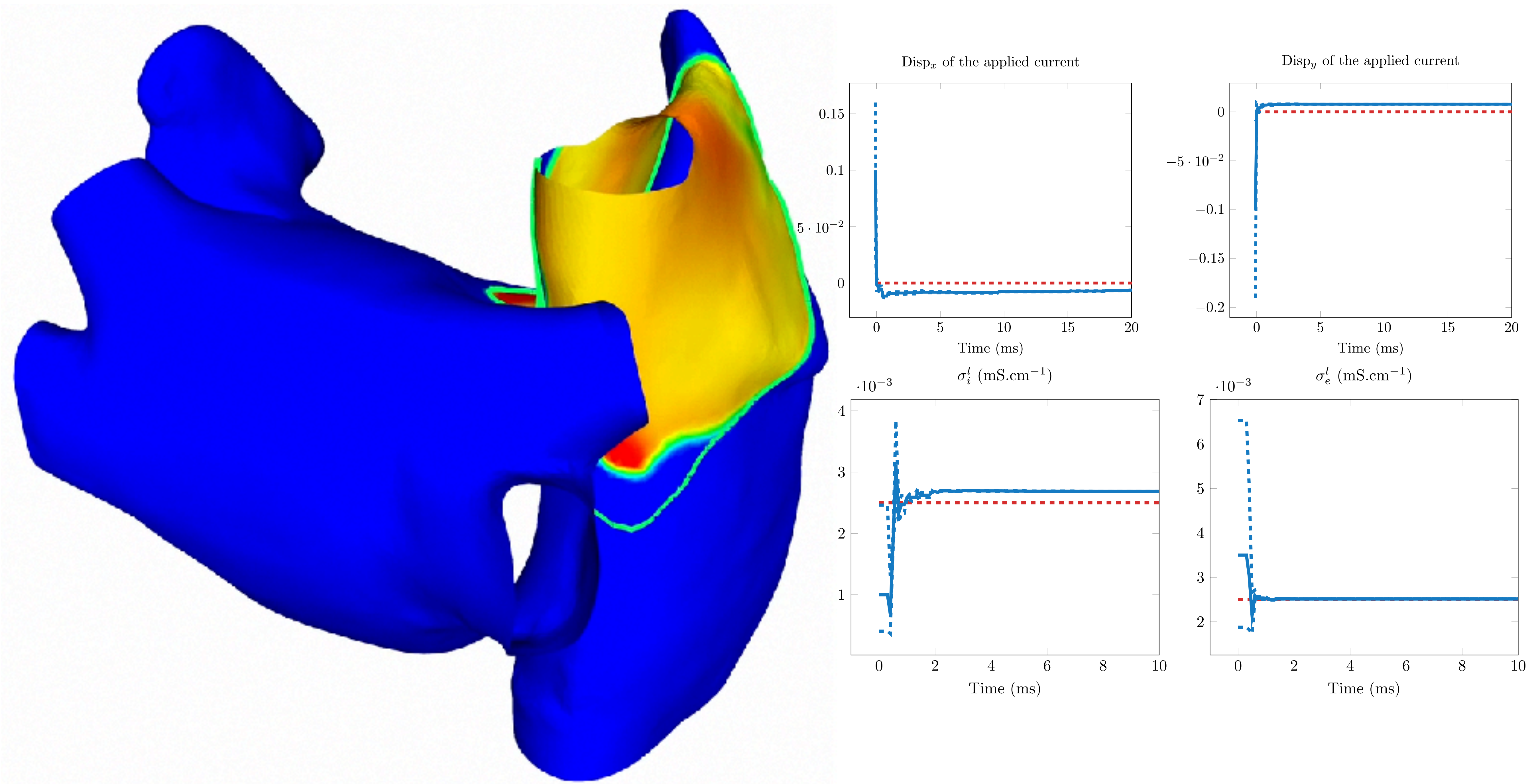
- Reaction diffusion system of Mitchell-Schaeffer type (similar to Fitzugh-Nagumo)

> A. Collin, D. Chapelle, P. Moireau, Sequential State Estimation for Electrophysiology Models with Front Level-Set Data Using Topological Gradient Derivations — FIMH 2015

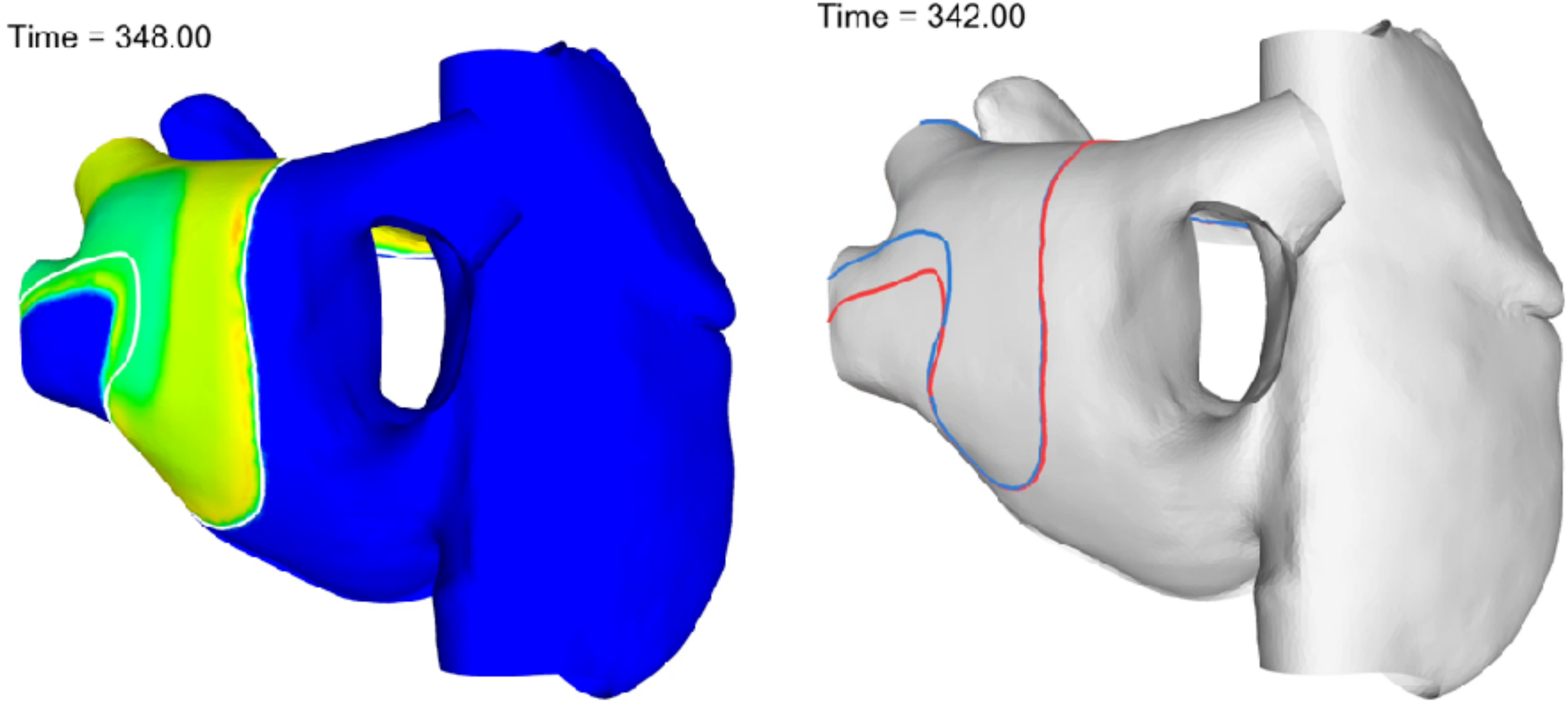


State and parameter estimation

- Model: bidomain surface model



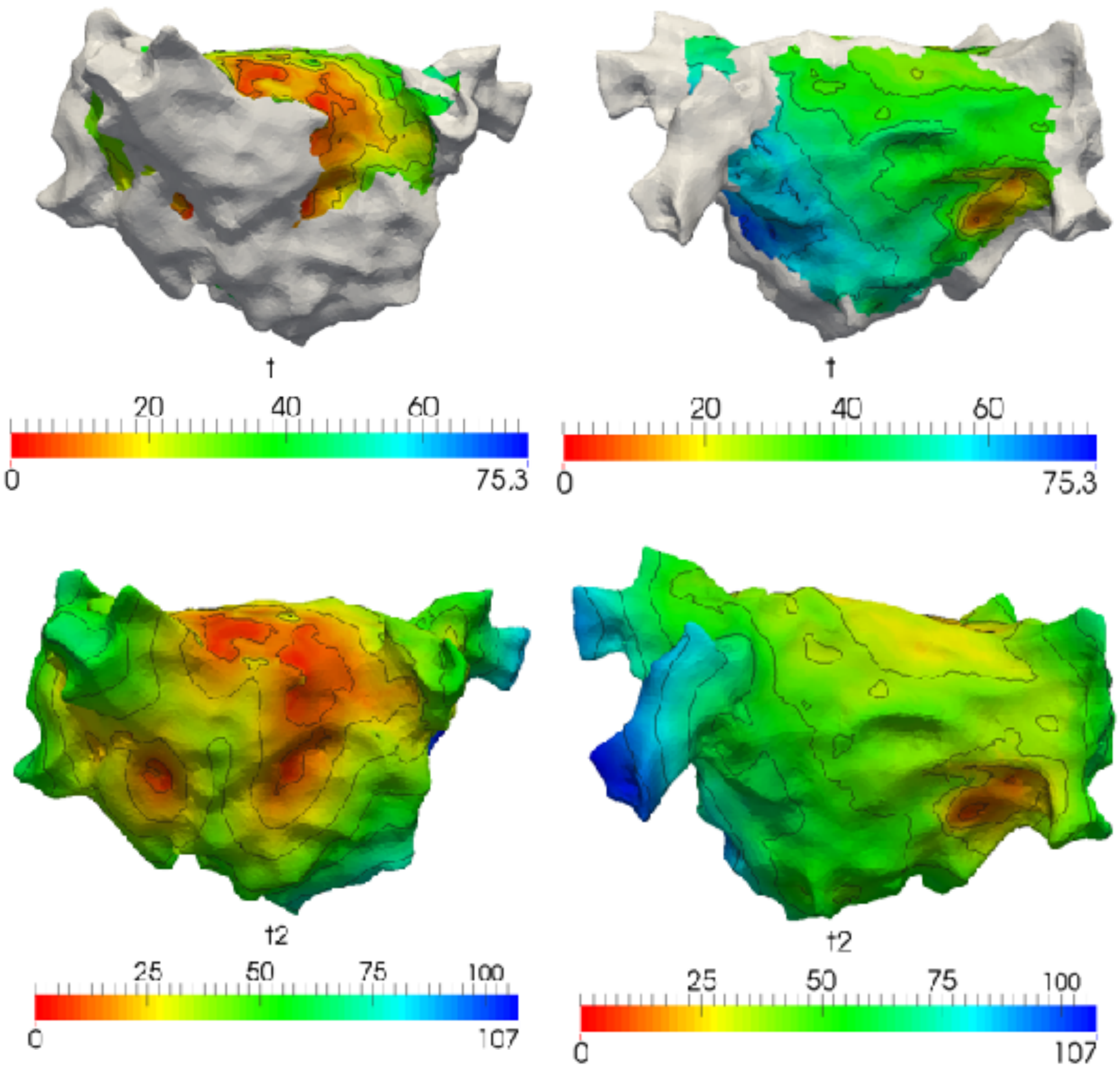
With topological gradient



Toward real data

- A data completion tool
  - different surface model

With A. Gérard, A. Collin, Y. Coudière

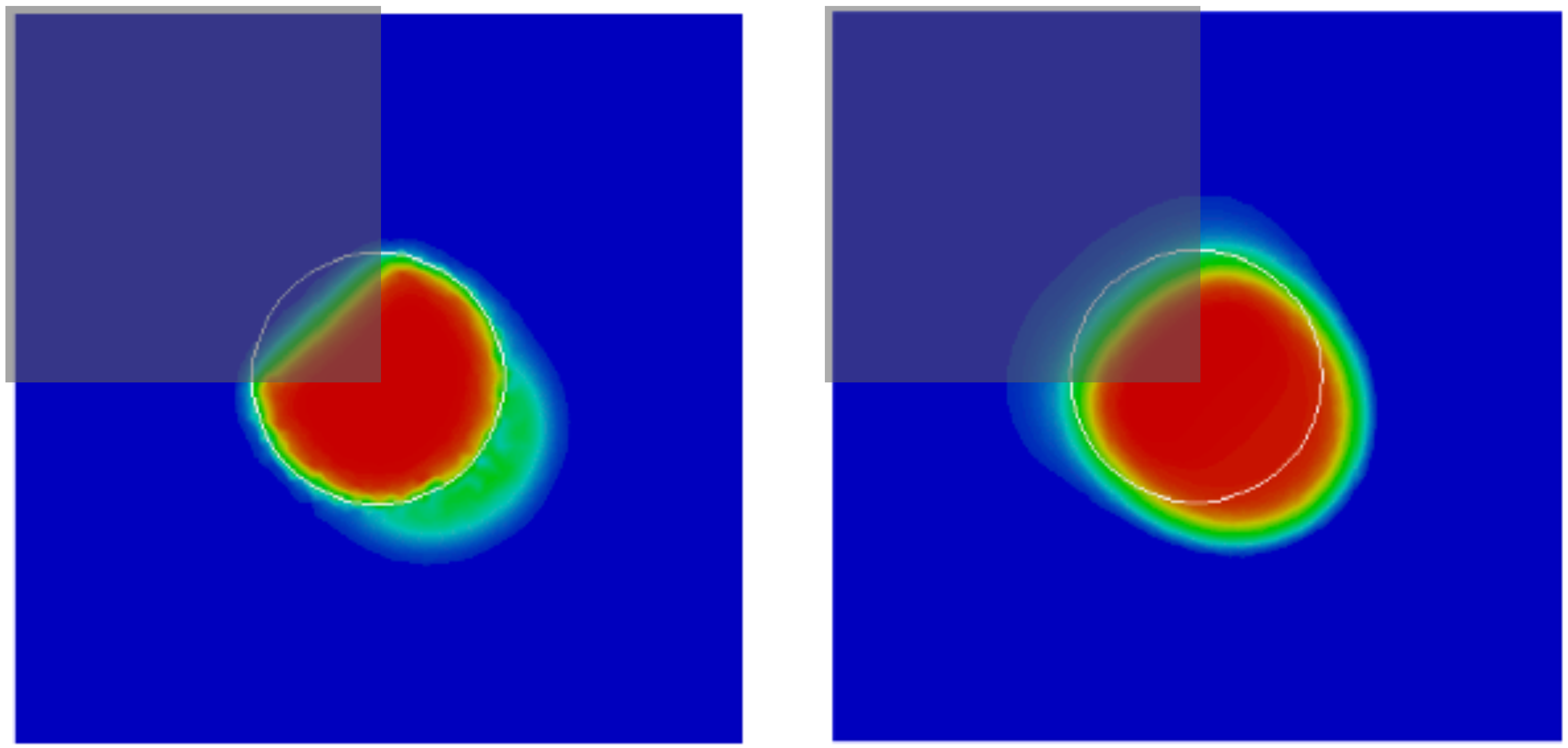


- But parameter estimation is also possible

Ingredients for real data robustness

With A. Collin

- Different cases
  - Simple pacing
  - Multiple pacing
  - Fibrilation ?

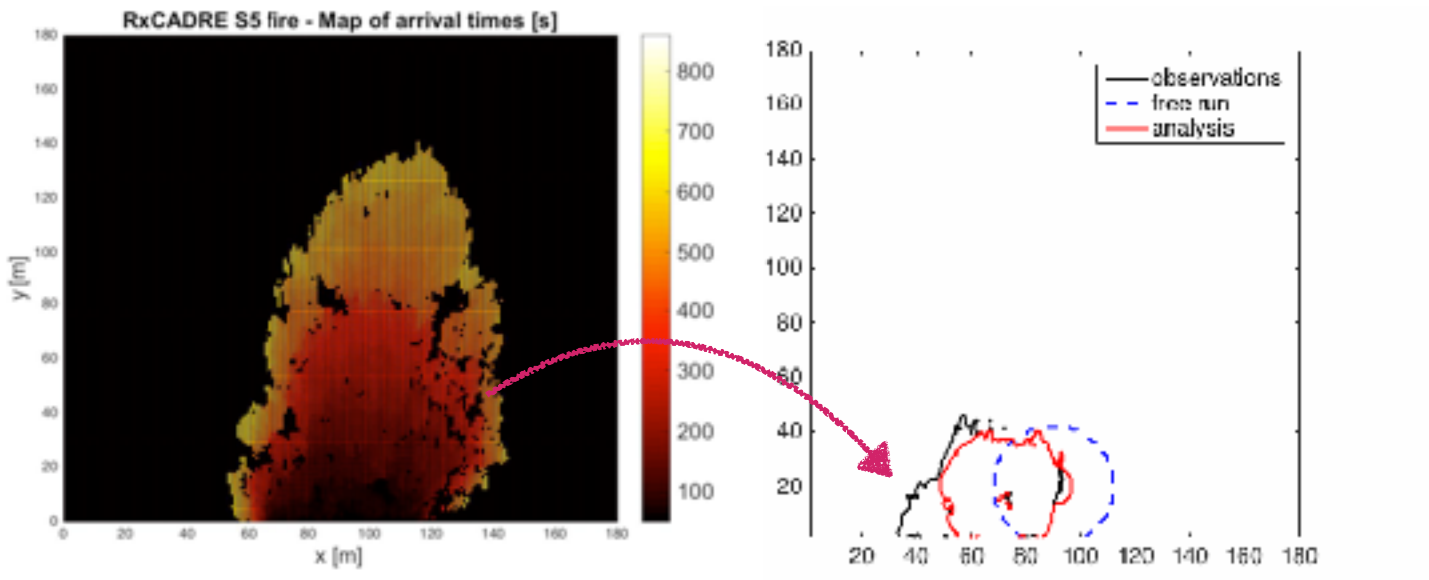


- Sampled data
  - interpolate the feedba
- Partial data
  - Correct where you have the data or find better correction based on H_1 Sobolev gradient concepts

P. Moireau, D. Chapelle, P. Le Tallec — Filtering for distributed mechanical systems using position measurements: perspectives in medical imaging — Inverse Problems 2009

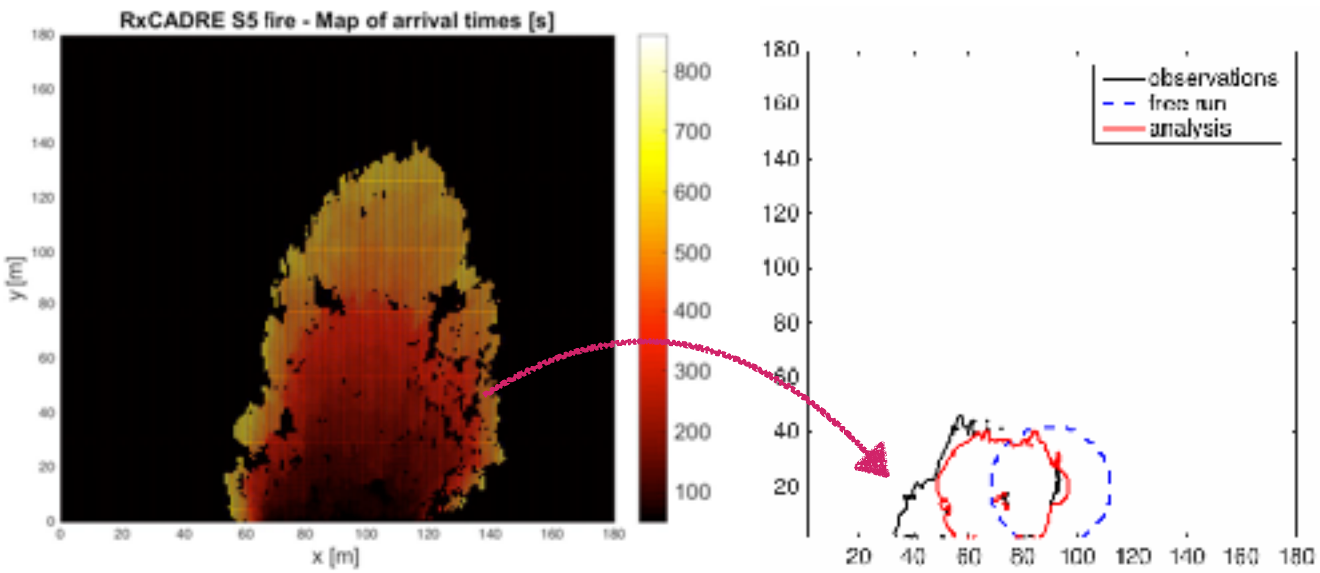
G. Charpiat, P. Maurel, J.P. Pons, R. Keriven, O. Faugeras — Generalized gradients: Priors on minimization flows, Int. J. Comput. Vision

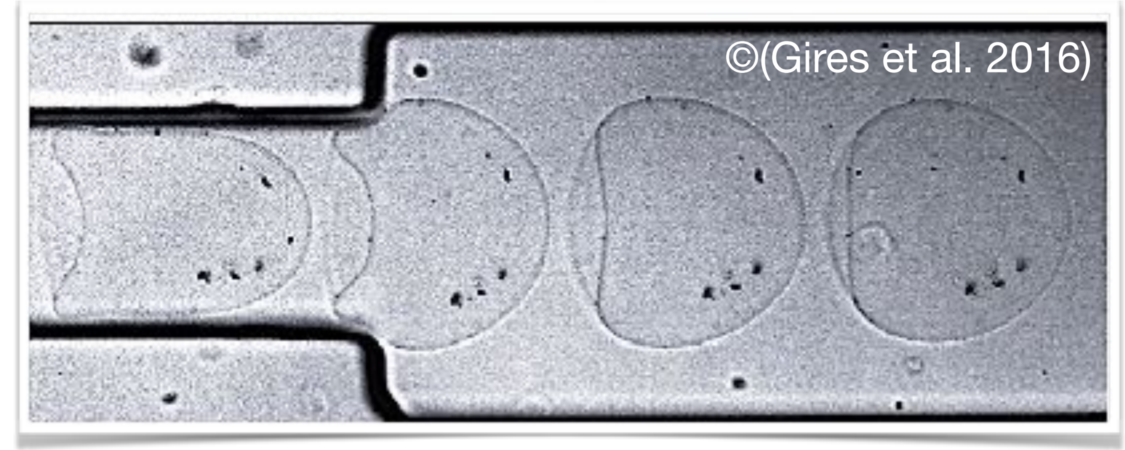
Other examples



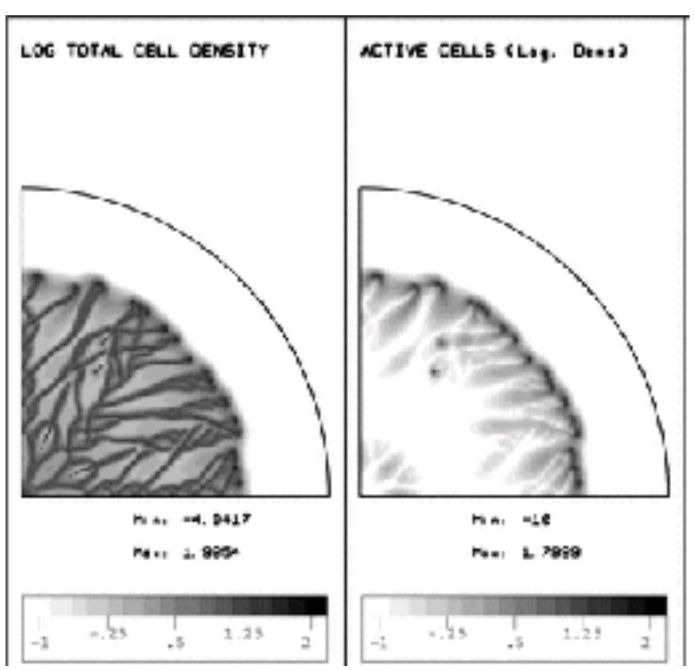
M.C. Rochoux, A. Collin, C. Zhang, A. Trouve, D. Lucor and P. Moireau — Front Shape Similarity Measure For Shape-Oriented Sensitivity Analysis And Data Assimilation For Eikonal Equation — Proceedings of the CEMRACS 2016

Other examples

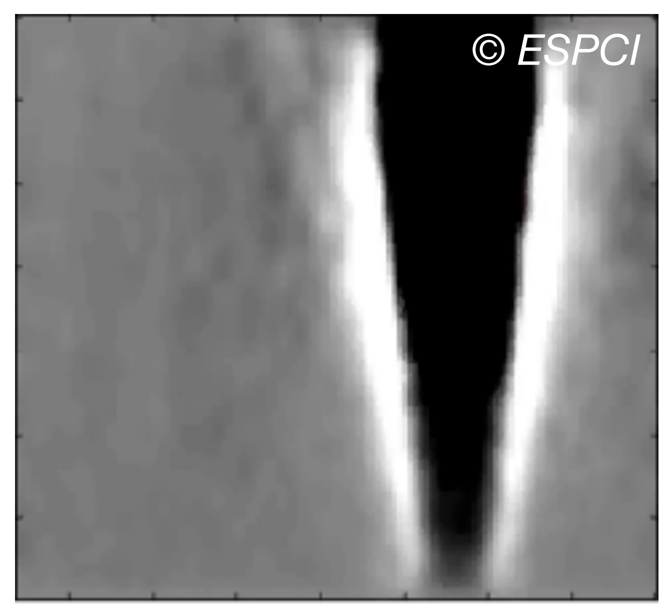


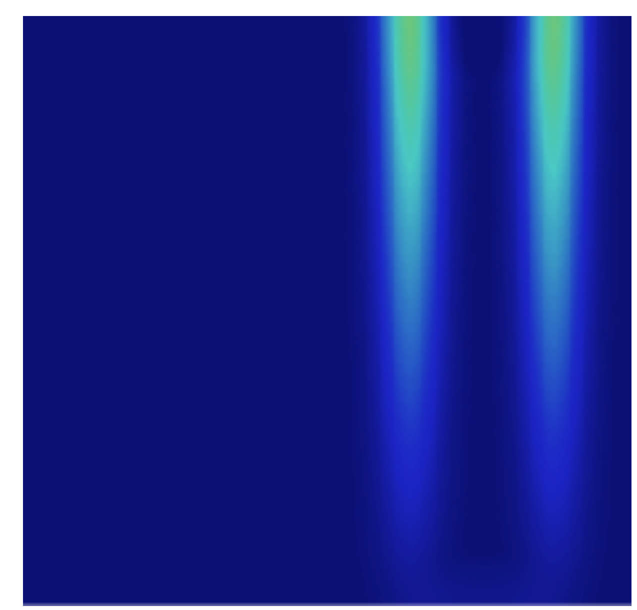


M.C. Rochoux, A. Collin, C. Zhang, A. Trouve, D. Lucor and P. Moireau — Front Shape Similarity Measure For Shape-Oriented Sensitivity Analysis And Data Assimilation For Eikonal Equation — Proceedings of the CEMRACS 2016



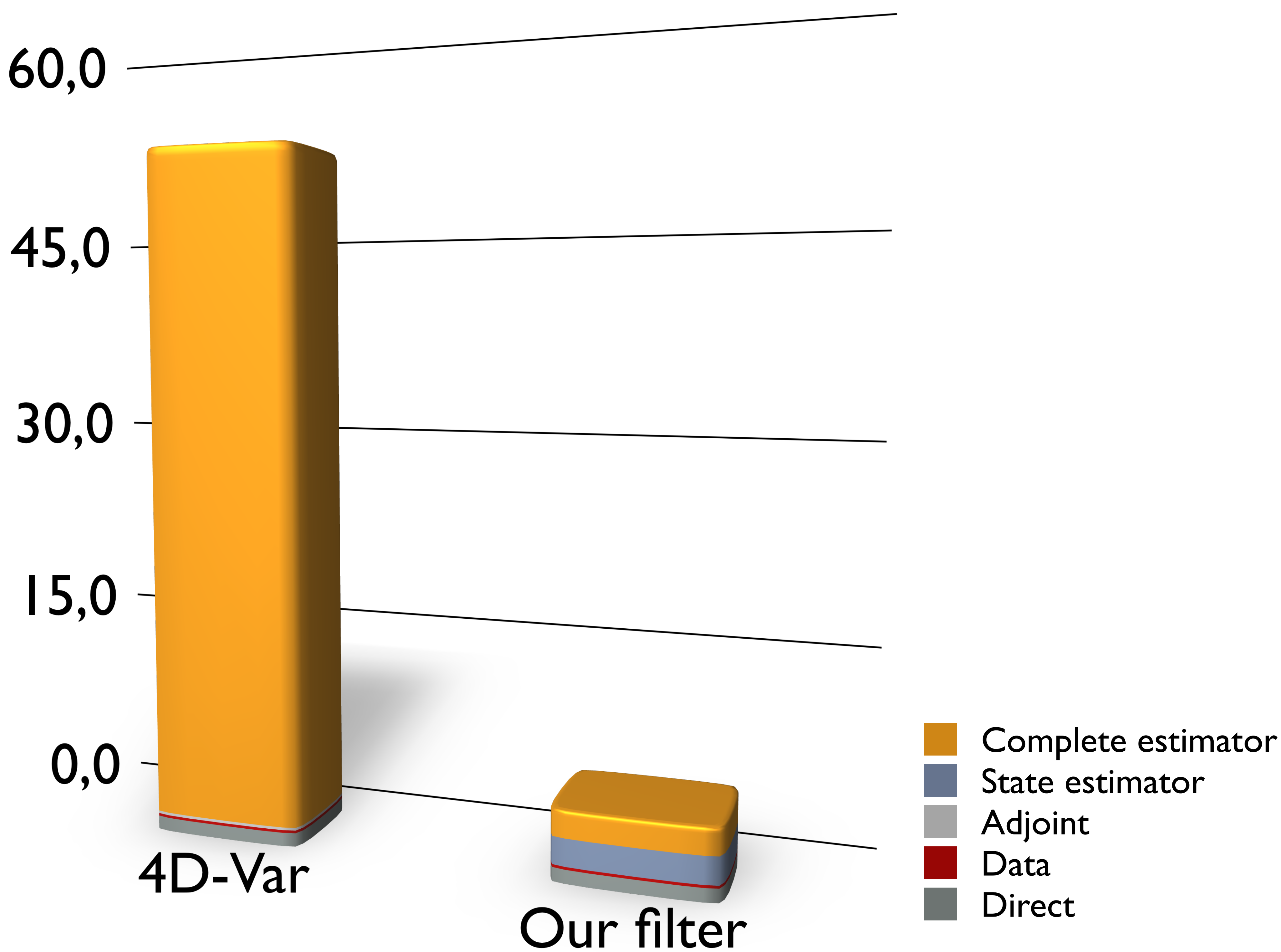
©(A. Marrocco)





Conclusion

It works and it can be efficient



The framework

For each physical model (hence evolution PDE) and measurements

- **Challenge 1:** Formulation of adequate similarity measures adapted to the system of interest and design of the control feedback;
- **Challenge 2:** Analysis of the resulting observer: existence, stability, convergence;
- **Challenge 3:** Joint identification of model parameters – namely, adaptive observer formulation – and analysis of identification properties; Statistics on shape for Kalman based formulations
- **Challenge 4:** Discretization, numerical strategies and numerical analysis of the observer formulation;
- **Challenge 5:** Applications with real data, in particular in the cardiovascular context, but not only

Challenges

- Non-linear systems in infinite dimension : theoretical aspects
- Shapes
- Constraints: theoretical aspects, and practical aspects (pressure measurements)
- link identifiability and parameter-observer convergence
- Combining Data assimilation and Learning

References

C. Bardos, G. Lebeau, and J. Rauch. Control and Stabilization of wave equation. In *Journée Équations aux dérivées partielles*, pages 1–15, July 1987.

A. Bensoussan, M. C. Delfour, G. Da Prato, and S. K. Mitter. *Representation and Control of Infinite Dimensional Systems*. Birkhauser Verlag, second edition edition, 2007. ISBN 3-7643-3641-2.

M Boutayeb, H Rafaralahy, and M Darouach. Convergence analysis of the extended Kalman filter used as an observer for nonlinear deterministic discrete-time systems. *IEEE Transactions on Automatic Control*, 42(4): 581–586, 1997.

N. Burq and P. Gérard. Contrôle optimal des equations aux derivees partielles. Cours de l'Ecole Polytechnique, 2002.

D. Chapelle, N. Cîndea, M. De Buhan, and P. Moireau. Exponential Convergence of an Observer Based on Partial Field Measurements for the Wave Equation. *Mathematical Problems in Engineering*, 2012:1–12, 2012.

A Haraux. Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps. *Portugal. Math.*, 46(3):245–258, 1989.

F J Kraus and V Kučera. Linear quadratic and pole placement iterative design. In *1999 European Control Conference (ECC*, pages 653–658. IEEE, 1999.

A.J. Krener. A Lyapunov theory of nonlinear observers. In G G Yin and Qing Zhang, editors, *Stochastic analysis, control, optimization and applications*, pages 409–420. Springer, 1998.

A.J. Krener. The convergence of the extended Kalman filter. In *Lecture Notes in Control and Inform. Sci.*, pages 173–182. Springer, Berlin, 2003.

P. Moireau and D. Chapelle. Reduced-Order Unscented Kalman Filtering with Application to Parameter Identification in Large-Dimensional Systems. *ESAIM Control Optim. Calc. Var.*, 17(2):380–405, April 2011.

P. Moireau, D. Chapelle, and P. Le Tallec. Joint state and parameter estimation for distributed mechanical systems. *Comput Method Appl M*, 197 (6-8):659–677, 2008.

K. Ramdani, M. Tucsnak, and G. Weiss. Recovering the initial state of an infinite-dimensional system using observers. *Automatica*, 46(10):1616–1625, 2010.

M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 2009. ISBN 978-3-7643-8993-2. doi: 10.1007/978-3-7643-8994-9. URL http://dx.doi.org/10.1007/978-3-7643-8994-9.

Qinghua Zhang. Adaptive observer for multiple-input-multiple-output (MIMO) linear time-varying systems. *IEEE Trans. Automat. Control*, 47(3):525–529, 2002.

Qinghua Zhang and Aiping Xu. Global adaptive observer for a class of nonlinear systems. *Decision and Control, 2001. Proceedings of the 40th IEEE Conference on DOI - 10.1109/.2001.980478*, 4:3360–3365 vol.4, 2001.