# Influence of the mode of reproduction on species invasion

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CEMRACS Project

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**Question**

How does a given mode of reproduction structure a biological invasion?

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# Biological background

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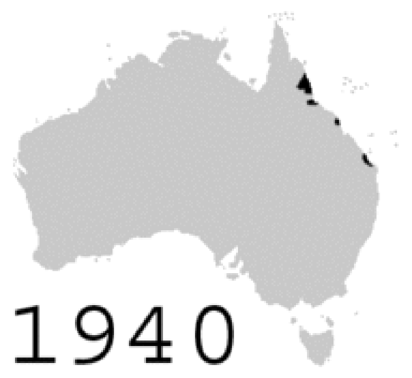
## Invasion of cane toad (*Rhinella Marina*) in Australia




Source: Wikipedia

- Sexual reproduction
- Influence of the phenotype (long legs) on the speed of propagation
- Acceleration of the front of propagation

# Presentation of the model

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$$\begin{cases} \partial_t f - \theta\,\Delta_x f = r\,(\mathcal{B}[f] - \rho\, f) & t > 0\,,\ x \in \mathbb{R}\,,\ \theta \in (1,+\infty), \\ \rho(t,x) = \displaystyle\int_1^\infty f(t,x,\theta')\,\mathrm{d}\theta' & t \geq 0\,,\ x \in \mathbb{R}, \\ f(0,x,\theta) = f_0(x,\theta) & x \in \mathbb{R}\,,\ \theta \in [1,+\infty), \\ \partial_\theta f(t,x,1) = 0 & t \geq 0\,,\ x \in \mathbb{R}. \end{cases} \tag{1}$$

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$$\mathcal{B}[f](t,x,\theta) := f(t,x,\theta) + \Delta_\theta f(t,x,\theta),$$

- Sexual reproduction and recombination

$$\mathcal{B}[f](t,x,\theta) := \frac{1}{\sqrt{\pi}} \iint_{(1,\infty)^2} \exp\left[-\left(\theta - \frac{\theta_1+\theta_2}{2}\right)^2\right] f(t,x,\theta_1) \frac{f(t,x,\theta_2)}{\rho(t,x)} \,\mathrm{d}\theta_1\,\mathrm{d}\theta_2.$$

# Sexual case: Reproduction term

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Chances of an individual of phenotype $\theta_1$ encountering another individual of phenotype $\theta_2$ at time $t$ and location $x$.

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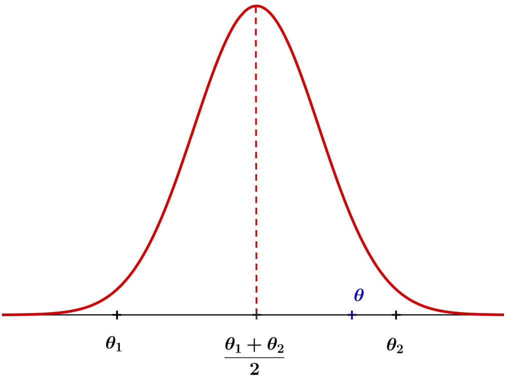


Random choice of the child phenotype $\theta$ with a Gaussian law centered in the mean value of the phenotypes of both parents.

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Let $f$ be the solution of the equation

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with a regular enough non-negative initial condition $f_0$ compactly supported in $(1,\infty)$, and uniformly exponentially decreasing in space.

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with a regular enough non-negative initial condition $f_0$ compactly supported in $(1,\infty)$, and uniformly exponentially decreasing in space. For all $t \geq 0$ and $x \in \mathbb{R}$, we define

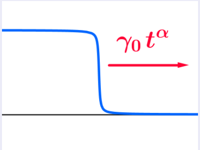
$$S(t,x) := \sup_\theta f(t,x,\theta)\,, \text{ and } \gamma_0 := \frac{2}{3}2^{1/4}.$$

Then we have for all $\gamma > \gamma_0$,

$$\lim_{t\to\infty} \sup_{x>\gamma t^{3/2}} S(t,x) = 0,$$

and for $\gamma < \gamma_0$,

$$\lim_{t\to\infty} \sup_{x<\gamma t^{3/2}} S(t,x) > 0.$$



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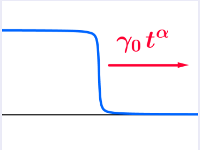


## Purpose

Formally and numerically determine the asymptotic speed of propagation of the population in the sexual case.

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# Sexual case: Formal analysis

## Hopf-Cole transformation

For $t \gg 1$, we consider the function $u$ such that:

$$f(t,x,\theta) = \exp\left[-t\,u\left(\log(t), \frac{x}{t^{\alpha}}, \frac{\theta}{t^{\beta}}\right)\right].$$

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Let $s = \log(t)$, $y = \frac{x}{t^\alpha}$, $\eta = \frac{\theta}{t^\beta}$, so $u(s,y,\eta)$ satisfies the equation

$$-u - \partial_s u + \alpha y \partial_y u + \beta \eta \partial_\eta u$$
$$= -\eta e^{(1-2\alpha+\beta)s} \Delta_y u + \eta e^{(2-2\alpha+\beta)s} (\partial_y u)^2 + r\left(\frac{\mathcal{B}[f]}{f} - \rho\right),$$

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with $\mathcal{B}[f]$ defined by

$$\frac{e^{2\beta s}}{\rho\sqrt{\pi}} \iint_{(e^{-\beta s},\infty)^2} \exp\left[-e^{2\beta s}\left(\eta - \frac{\eta_1+\eta_2}{2}\right)^2 - e^s u(s,\,y,\,\eta_1) - e^s u(s,\,y,\,\eta_2)\right] d\eta_1\, d\eta_2.$$

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We choose

$$(\alpha\,,\,\beta) \;=\; \left(\frac{5}{4}\,,\,\frac{1}{2}\right).$$

# Sexual case: Formal analysis

## Formal Taylor expansion of $u$

$$u(\log(t), y, \eta) = u_0(y, \eta) + \frac{1}{t} u_1(y, \eta) + \underset{t \to \infty}{o} \left( \frac{1}{t} \right),$$

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Formally, for $t \gg 1$, we get the stationary equation

$$\begin{aligned} & - u + \alpha y \partial_y u + \beta \eta \partial_\eta u \\ & = \eta (\partial_y u)^2 + r \left( \exp \left[ u_1(y, \, \eta) + u_1(y, a(y)) - 2 u_1 \left( y, \, \frac{\eta + a(y)}{2} \right) \right] - \rho_\infty(y) \right), \end{aligned}$$

where $\rho_\infty(y) = H(y_c - y)$ for a constant $y_c$ yet to be determined.

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where $\rho_\infty(y) = H(y_c - y)$ for a constant $y_c$ yet to be determined. Asymptotically

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## Sexual case: Formal analysis

Putting the Taylor expansion of $u$ into the stationary equation, we get

$$\begin{cases} -b(y) + \alpha y b'(y) &= r(1-\rho_\infty(y)) + a(y)(b'(y))^2, \\ -\alpha y a'(y) + \beta a(y) &= (b'(y))^2 - 2a(y)b'(y)a'(y). \end{cases}$$

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## Formal results

$$b(y) = \begin{cases} 0 & \text{if } y \leq y_c, \\ \left(\frac{9}{128}\right)^{1/3} y^{4/3} - r & \text{else,} \end{cases}$$

and

$$a(y) = \begin{cases} \left(\frac{3r}{2}\right)^{1/5} y^{2/5} & \text{if } y \leq y_c, \\ \left(\frac{3}{4}\right)^{1/3} y^{2/3} & \text{else,} \end{cases}$$

where $y_c := \left(\frac{128\, r^3}{9}\right)^{1/4} \approx 1.94\, r^{3/4}$.

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# Numerical scheme for $f$: Asexual case

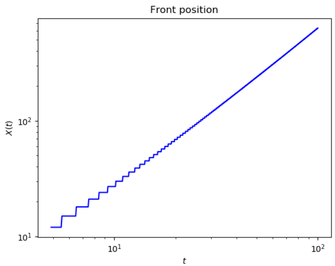
$$\begin{cases} \partial_t f \; - \; \theta \Delta_x f \; = \; r \Delta_\theta f \; + \; r\, f \,(1 \; - \; \rho)\,, \\ \\ f_0(x,\theta) \; = \; \sqrt{\frac{2}{\pi}}\, e^{-(x^2+(\theta-1)^2)/2}. \end{cases}$$

**Parameters:** $\Delta x = 3,\ \Delta\theta = 0.23,\ \Delta t = 0.05,\ r = 1$

## Evolution of $\rho(t,x)$

# Numerical scheme for $f$: Asexual case

## Front state




Front position (Log-Log scale)

- **Linear regression of the front position:** Speed of propagation of order $t^{1.48}$

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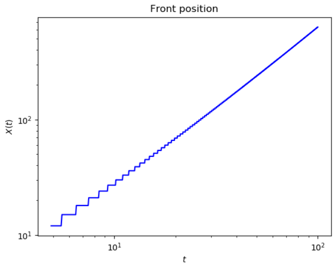


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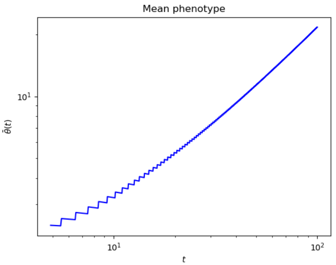


Mean phenotype (Log-Log scale)

- **Linear regression of the front position:** Speed of propagation of order $t^{1.48}$
- **Linear regression of the mean trait:** Speed of propagation of order $t^{0.96}$

# Numerical scheme for $f$: sexual case

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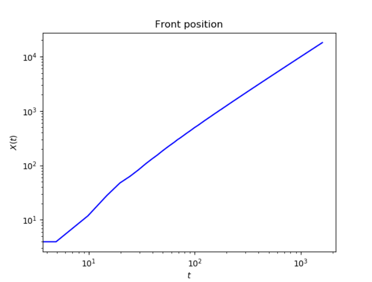
**Parameters:** $\Delta x = 4, \; \Delta\theta = 0.7, \; \Delta t = 0.05, \; r = 1$

## Evolution of $\rho(t,x)$

# Numerical scheme for $f$: Sexual case

## Front state

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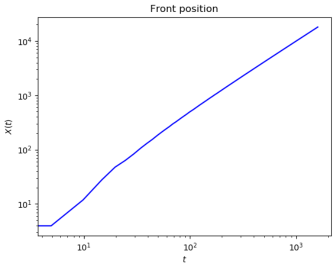
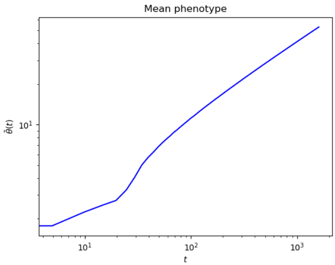
# Numerical scheme for $f$: Sexual case

## Front state




Front position (Log-Log scale)

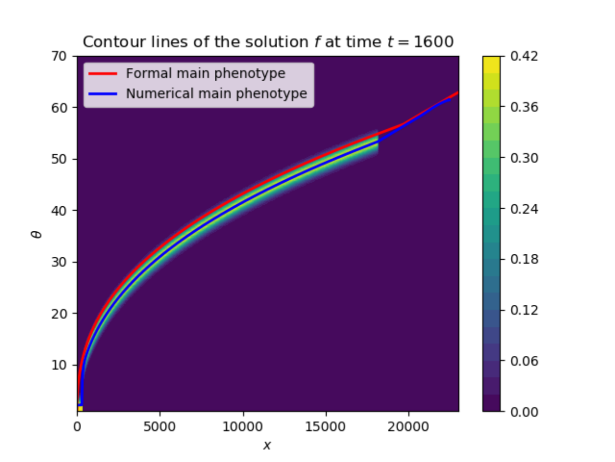



Mean phenotype (Log-Log scale)

- **Linear regression of the front position:** Speed of propagation of order $t^{1.27}$
- **Linear regression of the mean trait:** Speed of propagation of order $t^{0.53}$

# Numerical scheme for $f$: Sexual case

## Contour lines of the solution $f$




Contour lines at time $t = 1600$, with $r = 1$ and $f_0(x, \theta) = \sqrt{\frac{2}{\pi}} e^{-(x^2+(\theta-1)^2)/2}$

# Numerical scheme for $u$: Asexual case

**Hopf-Cole transformation:** We define $u$ such that

$$f(e^s, y, \eta) := \exp\left[-e^{-s}\, u(s, y, \eta)\right].$$

## Equation for the asexual reproduction case

$$\partial_s u + \eta e^{(2-2\alpha+\beta)s}(\partial_y u)^2 - \alpha y \partial_y u + r e^{(2-2\beta)s}(\partial_\eta u)^2 - \beta\eta\partial_\eta u$$
$$= -u + \eta e^{(1-2\alpha+\beta)s}\Delta_y u + r\left[e^{(1-2\beta)s}\Delta_\eta u - 1 + \rho\right].$$

Two hamiltonians

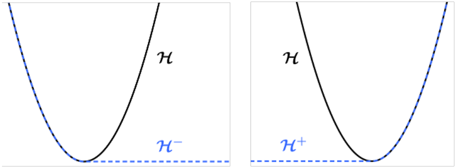
- $\mathcal{H}_1(s, \eta, y, u) := \eta e^{(2-2\alpha+\beta)s}(\partial_y u)^2 - \alpha y \partial_y u$
- $\mathcal{H}_2(s, \eta, y, u) := r e^{(2-2\beta)s}(\partial_\eta u)^2 - \beta\eta\partial_\eta u$

Numerical scheme for Hamilton-Jacobi equations: Crandall & Lions (1984)

## Purpose

Numerically show that if $(\alpha, \beta) \neq \left(\frac{3}{2}, 1\right)$, then $u$ converges towards a constant function or the indicator function of $\{0\}$

# Numerical scheme for $u$: Asexual case




$$\mathcal{H} = \max\left\{\mathcal{H}^-, \mathcal{H}^+\right\}$$

## Numerical scheme for $u_{i,j}^n \approx u(s^n, y_i, \eta_j)$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta s} + \max\left\{\mathcal{H}_1^-\left(\frac{u_{i+1,j}^n - u_{i,j}^n}{\Delta y}\right), \mathcal{H}_1^+\left(\frac{u_{i,j}^n - u_{i-1,j}^n}{\Delta y}\right)\right\}$$
$$+ \max\left\{\mathcal{H}_2^-\left(\frac{u_{i,j+1}^n - u_{i,j}^n}{\Delta \eta}\right), \mathcal{H}_2^+\left(\frac{u_{i,j}^n - u_{i,j-1}^n}{\Delta \eta}\right)\right\}$$
$$= -u_{i,j}^n + \eta_j e^{(1-2\alpha+\beta)s^n}(A_y u^n)_{i,j} + r\left[e^{(1-2\beta)s^n}(A_\eta u^n)_{i,j} - 1 + \rho_i^n\right],$$

where $A_y$ and $A_\eta$ are the discrete Laplace matrices with Neumann boundary conditions respectively in $y$ and $\eta$.

# Numerical scheme for $u$: Asexual case

**Parameters:** $\Delta y = 10^{-4}$, $\Delta \eta = 10^{-4}$, $\Delta s = 10^{-5}$,

$$u_0(y, \eta) = \frac{1}{2}\left(y^2 + \eta^2\right)$$

## Evolution of $\rho$ with rescaled variables and $\beta = 1$

$\alpha = 1$ $\alpha = 3/2$ $\alpha = 2$

- Confirmation of the propagation speed of order $1.3\, t^{3/2}$

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## Conclusion

- Numerical scheme for the PDEs satisfied by the density $f$ and the Hopf-Cole transformation for the asexual case

  $\rightsquigarrow$ Validation of the acceleration in power $3/2$ and of the acceleration constant
- Formal study of the propagation diffusion for the sexual case

$$x(t) = 1.94\, r^{3/4}\, t^{5/4}$$

- Numerical scheme for the PDE satisfied by the density $f$ for the sexual case

  $\rightsquigarrow$ Validation of the acceleration in power $5/4$

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## Prospects

- Scheme for the Hopf-Cole transformation $u$ for the sexual case
- Validation of the acceleration constant

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Thank You!


Thank you for
your attention!