Fluid-structure interaction in the cardiovascular system

Part 1 (continued)
Cardiac valves

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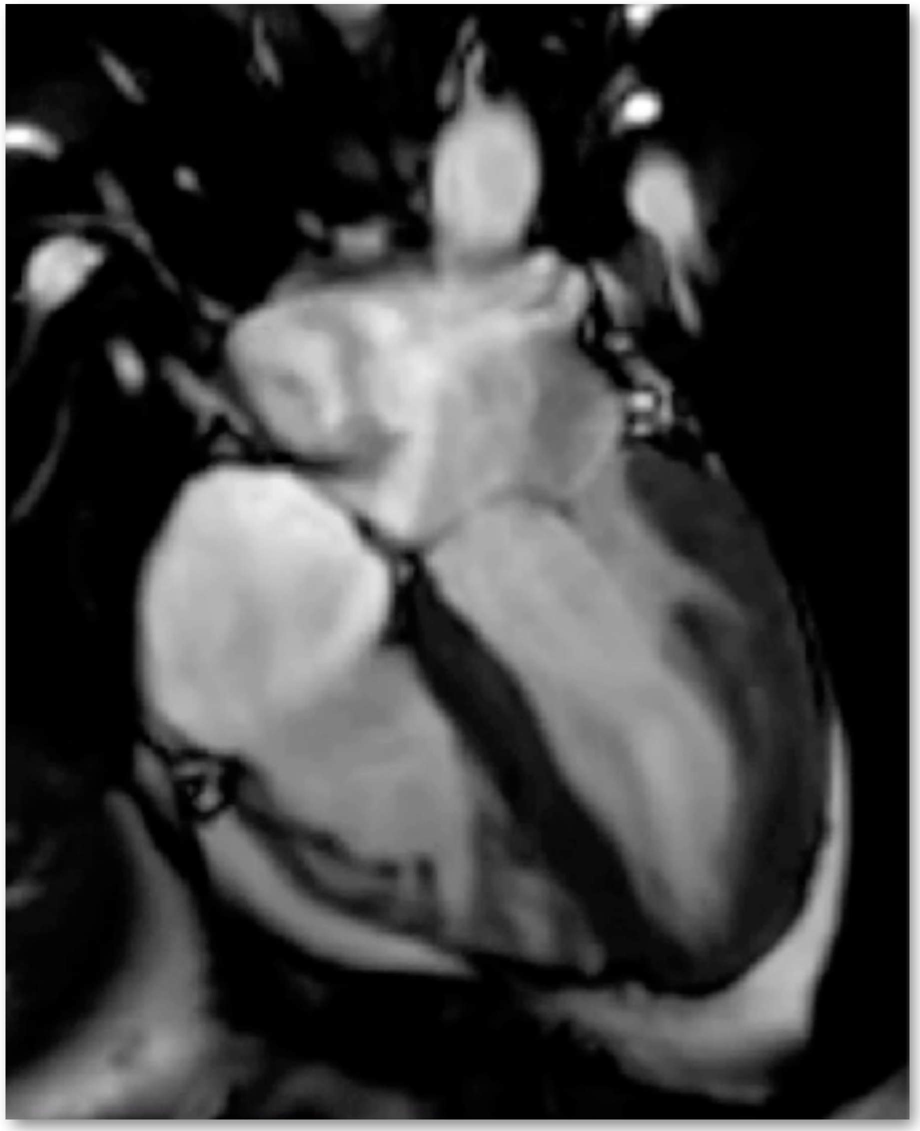
FSI in the heart

- Interaction blood/myocardium
- Interaction blood/valves
- Difficulties
  - large displacements
  - fast dynamics
  - high pressure drop
  - isovolumic phases
  - change of topology
  - …

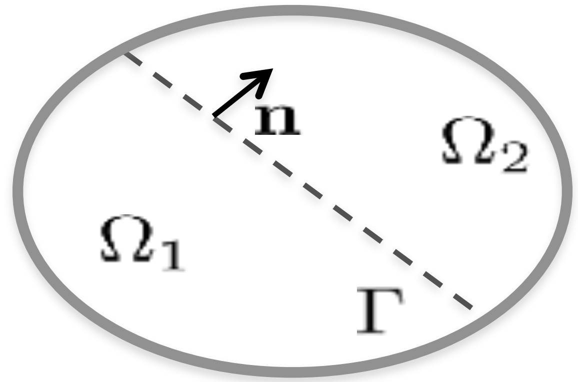


Outline

- Resistive Immersed Surface
- FSI with Lagrange multiplier
- A new FSI explicit scheme for valves

Resistive Immersed Surfaces

- Two subdomains & “resistive” (**porous**) interface :



$$\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$$

- Incompressible Navier-Stokes on each subdomain (in $\Omega_i, i = 1, 2$)

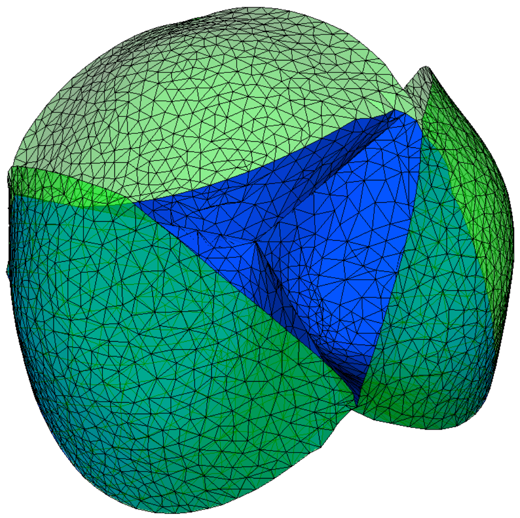
$$\begin{aligned} \rho_f(\partial_t \boldsymbol{u}_i + \boldsymbol{u}_i \cdot \boldsymbol{\nabla} \boldsymbol{u}_i) + \boldsymbol{\nabla} p_i - \mu \Delta \boldsymbol{u}_i &= 0 \\ \operatorname{div} \boldsymbol{u}_i &= 0 \end{aligned}$$

- Transmission conditions :

$$\llbracket 2\mu \boldsymbol{\epsilon}(\boldsymbol{u}) \cdot \mathbf{n} - p\mathbf{n} \rrbracket = -r_\Gamma \boldsymbol{u}$$
$$\llbracket \boldsymbol{u} \rrbracket = 0$$

Notation: $\boldsymbol{\epsilon}(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)$ and $\llbracket q\boldsymbol{n} \rrbracket = q_1|_\Gamma \boldsymbol{n}_1 + q_2|_\Gamma \boldsymbol{n}_2 = (q_1 - q_2)|_\Gamma \boldsymbol{n}$

Resistive Immersed Surfaces



- Simplified model with **resistive immersed surfaces**
- Idea: “0D dynamics” and “3D geometry”
  - Two “double-surfaces” (open / closed)
  - “Close” surface (blue)

```
If the valve is closed
    If (p^+ - p^-) < 0 then r_Γ = 0
Else
If ∫_Γ u·n < 0 then r_Γ = +∞
```

If the valve is closed

  If $(p^+ - p^-) < 0$ then $r_\Gamma = 0$

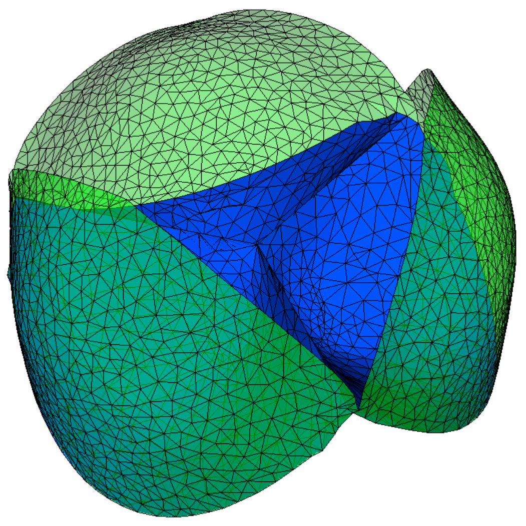
Else

If $\int_\Gamma \mathbf{u}\cdot\mathbf{n} < 0$ then $r_\Gamma = +\infty$

  - And the contrary for the “open” surface (green)

- **Time scheme :** the status of the valve is decided *before* the new time step solution and checked *a posteriori*. In case of an incompatibility, the time step is redone with the correct valve status.

Resistive Immersed Surfaces



- Cons
  - ⋆ no valve dynamics
  - ⋆ positions of valves have to be chosen
- Pros
  - ⋆ With respect to 3D FSI model of valves
    - CPU time, robustness
    - accurate trans-valvular pressure drop (systole & diastole)
  - ⋆ With respect to 0D models of valve
    - reasonable jet shape (“Effective Orifice Area”)
    - no boundary conditions artifacts
  - ⋆ Model complexity adapted to available data

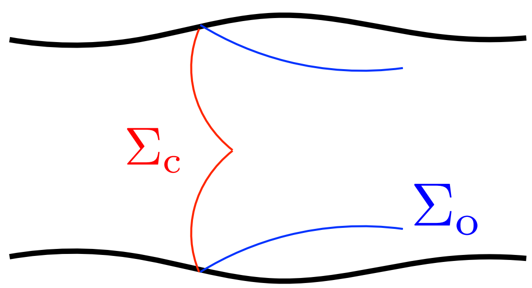
M Astorino, J Hamers, SC Shadden, JF Gerbeau. A robust and efficient valve model based on resistive immersed surfaces. *International Journal for Numerical Methods in Biomedical Engineering (2012), 28(9), 937-959.*

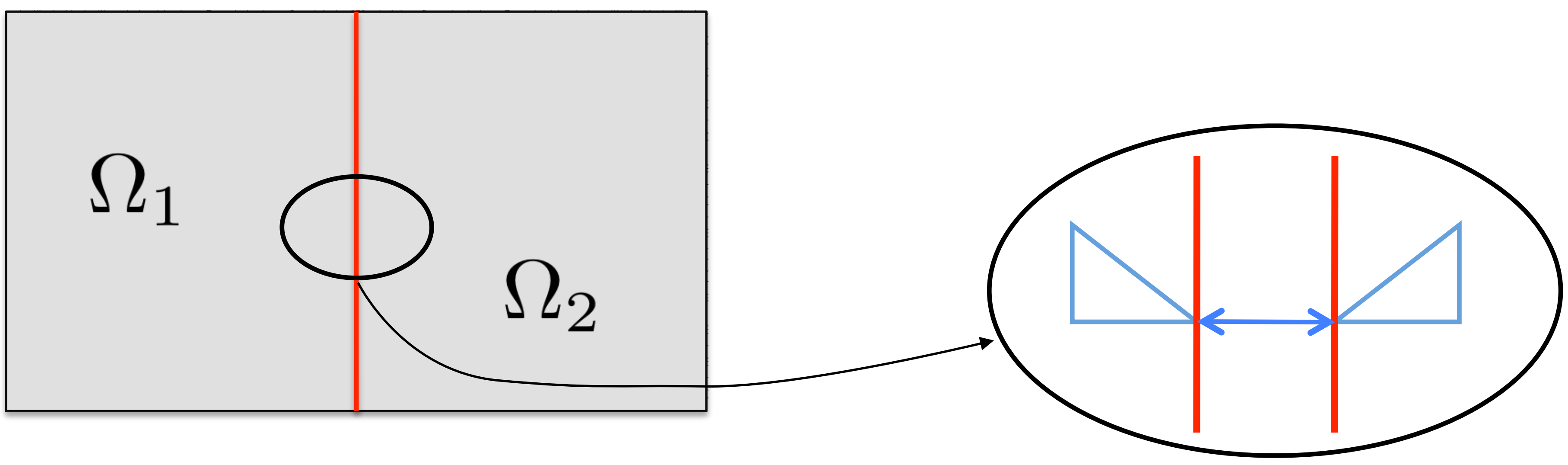
M Fedele, E Faggiano, L Dede’, A Quarteroni. A patient-specific aortic valve model based on moving resistive immersed implicit surfaces. *Biomechanics and Modeling in Mechanobiology (2017)*, 16(5), 1779-1803.

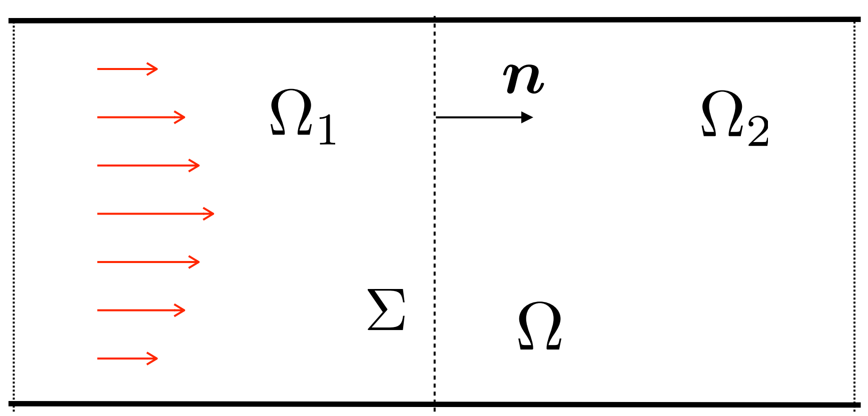
Resistive Immersed Surfaces

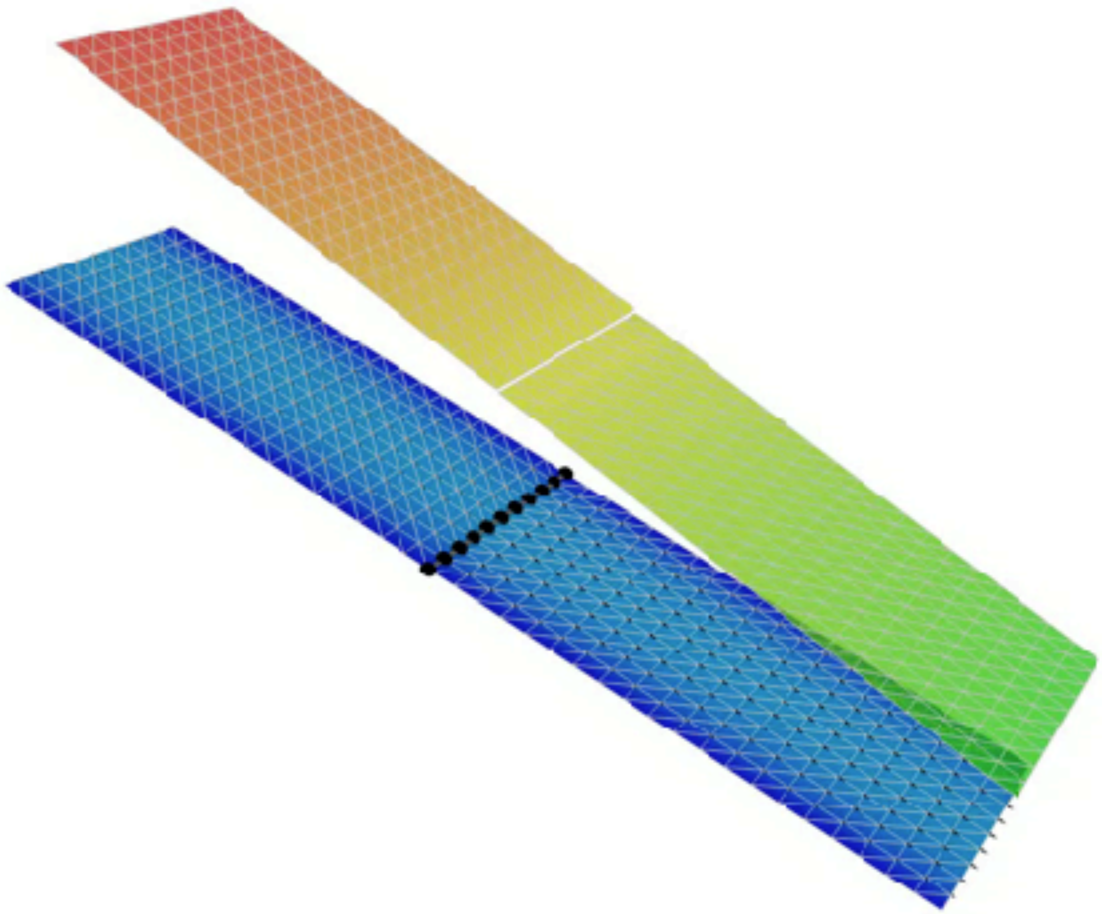
$$\begin{cases} \rho^{\mathrm{f}}\big(\partial_t|_{\mathcal{A}}\boldsymbol{u} + (\boldsymbol{u}-\boldsymbol{w})\cdot\nabla\boldsymbol{u}\big) - \mathbf{div}\boldsymbol{\sigma}(\boldsymbol{u},p) + R(\boldsymbol{u}-\boldsymbol{w})\delta_{\Sigma(t)} = \mathbf{0} & \text{in} \quad \Omega(t) \\ \mathrm{div}\boldsymbol{u} = 0 & \text{in} \quad \Omega(t) \end{cases}$$

$$\begin{cases} -[\![\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n}]\!] = R\boldsymbol{u} & \text{on} \quad \Sigma_{\mathrm{c}} \\ -[\![\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n}]\!] = \big(R_{\mathrm{closed}} - R\big)\boldsymbol{u} & \text{on} \quad \Sigma_{\mathrm{o}} \end{cases}$$



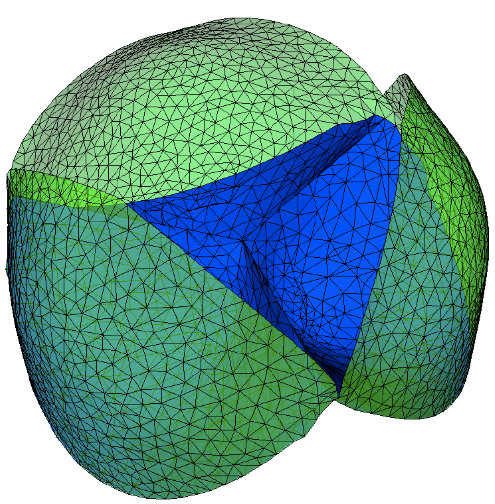



Ω_1
$\boldsymbol{n}$
Ω_2
Σ
Ω

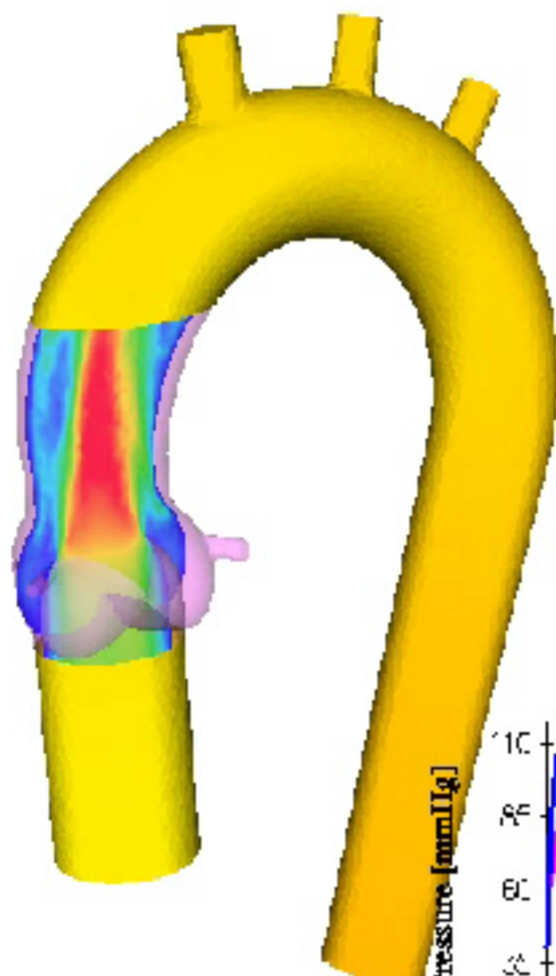


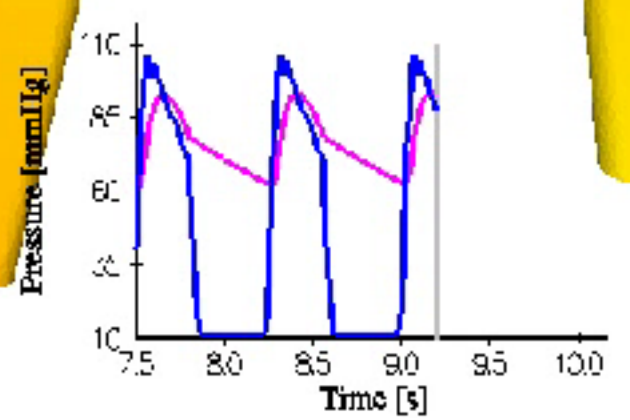
Normal and “Stenotic” Aortic Valve

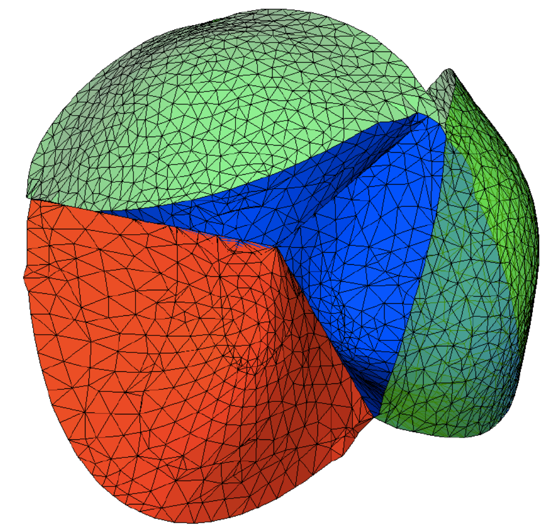
normal stenotic



Normal configuration



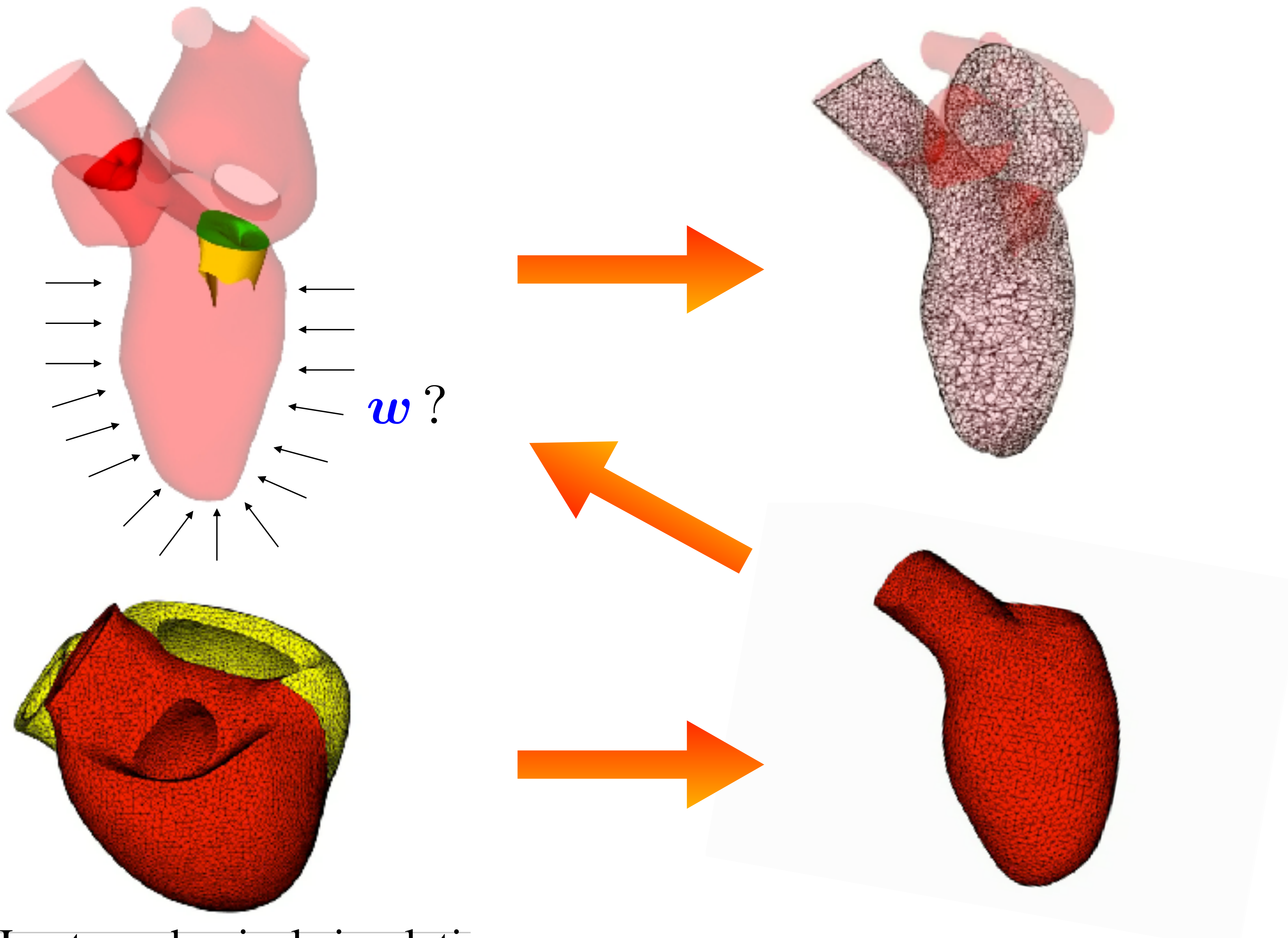




“Stenotic” configuration

Simulations: M. Astorino

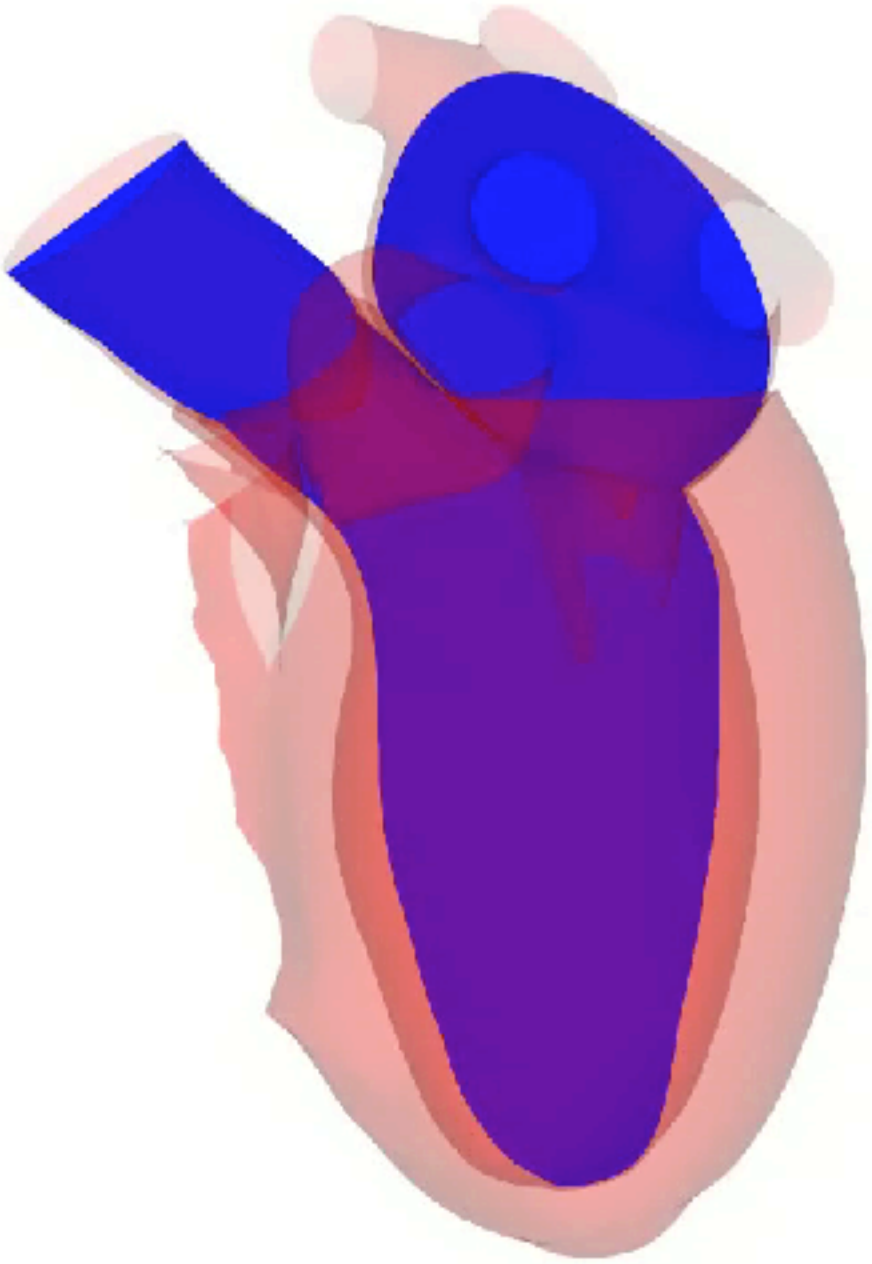
Resistive Immersed Surfaces



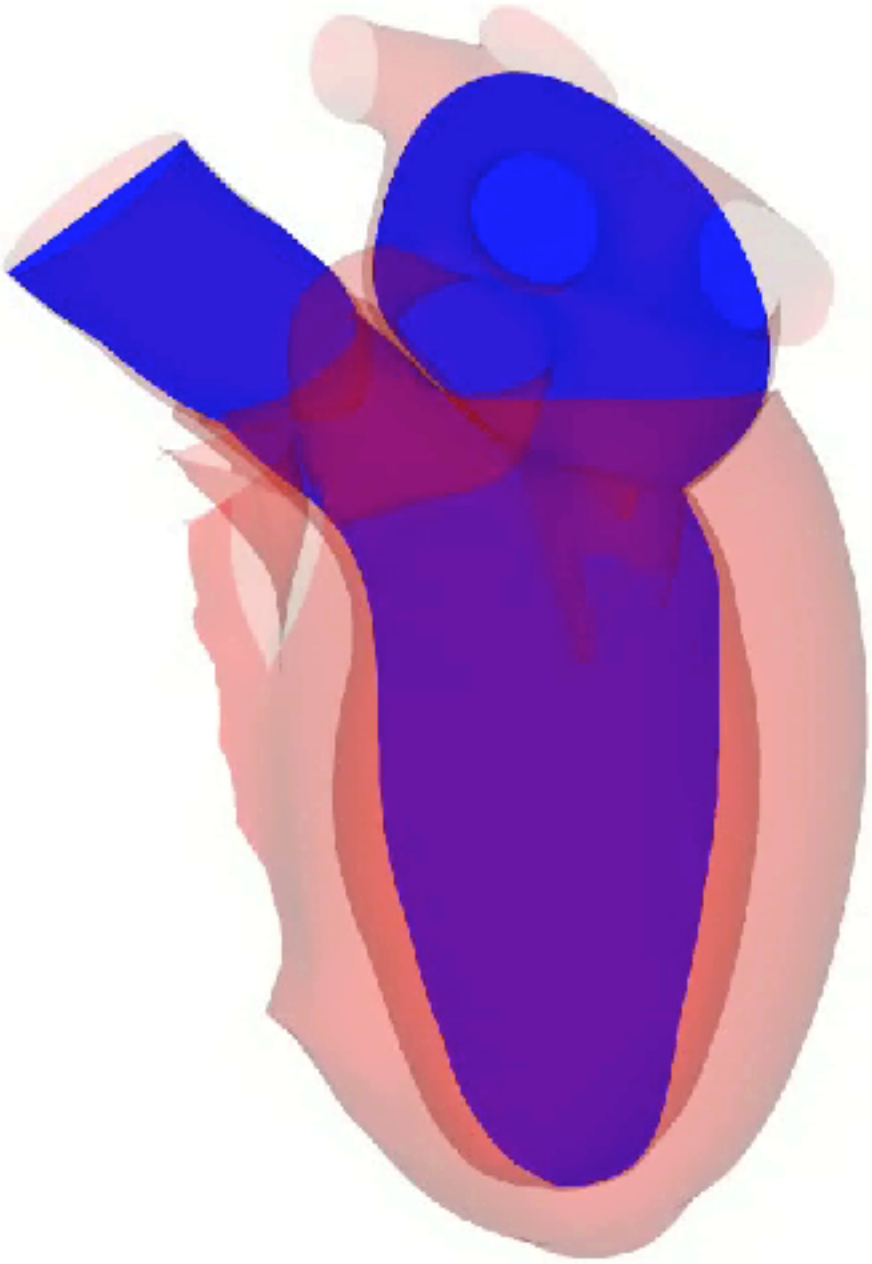
Heart mechanical simulation
with reduced fluid model

(G. Bureau, D. Chapelle, P. Moireau, INRIA)

Resistive Immersed Surfaces



Normal condition



Mitral regurgitation

(Simulations by L. Boilevin-Kayl & A. This)

Outline

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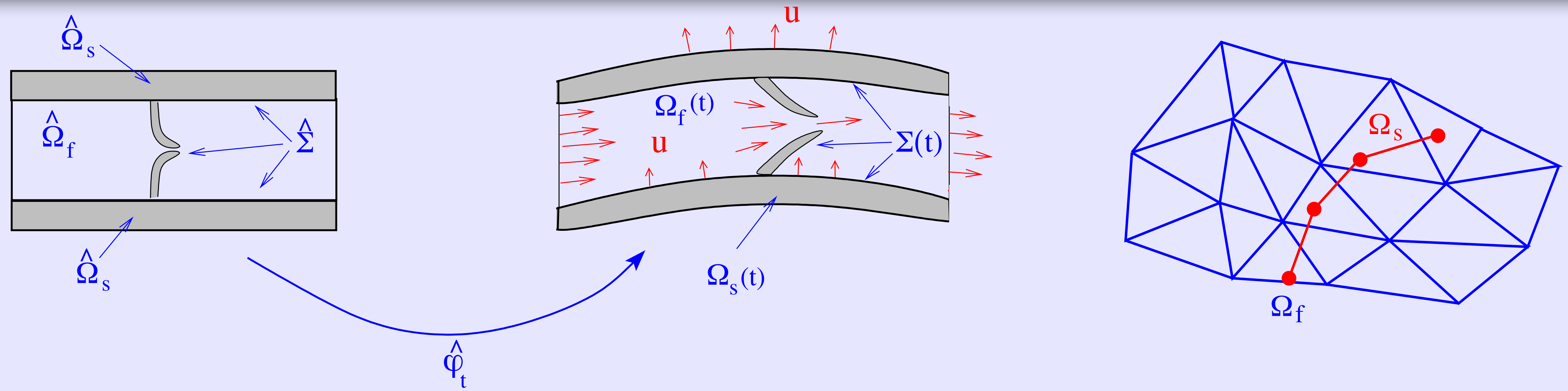
- FSI with Lagrange multiplier
- A new FSI explicit scheme for valves

FSI with Lagrange multipliers

Valve : “Fictitious Domains” (FD)



Pros:

- large displacement without re-meshing
- change of topology

Cons:

- difficult to avoid leaks, to capture the pressure drop (closed valves)
- until recently: efficient partitioned coupling scheme was missing

Lagrange multiplier approach

Basic idea:

- Write the kinematics relation:

$$\boldsymbol{u} = \dot{\boldsymbol{d}}, \qquad \text{on } \Sigma$$

as a constraint imposed on the coupled system, under the weak form:

$$b(\boldsymbol{\mu}, \boldsymbol{u}) = b(\boldsymbol{\mu}, \dot{\boldsymbol{d}}), \quad \forall \boldsymbol{\mu} \in \Lambda$$

$$b : \boldsymbol{\Lambda} \times [H^1(\Sigma)]^d \to \mathbb{R}$$

$$b(\boldsymbol{\mu}, \boldsymbol{z}) = 0, \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda} \implies \boldsymbol{z} = \mathbf{0} \text{ on } \Sigma.$$

- The Lagrange multiplier corresponding to this constraint is:

$$\boldsymbol{\lambda} = -\boldsymbol{\sigma}^f \cdot \boldsymbol{n}^f = \boldsymbol{\sigma}^s \cdot \boldsymbol{n}^s$$

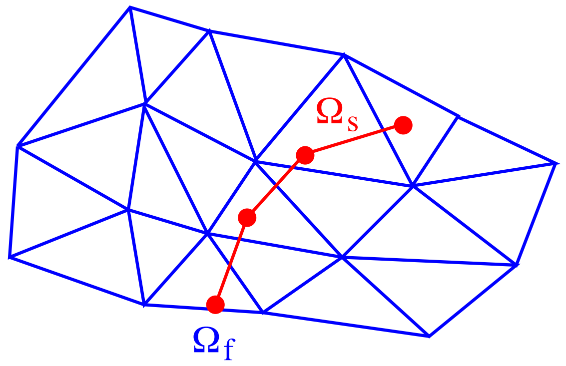
Lagrange multiplier approach

Discretization of the Lagrange multiplier space:

- **Choice 1:**

$$\Lambda_h = \left\{ \mu_h \in L^2(\Sigma_h) \Big/ \ \mu_h \text{ piecewise polynomial} \right\}$$

$$b(\boldsymbol{\mu}, \boldsymbol{z}) = \int_{\Sigma_h} \boldsymbol{\mu} \cdot \boldsymbol{z} \, ds$$



F. Baaijens. “A fictitious domain/mortar element method for fluid–structure interaction”, *Int. J. Num. Meth. Fluids* (2001), 35, 743-761

Girault, V., Glowinski, R., López, H., & Vila, J. P. “A boundary multiplier/fictitious domain method for the steady incompressible Navier-Stokes equations.”, *Numerische Mathematik* (2001), 88(1), 75-103.

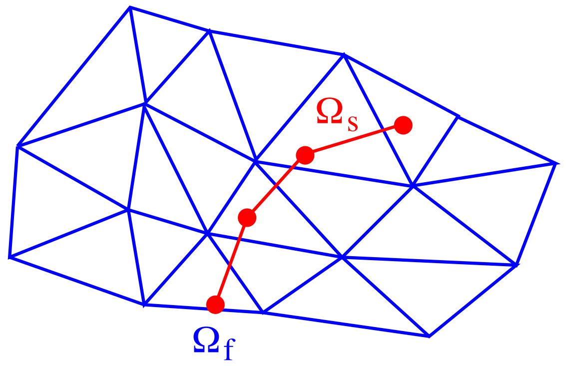
Lagrange multiplier approach

Discretization of the Lagrange multiplier space:

- **Choice 2:**

$$\Lambda_h = \left\{ \boldsymbol{\mu}_h = \sum_{i=1}^{N_h^{\mathrm{s}}} \boldsymbol{\mu}_i \delta_{\boldsymbol{x}_i^{\mathrm{s}}} \Big/ \boldsymbol{\mu}_i \in \mathbb{R}^d, \quad i = 1, \ldots, N_h^{\mathrm{s}} \right\}$$

$$b(\boldsymbol{\mu}, \boldsymbol{z}) = \langle \boldsymbol{\mu}, \boldsymbol{z} \rangle = \sum_{i=1}^{N_j^{\mathrm{s}}} \boldsymbol{\mu}_i \cdot \boldsymbol{z}(x_i^{\mathrm{s}})$$



N. Dos Santos, J. F. Gerbeau, J.F. Bourgat. A partitioned fluid–structure algorithm for elastic thin valves with contact. *Computer Methods in Applied Mechanics and Engineering* (2008), 197(19), 1750-1761.

B. Fabrèges, B. Maury, "Approximation of single layer distributions by Dirac masses in Finite Element computations". *Journal of Scientific Computing* (2014), 58(1), 25-40.

Dirac Lagrange multipliers

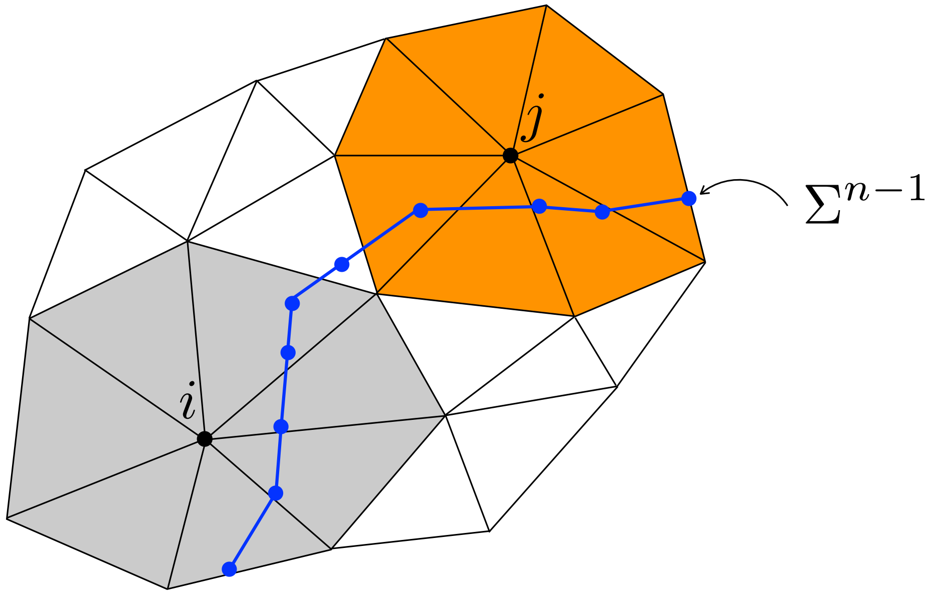
- Kinematic constraint with Dirac Lagrange multipliers

$$b(\boldsymbol{\mu}, \boldsymbol{u}_f - \boldsymbol{u}_s) = \sum_{i=1}^{N_j^{\mathrm{s}}} \boldsymbol{\mu}_i \cdot (\boldsymbol{u}_f(x_i^{\mathrm{s}}) - \boldsymbol{u}_s(x_i^{\mathrm{s}})) = 0, \quad \forall \boldsymbol{\mu}_i \in \mathbb{R}^d$$

$$\Longrightarrow \ \boldsymbol{u}_f(x_i^{\mathrm{s}}) = \boldsymbol{u}_{s,i}$$

- **Algebraic form:** let K be the fluid-to-solid interpolation matrix

$$\mathrm{K}\mathrm{u}_f^n = \mathrm{u}_s^n$$



Dirac Lagrange multipliers

- **To simplify the presentation:** omit the pressure & treat the velocity as a scalar.

$$\begin{bmatrix} \mathrm{A}^{\mathrm{f}} & \mathrm{K}^{\mathrm{T}} \\ \mathrm{K} & 0 \end{bmatrix} \begin{bmatrix} \mathrm{u}_f^n \\ \lambda^n \end{bmatrix} = \begin{bmatrix} \mathrm{b}^{n-1} \\ \mathrm{u}_s^n \end{bmatrix},$$

- **Difficulties:**
  - saddle point problem
  - the pattern of matrix changes at every time step

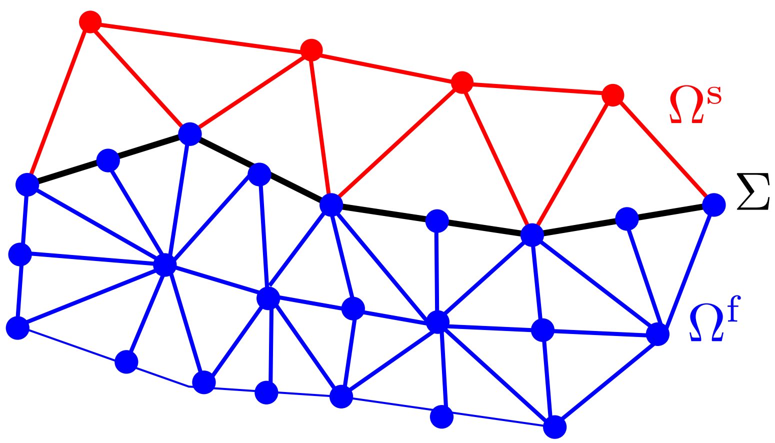
Side remark:

- **Reminder:** conservative computation of the load:

Variational residual: $\mathrm{R}^{\mathrm{f}} = \mathrm{F}^{\mathrm{f}} - \mathrm{A}^{\mathrm{f}}\,\mathrm{U}^{\mathrm{f}}$ → approx. of $-\left(\int_{\Sigma} \frac{\partial u^{\mathrm{f}}}{\partial \boldsymbol{n}^{\mathrm{f}}} v_i^{\mathrm{f}}\right)_{i=1..n^{\mathrm{f}}}$

Interpolation: $\mathrm{U}^{\mathrm{f}}_{\Sigma} = \mathrm{K}_{\mathrm{s}}\,\mathrm{U}^{\mathrm{s}}_{\Sigma}$

Solid: $\mathrm{A}^{\mathrm{s}}\mathrm{U}^{\mathrm{s}} = \mathrm{F}^{\mathrm{s}} + \mathrm{K}_{\mathrm{s}}^{T}\mathrm{R}^{\mathrm{f}}$

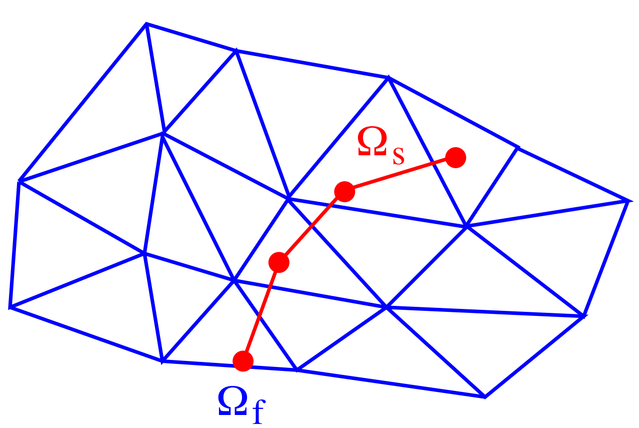


Thus:

$$\left\{\begin{array}{rcl} \mathrm{A}^{\mathrm{f}}\mathrm{U}^{\mathrm{f}} + \mathrm{R}^{\mathrm{f}} & = & \mathrm{F}^{\mathrm{f}} \\ \mathrm{U}^{\mathrm{f}}_{\Sigma} - \mathrm{K}_{\mathrm{s}}\,\mathrm{U}^{\mathrm{s}}_{\Sigma} & = & 0 \\ \mathrm{A}^{\mathrm{s}}\mathrm{U}^{\mathrm{s}} - \mathrm{K}_{\mathrm{s}}^{T}\mathrm{R}^{\mathrm{f}} & = & \mathrm{F}^{\mathrm{s}} \end{array}\right.$$

- **To be compared with**

$$\left\{\begin{array}{rcl} \mathrm{A}^{\mathrm{f}}\mathrm{U}^{\mathrm{f}} + \mathrm{K}_{\mathrm{f}}^{T}\Lambda^{\mathrm{f}} & = & \mathrm{F}^{\mathrm{f}} \\ \mathrm{K}_{\mathrm{f}}\mathrm{U}^{\mathrm{f}}_{\Sigma} - \mathrm{U}^{\mathrm{s}}_{\Sigma} & = & 0 \\ \mathrm{A}^{\mathrm{s}}\mathrm{U}^{\mathrm{s}} - \Lambda^{\mathrm{f}} & = & \mathrm{F}^{\mathrm{s}} \end{array}\right.$$



- **A first approach:** regularization by penalization

$$\begin{bmatrix} \mathrm{A}^{\mathrm{f}} & \mathrm{K}^{\mathrm{T}} \\ \mathrm{K} & -\varepsilon I \end{bmatrix} \begin{bmatrix} \mathrm{u}_f^n \\ \lambda^n \end{bmatrix} = \begin{bmatrix} \mathrm{b}^{n-1} \\ \mathrm{u}_s^n \end{bmatrix},$$

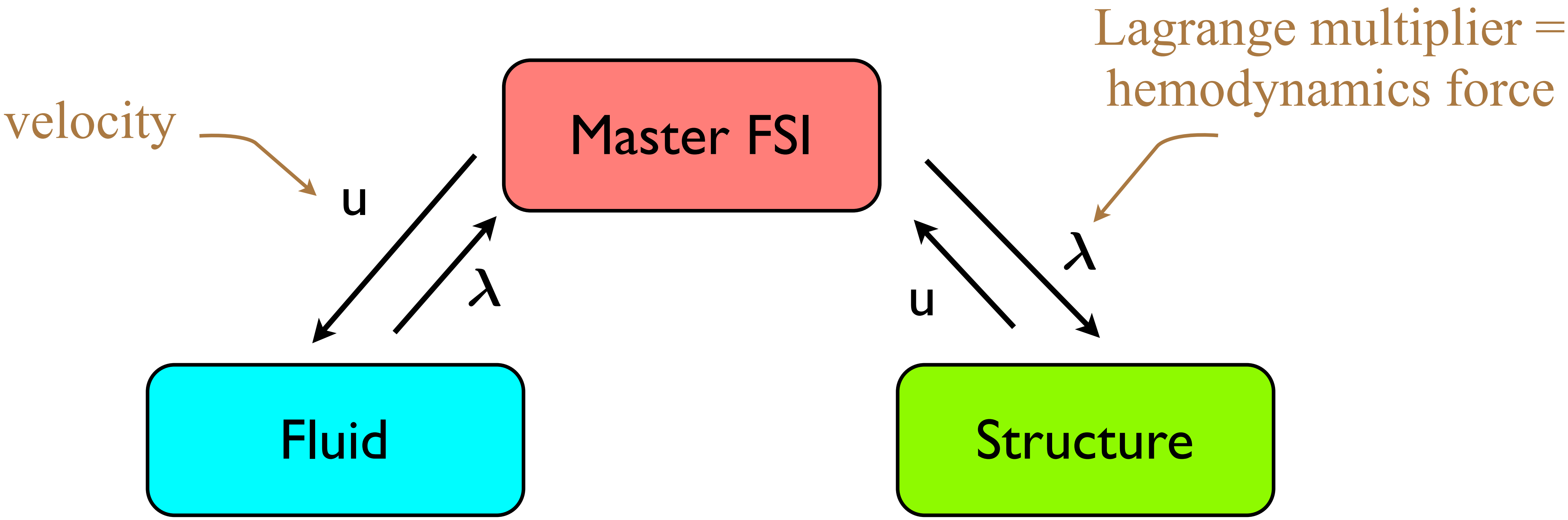
- Eliminating the Lagrange multiplier:

$$\left(\mathrm{A}^{\mathrm{f}} + \frac{1}{\varepsilon}\mathrm{K}^{\mathrm{T}}\mathrm{K}\right)\mathrm{u}_f^n = \mathrm{b}^{n-1} + \frac{1}{\varepsilon}\mathrm{K}^{\mathrm{T}}\mathrm{u}_s^n$$

the non-zero entries of $\mathrm{K}^{\mathrm{T}}\mathrm{K}$ are in the sparse pattern of A

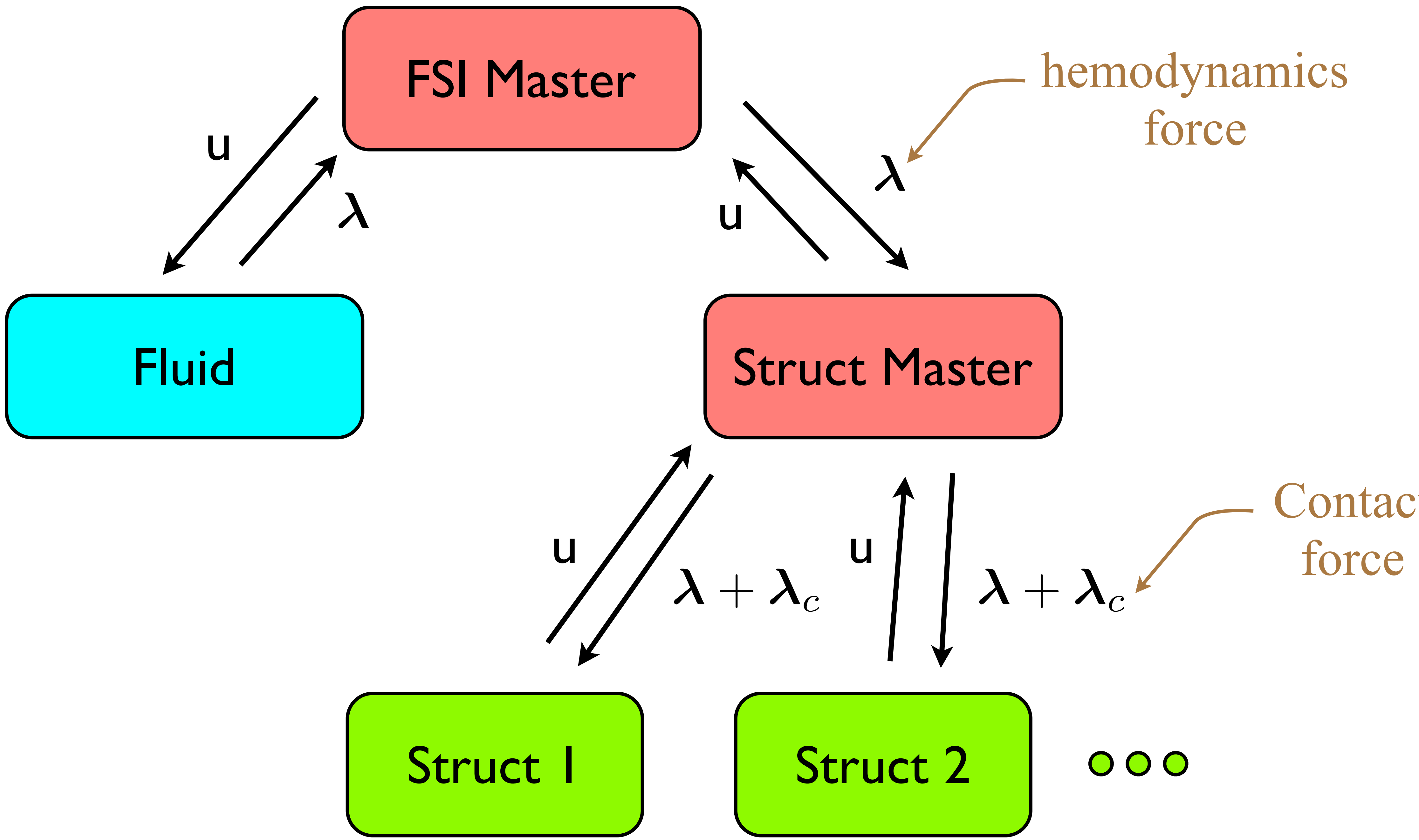
- But:
  - ill-conditioning
  - an efficient coupling scheme with the solid still has to be devised

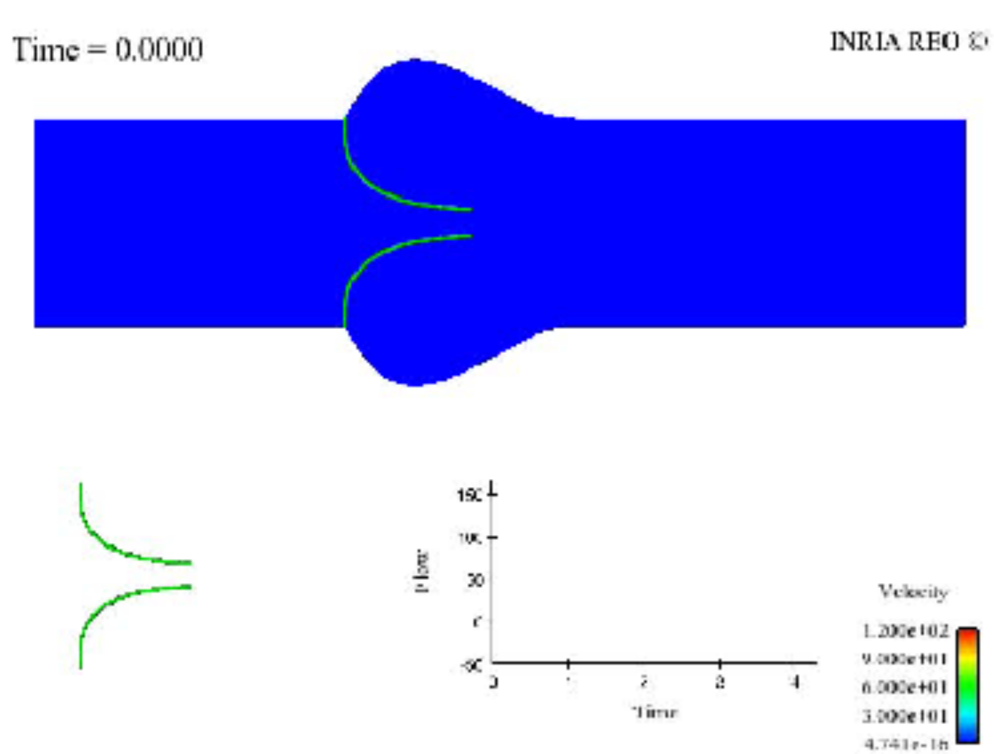
Implementation

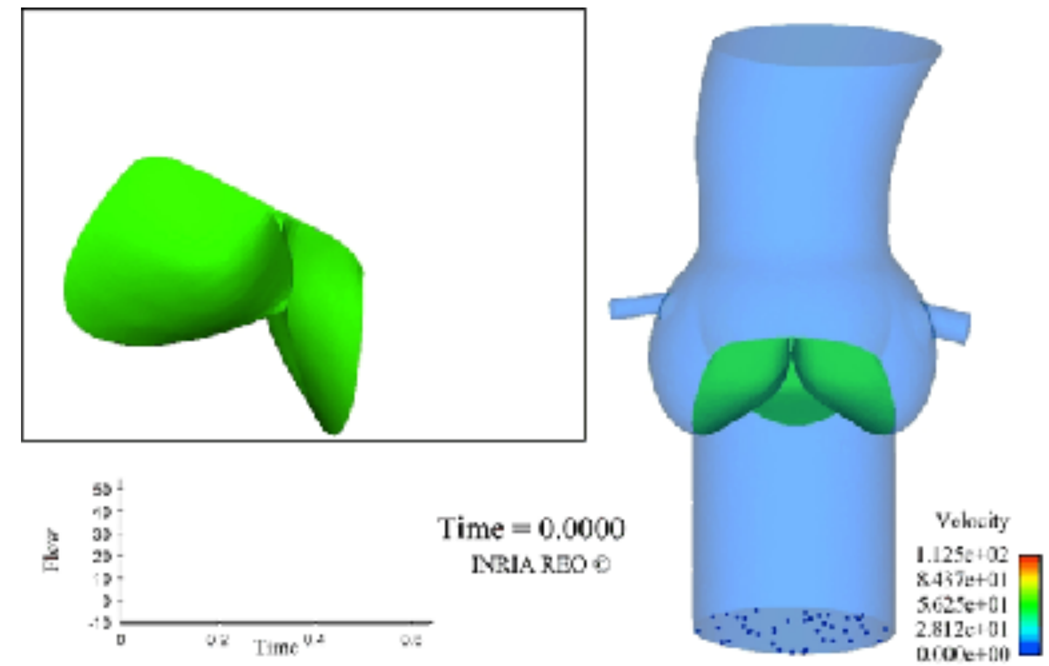


Implementation with contact

- The contact algorithm fits in the FSI framework (same data exchanged)
- The solid does not handle the contact by itself (very convenient!)







- Huge computational cost
- Difficulty to handle the correct pressure drop

Astorino, JFG, Pantz, Traore, *CMAME 09*

Outline

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- FSI with Lagrange multiplier
- FSI & contact
- A new FSI explicit scheme for valves

An explicit scheme for immersed solid

Implicit coupling with the Lagrange multiplier formulation:

$$\left\{\begin{array}{rcl} A_f u_f^n + K^T \lambda^n & = & b^{n-1} \\ K u_f^n & = & u_s^n \\ \rho_{\mathrm{s}} h_{\mathrm{s}} M_s \dfrac{u_s^n - u_s^{n-1}}{\tau} + A_s d^n & = & \lambda^n \end{array}\right.$$

Reminder: the Robin-Neumann explicit scheme

- Idea: only solid inertia needs to be implicitly coupled

- Fluid

$$\begin{cases} \rho^{\mathrm{f}} \partial_t \boldsymbol{u} - \mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{u}, p) = \mathbf{0} & \text{in} \quad \Omega^{\mathrm{f}} \\ \mathrm{div}\, \boldsymbol{u} = 0 & \text{in} \quad \Omega^{\mathrm{f}} \\ \boldsymbol{u} = \dot{\boldsymbol{d}} & \text{on} \quad \Sigma \end{cases}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} + \rho^{\mathrm{s}} \epsilon \partial_t \boldsymbol{u} = -\boldsymbol{L}^{\mathrm{e}} \boldsymbol{d} \quad \text{on} \quad \Sigma$$

- Thin solid

$$\begin{cases} \rho^{\mathrm{s}} \epsilon \partial_t \dot{\boldsymbol{d}} + \boldsymbol{L}^{\mathrm{e}} \boldsymbol{d} = -\boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} & \text{on} \quad \Sigma \\ \dot{\boldsymbol{d}} = \partial_t \boldsymbol{d} & \text{on} \quad \Sigma \end{cases}$$

$$\boxed{\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} + \frac{\rho^{\mathrm{s}} \epsilon}{\tau} \boldsymbol{u}^n = \frac{\rho^{\mathrm{s}} \epsilon}{\tau} \dot{\boldsymbol{d}}^{n-1} - \boldsymbol{L}^{\mathrm{e}} \boldsymbol{d}^\star} \quad \text{on} \quad \Sigma, \quad \boldsymbol{d}^\star = \left| \begin{array}{l} 0 \\ \boldsymbol{d}^{n-1} \\ \boldsymbol{d}^{n-1} + \tau \dot{\boldsymbol{d}}^{n-1} \end{array} \right.$$

- Added-mass free *and* parameter free

M Fernández. Incremental displacement-correction schemes for incompressible fluid-structure interaction: stability and convergence analysis, *Numerische Math.* (2012), vol. 123, p. 21-65.

Extension to the Lagrange multiplier formulation

Provisional step: intermediate solid velocity

$$
\text{(F)} \qquad \left\{ \begin{array}{rcl} A_f u_f^n + K^T \lambda^n & = & b^{n-1} \\ K u_f^n & = & \tilde{u}_s^n \\ \frac{\rho_s h_s}{\tau} M_s \tilde{u}_s^n - \lambda^n & = & \frac{\rho_s h_s}{\tau} M_s u_s^{n-1} - A_s d^{n-1} \end{array} \right.
$$

$$
\text{(S)} \qquad \rho_s h_s M_s \frac{u_s^n - u_s^{n-1}}{\tau} + A_s d^n = \lambda^n
$$

Elimination of the intermediate velocity:

$$
\lambda^n = \frac{\rho_s h_s}{\tau} M_s K u_f^n - \frac{\rho_s h_s}{\tau} M_s u_s^{n-1} + A_s d^{n-1}
$$

An explicit scheme for immersed thin solid

– Eliminating d^{n-1} and using the solid equation at time t^{n-1}:

$$\lambda^n = \frac{\rho_s h_s}{\tau} M_s (K u_f^n - 2u_s^{n-1} + u_s^{n-2}) + \lambda^{n-1}$$

– Eliminating λ^n in the fluid:

$$\left(A_f + \frac{\rho_s h_s}{\tau} K^T M_s K \right) u_f^n = b^{n-1} + 2K^T u_s^{n-1} - K^T u_s^{n-2} - K^T \lambda^{n-1}$$

Better conditioned than penalization

Preserve the sparsity if the mass matrix is lumped

– Solid resolution: $$\rho_s h_s M_s \frac{u_s^n - u_s^{n-1}}{\tau} + A_s d^n = \lambda^n$$

A loosely coupled scheme for fictitious domain approximations of fluid-structure interaction problem with immersed thin-walled structures, *L. Boilevin-Kayl, M. Fernández, J.F. Gerbeau, (2018)*, https://hal.inria.fr/hal-01811290

Proposition *(L. Boilevin-Kayl, JFG, M. Fernández, 2018)*

- Let $\{(\boldsymbol{u}_h^n, p_h^n, \dot{\boldsymbol{d}}_h^n, \boldsymbol{d}_h^n)\}_{n\geq 1}$ be given by the loosely coupled algorithm

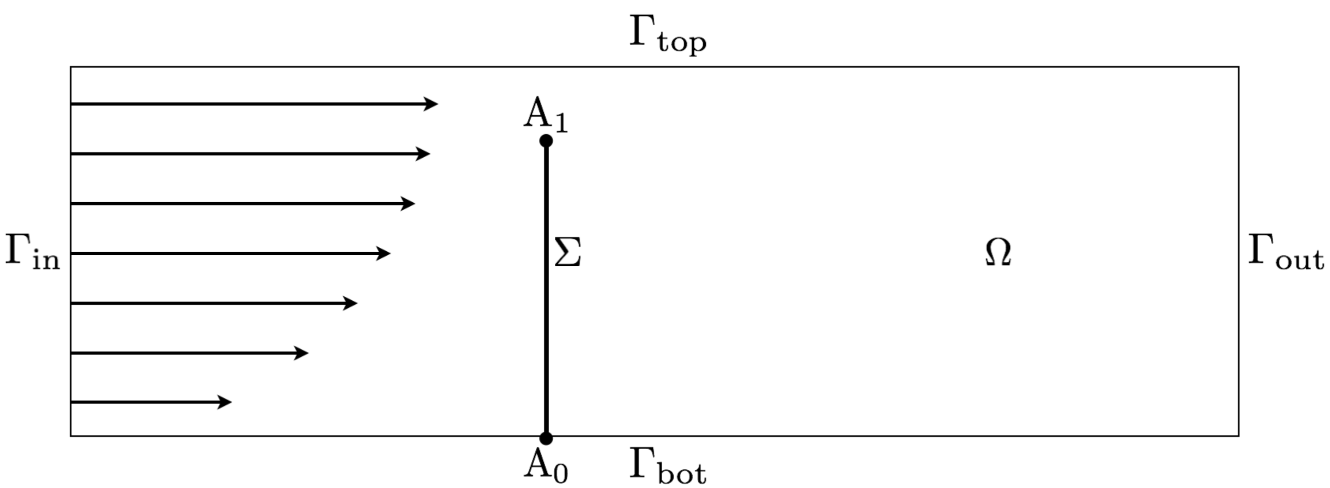
- We have the stability estimate:

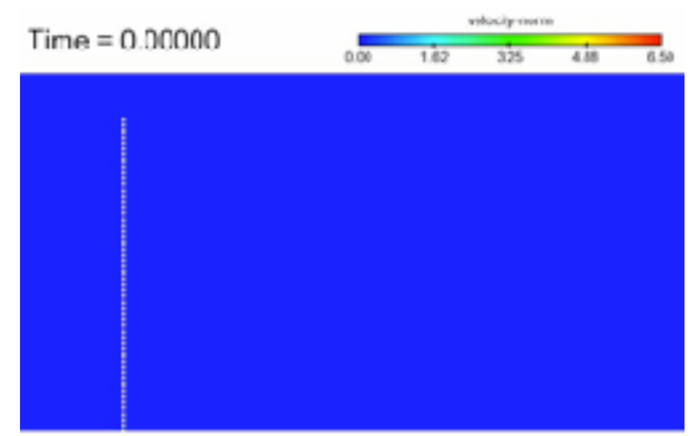
$$E^n \lesssim E^0 + \tau^2 \|\dot{\boldsymbol{d}}_h^0\|_{\mathrm{s}}^2 + \frac{\tau^2}{\rho^{\mathrm{s}}\epsilon^{\mathrm{s}}} \|\boldsymbol{L}_h^{\mathrm{s}} \boldsymbol{d}_h^0\|_{0,\Sigma}^2$$

with $E^n = \dfrac{\rho^{\mathrm{f}}}{2} \|\boldsymbol{u}_h^n\|_{0,\Omega}^2 + \dfrac{\rho^{\mathrm{s}}\epsilon^{\mathrm{s}}}{2} \|\dot{\boldsymbol{d}}_h^n\|_{0,\Sigma}^2 + \dfrac{1}{2} \|\boldsymbol{d}_h^n\|_{\mathrm{s}}^2.$

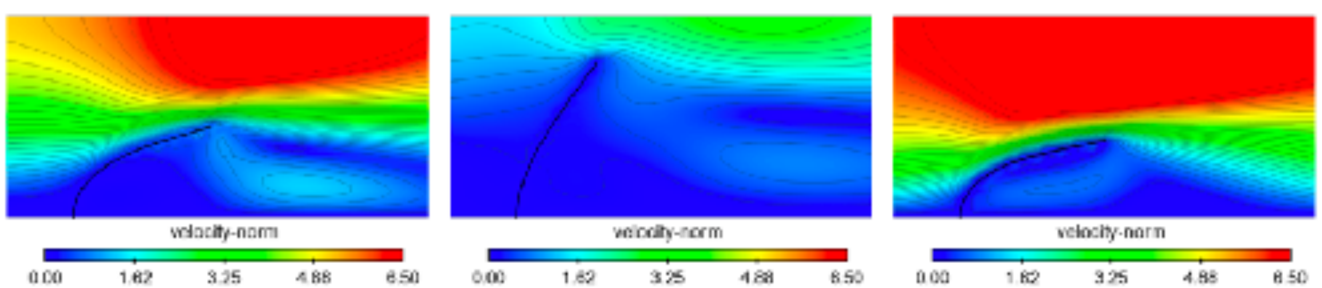
A loosely coupled scheme for fictitious domain approximations of fluid-structure interaction problem with immersed thin-walled structures, *L. Boilevin-Kayl, M. Fernández, J.F. Gerbeau, (2018),* https://hal.inria.fr/hal-01811290

1st test case: comparison with implicit scheme



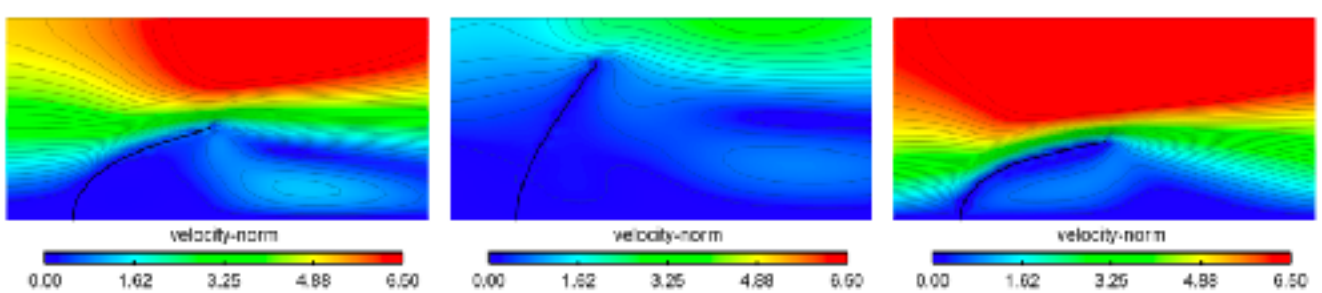


Implicit:

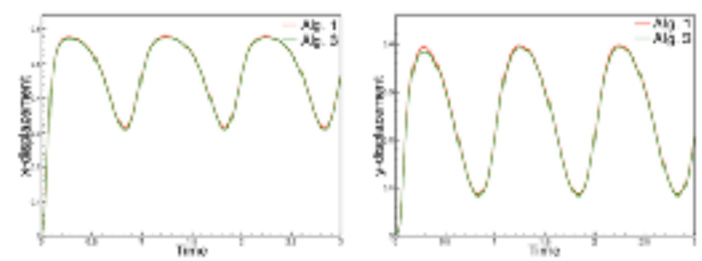


(a) Algorithm 4.1: $t = 0.45$. (b) Algorithm 4.1: $t = 0.85$. (c) Algorithm 4.1: $t = 1.25$.

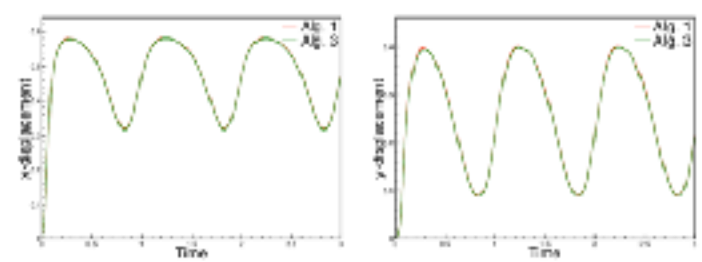
Explicit:



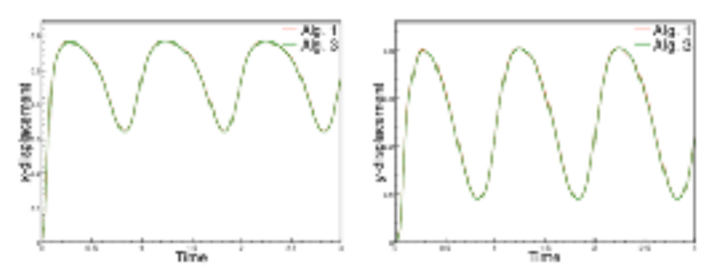
(d) Algorithm 4.3: $t = 0.45$. (e) Algorithm 4.3: $t = 0.85$. (f) Algorithm 4.3: $t = 1.25$.



(a) $\tau = 10^{-3}$. (b) $\tau = 10^{-3}$.



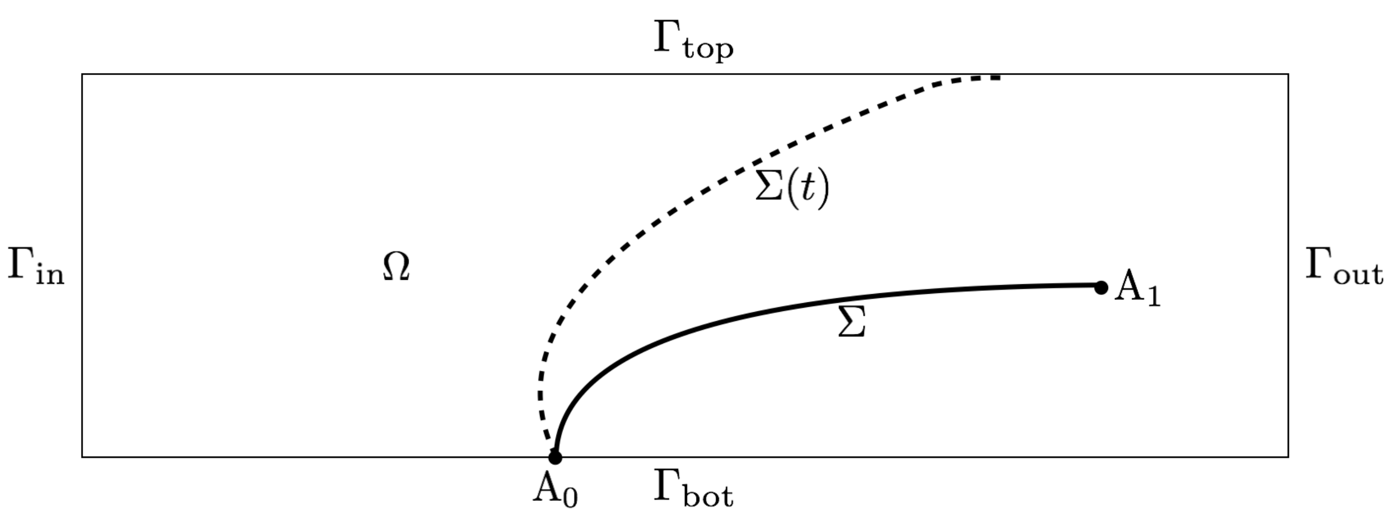
(c) $\tau = 5 \cdot 10^{-4}$. (d) $\tau = 5 \cdot 10^{-4}$.

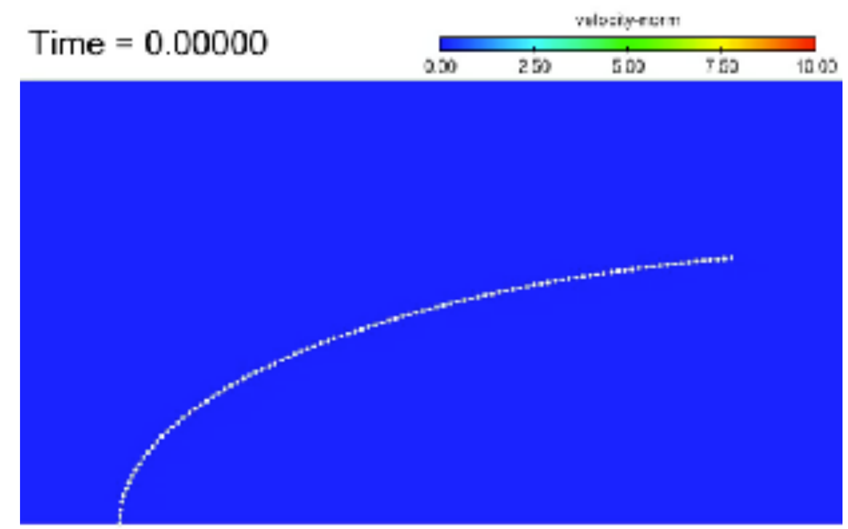


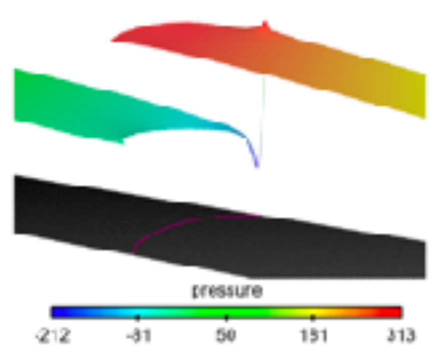
(e) $\tau = 2.5 \cdot 10^{-4}$. (f) $\tau = 2.5 \cdot 10^{-4}$.

Tip position

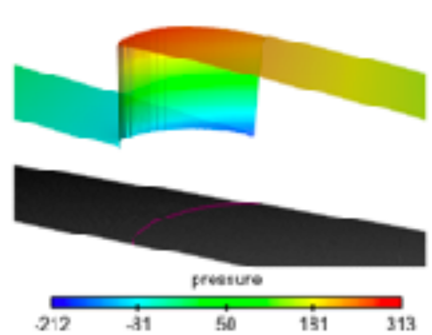
2d test case - comparison with NXFEM





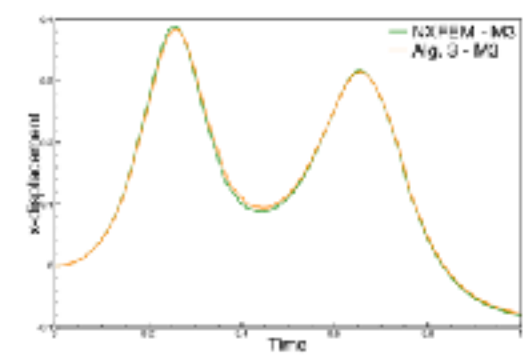


(c) Nitsche-XFEM: M_3.

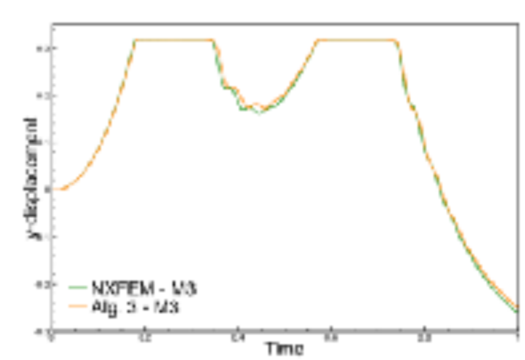


(f) Algorithm 4.3: M_3.

Pressure drop

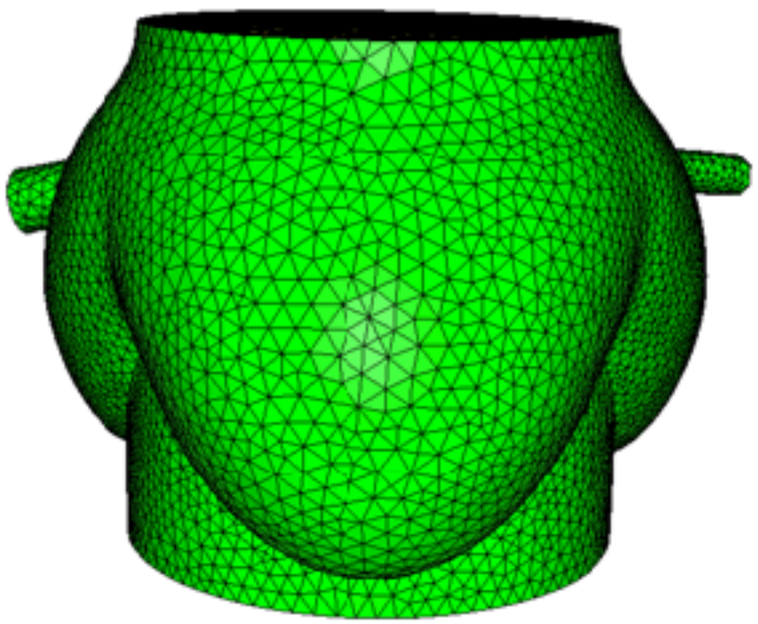


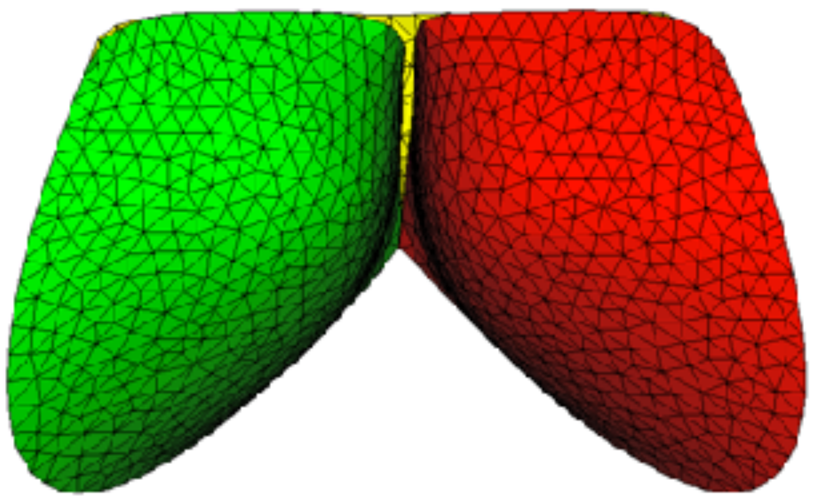
(e) M_3.

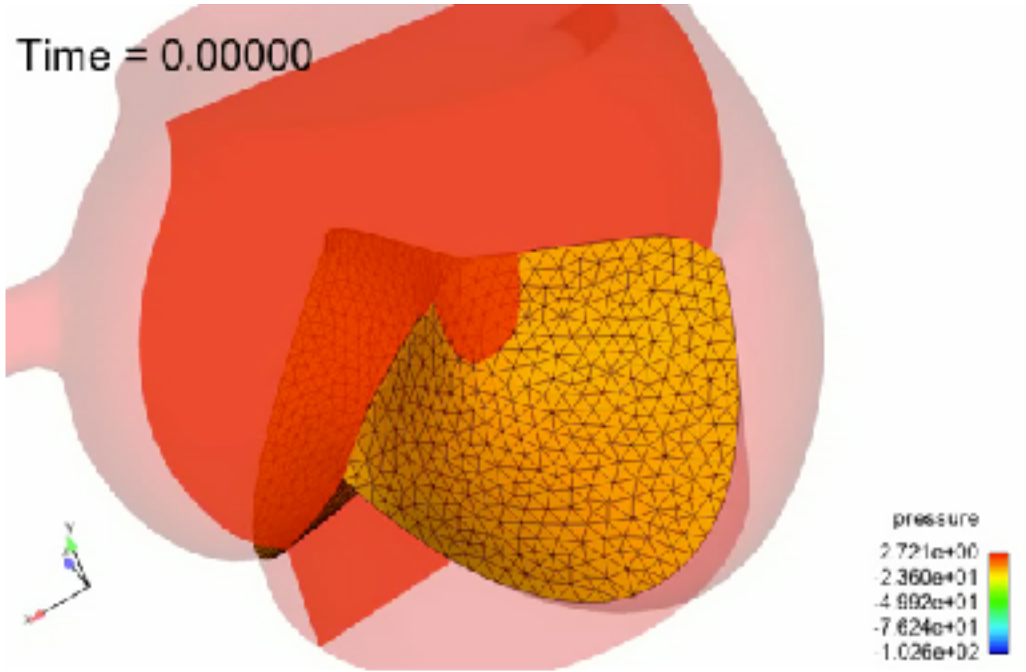


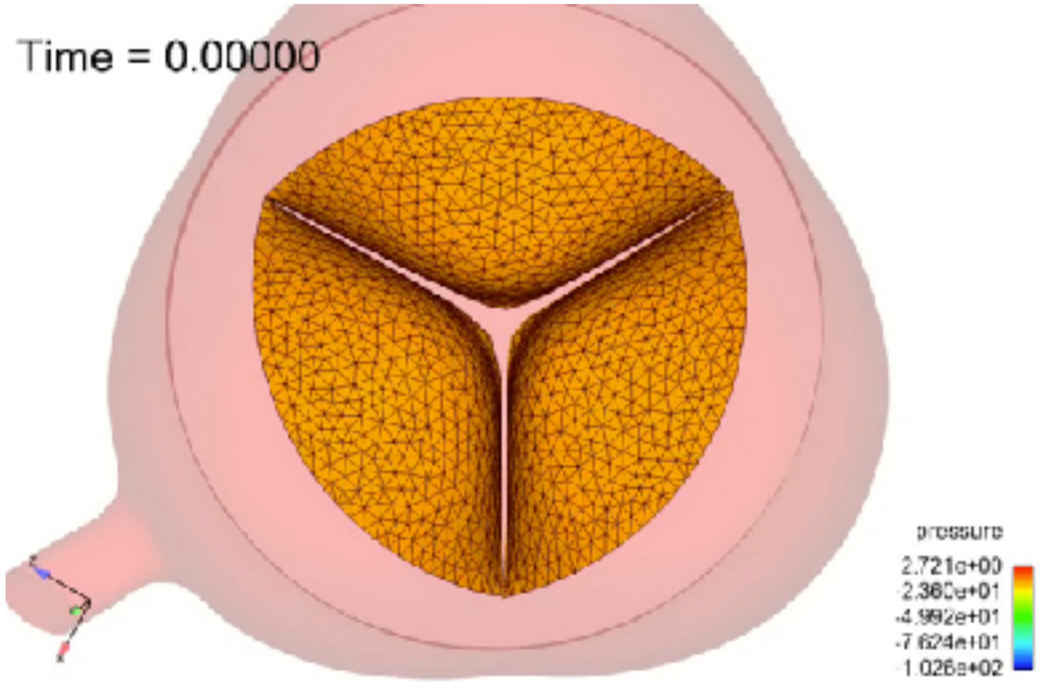
(f) M_3.

Tip position

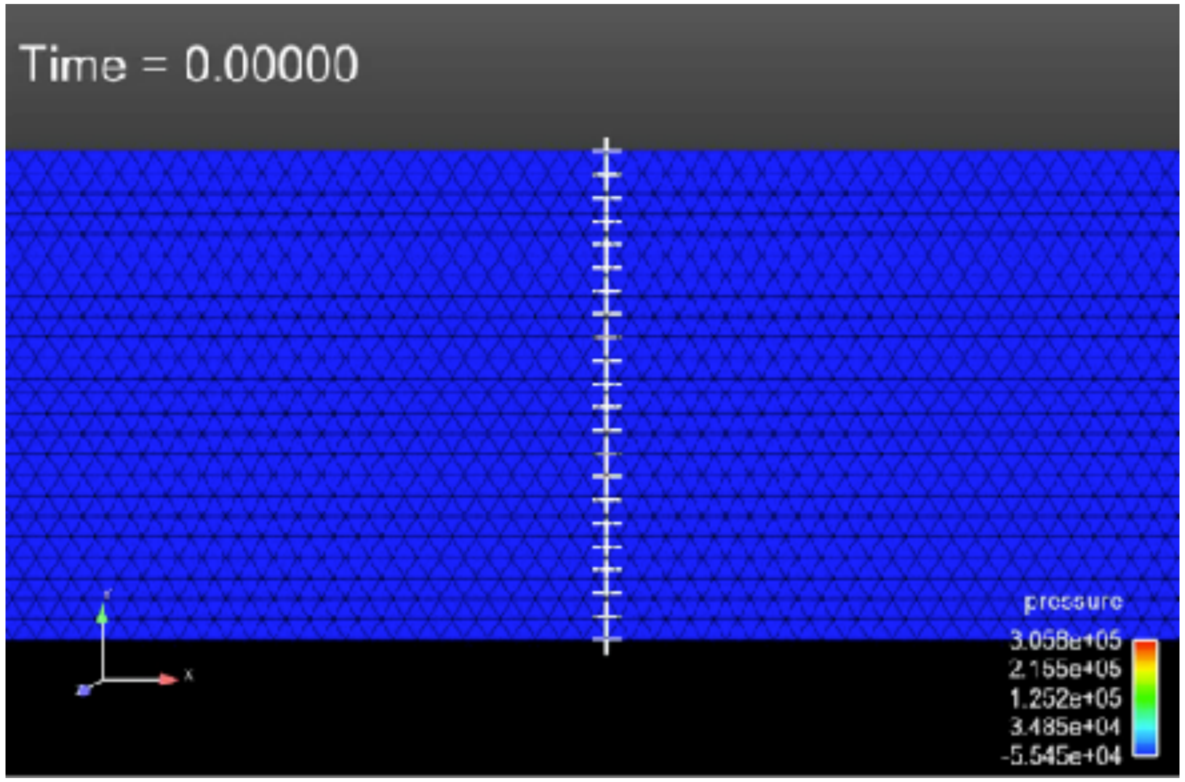








Not yet with the physiological values…



Remark on mass conservation:

- we boost the grad-div stabilization near the valve as proposed in *D. Kamensky et al. CMAME 2015*
- Raise conditioning issues... on going work !