Fluid-structure interaction in the cardiovascular system

Part 1 -
Forward problems

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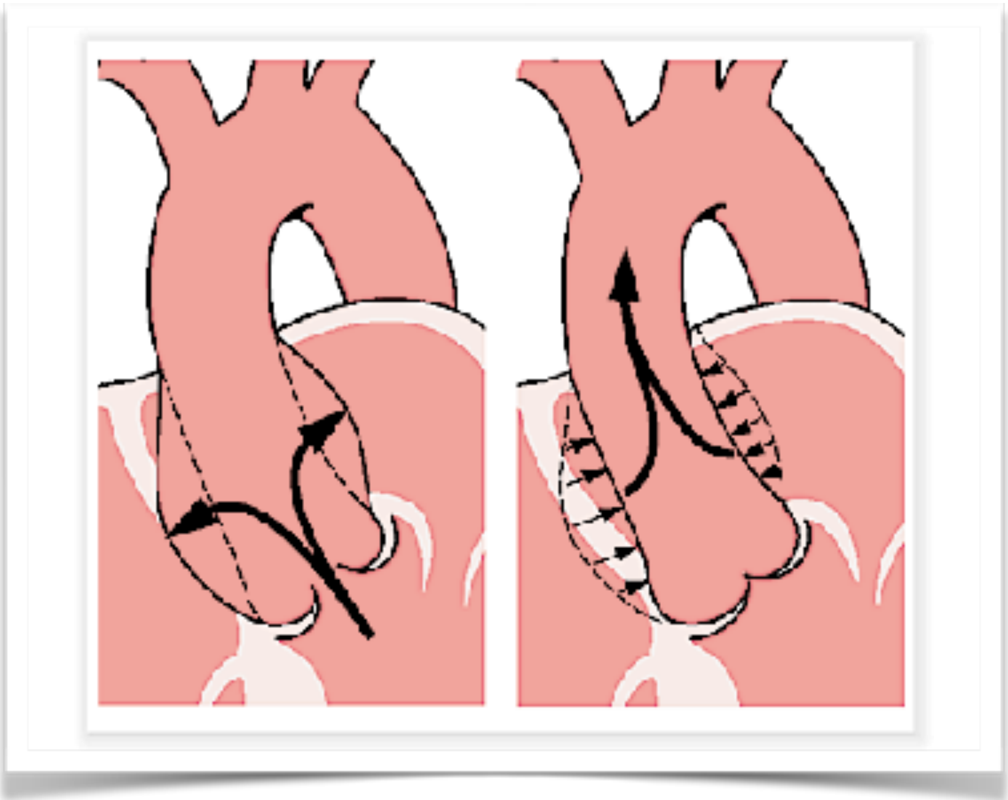
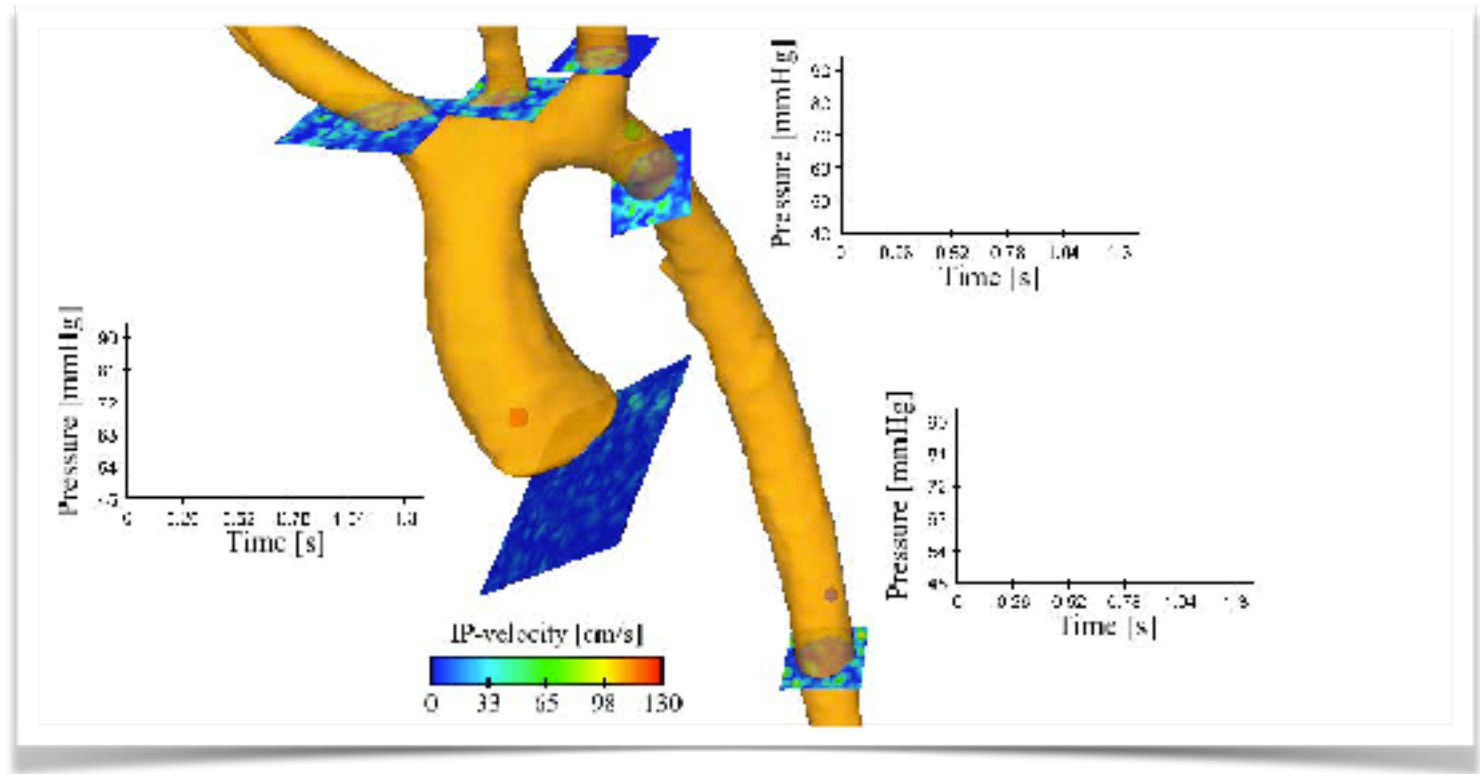
Fluid-Structure interaction in large vessels



Windkessel effect

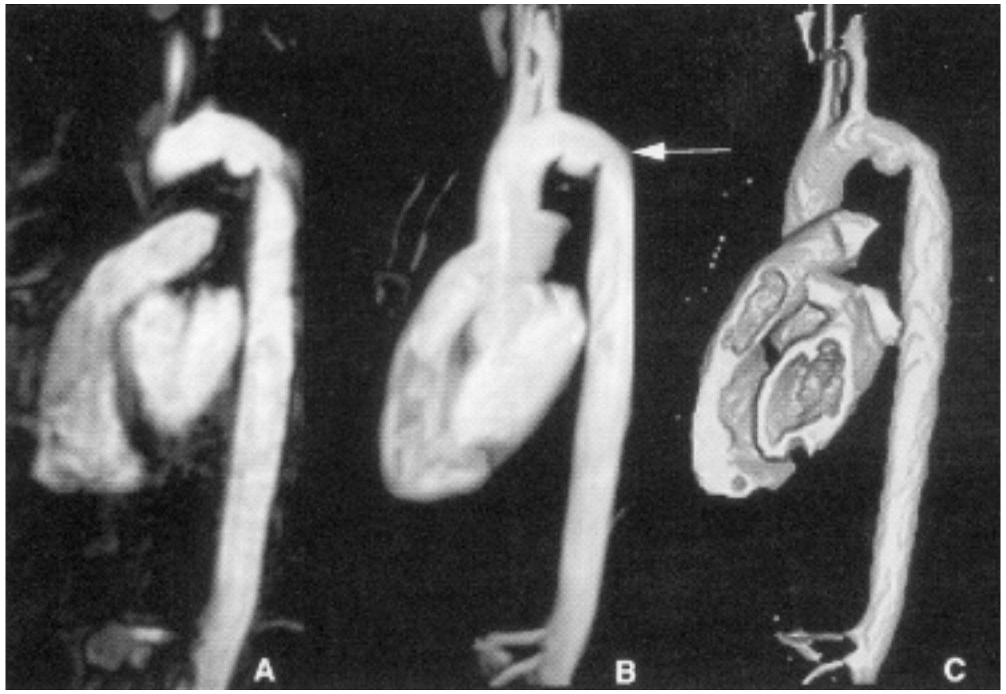


Aorta

KCL (euHeart)

- Large arteries distend during systole, recoil during diastole
- This damps the pressure fluctuation; helps a constant perfusion of organs during diastole
- Distensibility is responsible for wave propagation

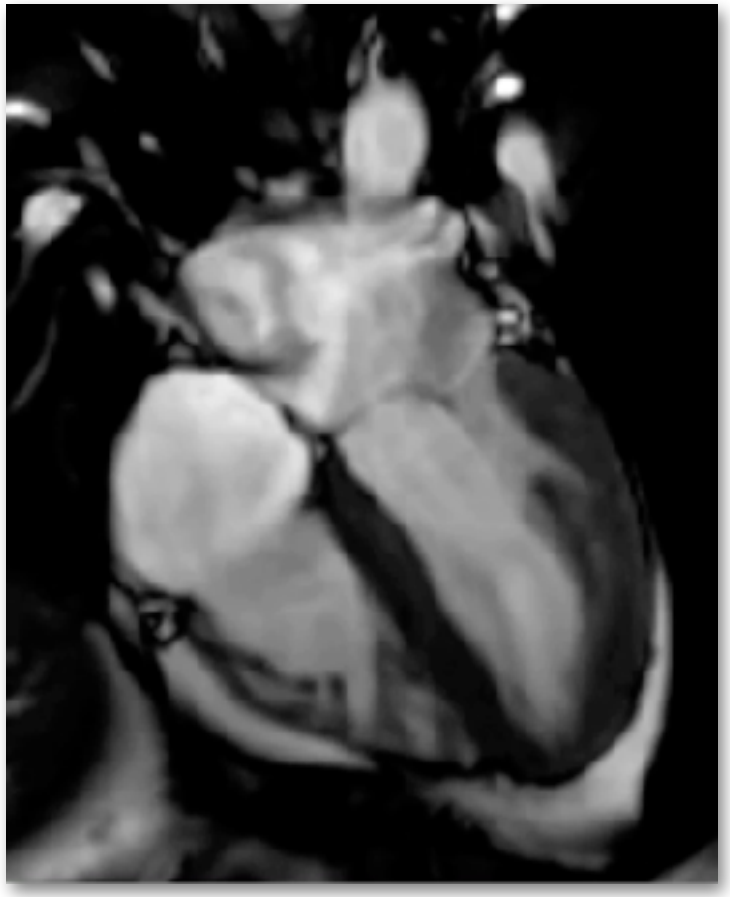
Fluid-Structure interaction in large vessels

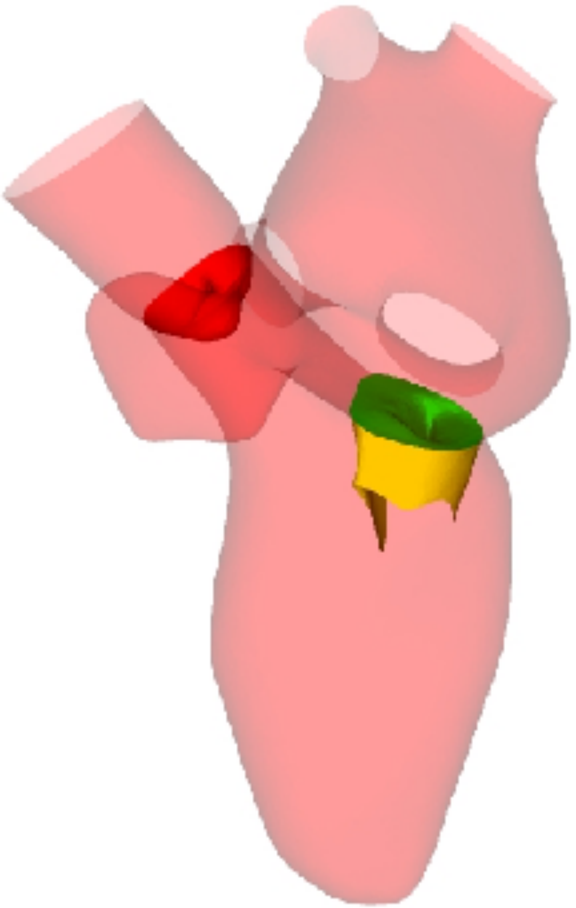


Example: aortic coartaction

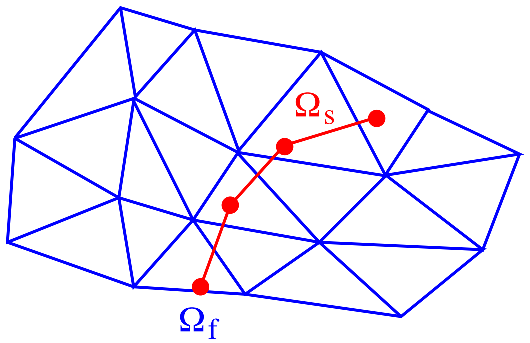
- After surgical repair, patients must be followed on a regular basis
- Exercise test is often necessary to assess the patient condition
- **Question:** With computer simulations, can we extrapolate the rest test to avoid the stress test ?
- Maybe... if we are able to personalize the model with the available medical data (arterial stiffness, etc.)

Fluid-Structure interaction for cardiac valves

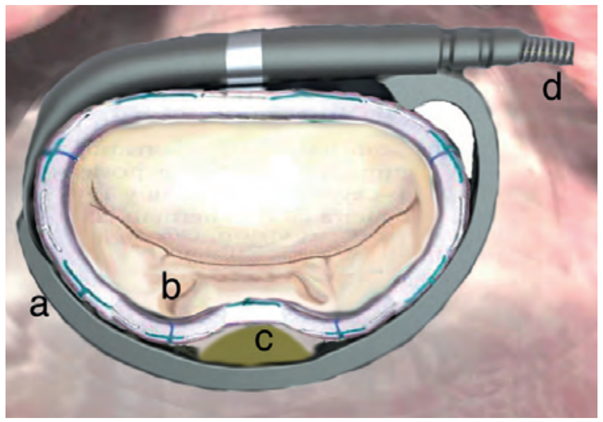


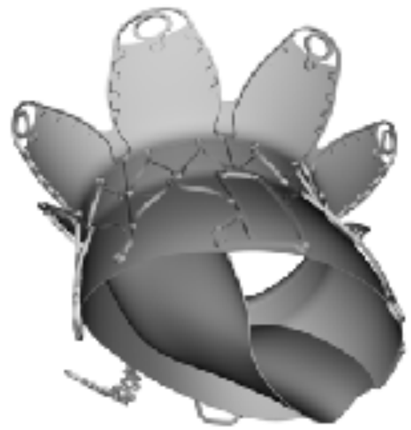


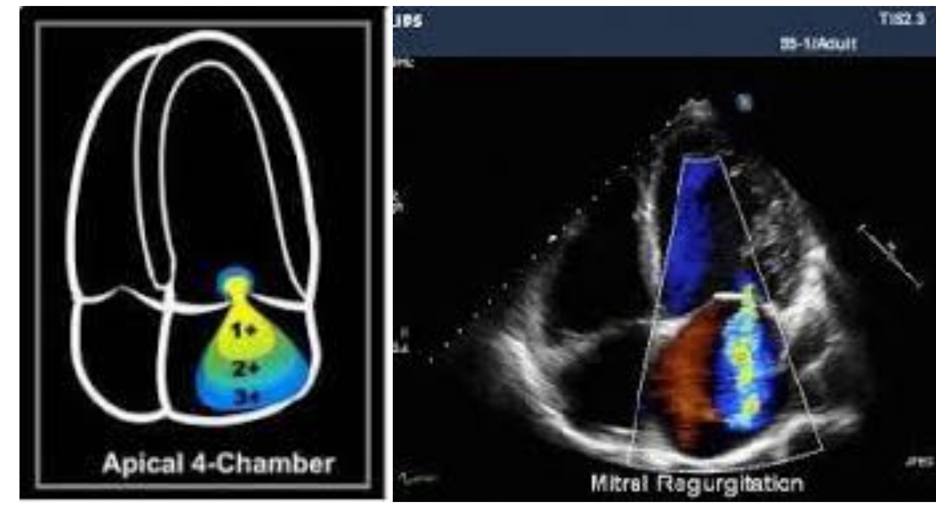
- Difficulties
  - very large displacements
  - fast dynamics
  - high pressure drop
  - isovolumic phases
  - change of topology (contact)
  - …



Industrial applications







- **Goal of simulation**
  - virtual assays
  - limit, or better prepare, animal experiments

Ongoing PhD thesis (L. Boilevin-Kayl)

- **Goal of simulation**
  - provide a “ground truth” to assess clinical estimation of regurgitation

Ongoing PhD thesis (A.This)

Outline

- Generality: energy balance, load computation, coupling
- Added mass effect and incompressibility
- Semi-implicit coupling *via* projection schemes
- Explicit coupling *via* Nitsche and Robin formulations
- Explicit coupling for thin structures

C Farhat, M Lesoinne, P Le Tallec, "Load and motion transfer algorithms for fluid/structure interaction problems with non-matching discrete interfaces" *Comp. Meth. Applied Mech. and Engng* (1998) 157(1), 95-114.

P Le Tallec, J Mouro, "Fluid structure interaction with large structural displacements" *Comp. Meth. Applied Mech. and Engng* (2001) 190, 3039-3067.

Linear FSI equations

Newton's law:

Velocity Cauchy stress tensor

$$\rho \frac{D\boldsymbol{u}}{Dt} - \mathrm{div}\boldsymbol{\sigma} = 0 \qquad \int_{\Sigma} \boldsymbol{\sigma} \cdot \boldsymbol{n} = \text{force on surface } \Sigma$$

Main assumptions in the fluid:

- Linearization: $\dfrac{D\boldsymbol{u}}{Dt} = \dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \approx \dfrac{\partial \boldsymbol{u}}{\partial t}$
- Fixed domain
- Incompressible fluid: $\mathrm{div}\, \boldsymbol{u} = 0$
- Newtonian fluid: $\boldsymbol{\sigma}^f = \boldsymbol{\sigma}(\boldsymbol{u}, p) = -p\mathbb{I} + 2\mu\boldsymbol{\epsilon}(\boldsymbol{u})$

Viscosity $\qquad \boldsymbol{\epsilon}(\cdot) = \frac{1}{2}\left(\nabla \cdot + \nabla \cdot^T\right)$

Main assumptions in the solid:

- Infinitesimal displacements
- Isotropic linear elasticity: $\boldsymbol{\sigma}^s = \boldsymbol{\sigma}(\boldsymbol{d}) = 2\mu^s\boldsymbol{\epsilon}(\boldsymbol{d}) + \lambda\, \mathrm{Tr}(\boldsymbol{\epsilon}(\boldsymbol{d}))\mathbb{I}$

Displacements Lamé coefficients

Linear FSI equations

- **Fluid equations (Eulerian formulation, transient Stokes):**

$$\rho^{\mathrm{f}}\frac{\partial \boldsymbol{u}}{\partial t} - \mathbf{div}(2\mu\boldsymbol{\epsilon}(\boldsymbol{u})) + \boldsymbol{\nabla} p = \mathbf{0}, \quad \text{in } \Omega^{\mathrm{f}}$$

$$\operatorname{div} \boldsymbol{u} = 0, \quad \text{in } \Omega^{\mathrm{f}}$$

- **Solid equations (Lagrangian formulation, linear elasticity):**

$$\rho^{\mathrm{s}}\frac{\partial^2 \boldsymbol{d}}{\partial t^2} - \mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{d}) = \mathbf{0}, \quad \text{in } \widehat{\Omega}^{\mathrm{s}}$$

- **Coupling conditions:**

$$\left\{ \begin{array}{rcll} \boldsymbol{\sigma}^f \cdot \boldsymbol{n}^f + \boldsymbol{\sigma}^s \cdot \boldsymbol{n}^s & = & 0, & \text{on } \Sigma \\ \boldsymbol{u} & = & \dot{\boldsymbol{d}}, & \text{on } \Sigma \end{array} \right.$$

where Σ is the fluid-structure interface and $\dot{\boldsymbol{d}} = \dfrac{\partial \boldsymbol{d}}{\partial t}$

Linear FSI equations

Energy estimate

If the system is isolated, then:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\Bigg[\underbrace{\int_{\Omega^{\mathrm{f}}}\frac{\rho^{\mathrm{f}}}{2}|\boldsymbol{u}|^2+\int_{\widehat{\Omega}^{\mathrm{s}}}\frac{\rho^{\mathrm{s}}}{2}|\dot{\boldsymbol{d}}|^2}_{\text{Kinetic energy}}+\underbrace{\int_{\widehat{\Omega}^{\mathrm{s}}}W\big(\boldsymbol{\epsilon}(\boldsymbol{d})\big)}_{\text{Elastic potential energy}}\Bigg]+\underbrace{\int_{\Omega^{\mathrm{f}}}2\mu|\boldsymbol{\epsilon}(\boldsymbol{u})|^2}_{\text{viscous dissipation}}=0$$

with $\quad W(\boldsymbol{\epsilon}(\boldsymbol{d}))=\mu^s[\boldsymbol{\epsilon}(\boldsymbol{d})]^2+\dfrac{\lambda}{2}\left[\mathrm{Tr}(\boldsymbol{\epsilon}(\boldsymbol{d}))\right]^2$

- Multiply the fluid equations by $\boldsymbol{u}$ and integrate by parts:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_{\Omega^{\mathrm{f}}}\frac{\rho^{\mathrm{f}}}{2}|\boldsymbol{u}|^2+\int_{\Omega^{\mathrm{f}}}2\mu|\boldsymbol{\epsilon}(\boldsymbol{u})|^2-\int_{\Sigma}\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n}^f\cdot\boldsymbol{u}=0$$

- Multiply the solid equations by $\dot{\boldsymbol{d}}$ and integrate by parts:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left[\int_{\widehat{\Omega}^{\mathrm{s}}}\frac{\rho^{\mathrm{s}}}{2}|\dot{\boldsymbol{d}}|^2+\int_{\widehat{\Omega}^{\mathrm{s}}}W(\boldsymbol{\epsilon}(\boldsymbol{d}))\right]-\int_{\Sigma}\boldsymbol{\sigma}(\boldsymbol{d})\cdot\boldsymbol{n}^s\cdot\dot{\boldsymbol{d}}=0.$$

- With the coupling conditions, the interface terms cancels

Not addressed in these lectures:
ALE and non-linearities

- **Fluid equations (Moving domain: ALE formulation, Navier-Stokes)**

$$\rho^{\mathrm{f}}\left(\frac{\partial \boldsymbol{u}}{\partial t}_{|\widehat{\boldsymbol{x}}} + (\boldsymbol{u}-\boldsymbol{w})\cdot\boldsymbol{\nabla}\boldsymbol{u}\right) - \mathbf{div}(2\mu\boldsymbol{\epsilon}(\boldsymbol{u})) + \boldsymbol{\nabla} p = \boldsymbol{0}, \text{in } \Omega^{\mathrm{f}}(t)$$

$$\operatorname{div}\boldsymbol{u} = 0, \text{in } \Omega^{\mathrm{f}}(t)$$

- **Solid equations (Lagrangian formulation, hyperelasticity):**

$$\rho^{\mathrm{s}}\frac{\partial^2 \boldsymbol{d}}{\partial t^2} - \mathbf{div}\big(\boldsymbol{F}(\boldsymbol{d})\boldsymbol{S}(\boldsymbol{d})\big) = \boldsymbol{0}, \quad \text{in } \widehat{\Omega}^{\mathrm{s}}$$

$$\text{2d Piola-Kirchhoff tensor: } \boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{E}}(\boldsymbol{E})$$

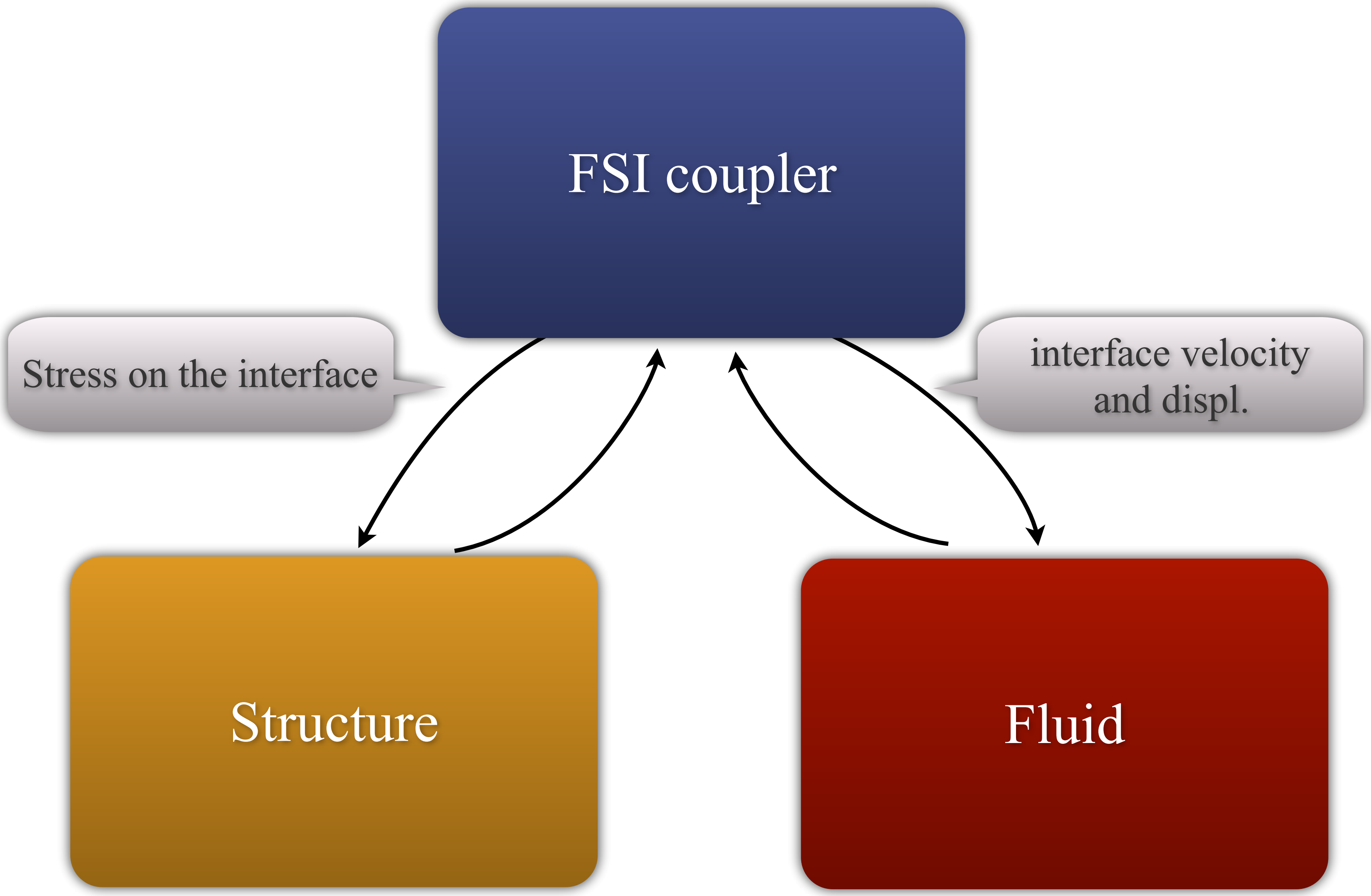
- **See e.g.:**

M. Fernández, L. Formaggia, J.-F. Gerbeau, A. Quarteroni. *The derivation of the equations for fluids and structures*. In Cardiovascular mathematics, volume 1 of MS&A. Model. Simul. Appl., Chap. 3, pages 77–121. Springer, 2009.

M. Fernández, J.-F. Gerbeau. *Algorithms for fluid-structure interaction problems*. In Cardiovascular mathematics, volume 1 of MS&A. Model. Simul. Appl., Chap. 9, pages 307–346. Springer, 2009.

Fluid-Structure coupling

- **Monolithic** schemes: fluid and structure simultaneously
  (not addressed in these lectures)
- **Partitioned** schemes**:** 2 solvers



  - **Explicit** schemes : one fluid/structure iteration at each time step
  - **Implicit** schemes : many fluid/structure iterations at each time step

Dirichlet-Neumann iterations

- **Fluid sub-problem** (Dirichlet boundary conditions)**:**

$$\text{Find } \boldsymbol{u} \text{ with } \boldsymbol{u}_{|\Sigma} = \frac{\partial \boldsymbol{d}_{|\Sigma}}{\partial t} \text{ and } p \text{ such that:}$$

$$\begin{aligned} \int_{\Omega^{\mathrm{f}}} \rho^{\mathrm{f}} \frac{\partial \boldsymbol{u}}{\partial t} \cdot \boldsymbol{v}^{\mathrm{f}} + 2\mu \int_{\Omega^{\mathrm{f}}} \boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v}^{\mathrm{f}}) - \int_{\Omega^{\mathrm{f}}} p \operatorname{div} \boldsymbol{v}^{\mathrm{f}} &= \int_{\Gamma^{\mathrm{f}}_{\mathrm{N}}} \boldsymbol{g} \cdot \boldsymbol{v}^{\mathrm{f}} \\ \int_{\Omega^{\mathrm{f}}} q \operatorname{div} \boldsymbol{u} &= 0 \end{aligned}$$

- **Solid sub-problem** (Neumann boundary condition)**:**

$$\text{Find } \boldsymbol{d} \text{ such that:}$$

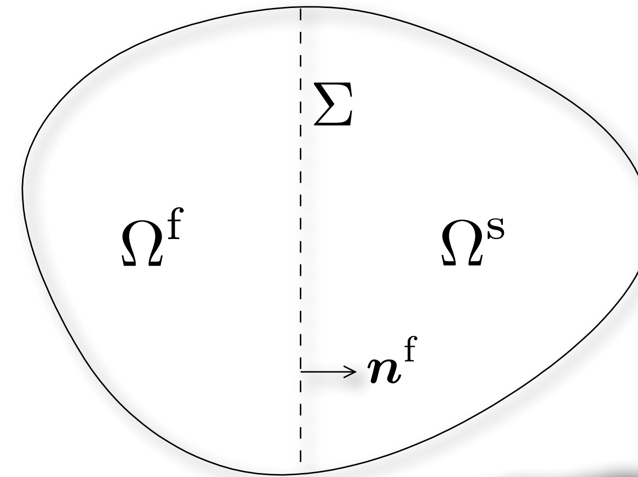
$$\int_{\widehat{\Omega}^{\mathrm{s}}} \rho^{\mathrm{s}} \frac{\partial^2 \boldsymbol{d}}{\partial t^2} \cdot \boldsymbol{v}^{\mathrm{s}} + \int_{\widehat{\Omega}^{\mathrm{s}}} \boldsymbol{\sigma}(\boldsymbol{d}) : \boldsymbol{\nabla} \boldsymbol{v}^{\mathrm{s}} = -\int_{\Sigma} \boldsymbol{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n}^{\mathrm{f}} \cdot \boldsymbol{v}^{\mathrm{s}}.$$

- How to compute the load?

$$\boxed{\int_{\Sigma} \boldsymbol{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n}^{\mathrm{f}} \cdot \boldsymbol{v}^{\mathrm{s}} = \int_{\Sigma} (-p\mathbb{I} + 2\mu \boldsymbol{\epsilon}(\boldsymbol{u})) \cdot \boldsymbol{n}^{\mathrm{f}} \cdot \boldsymbol{v}^{\mathrm{s}}}$$

Load computation: a toy example

- Dirichlet: $\begin{cases} -\Delta u^{\mathrm{f}} = f^{\mathrm{f}}, & \text{on} \quad \Omega^{\mathrm{f}}, \\ u^{\mathrm{f}} = u_{\mathrm{D}}, & \text{on} \quad \Sigma \end{cases}$



- Neumann: $\begin{cases} -\Delta u^{\mathrm{s}} = f^{\mathrm{s}}, & \text{in} \quad \Omega^{\mathrm{s}}, \\ \dfrac{\partial u^{\mathrm{s}}}{\partial \boldsymbol{n}^{\mathrm{s}}} = -\dfrac{\partial u^{\mathrm{f}}}{\partial \boldsymbol{n}^{\mathrm{f}}}, & \text{on} \quad \Sigma \end{cases}$

compute $\langle \dfrac{\partial u^{\mathrm{f}}}{\partial \boldsymbol{n}^{\mathrm{f}}}, v \rangle$ instead of $\dfrac{\partial u^{\mathrm{f}}}{\partial \boldsymbol{n}^{\mathrm{f}}}$

- The condition $u^{\mathrm{f}}_{|\Sigma} = u_{\mathrm{D}}$ has a sense for $u^{\mathrm{f}} \in H^1(\Omega^{\mathrm{f}})$
- On the contrary, $\dfrac{\partial u^{\mathrm{f}}}{\partial \boldsymbol{n}^{\mathrm{f}}}$ is not defined for all $u^{\mathrm{f}} \in H^1(\Omega^{\mathrm{f}})$
- Note that for u^{f} and v smooth ($v_{|\partial\Omega^{\mathrm{f}} \setminus \Sigma} = 0$):

$$\int_{\Sigma} \frac{\partial u^{\mathrm{f}}}{\partial \boldsymbol{n}^{\mathrm{f}}} v = \int_{\Omega^{\mathrm{f}}} \mathrm{div}(\boldsymbol{\nabla} u^{\mathrm{f}}) v + \int_{\Omega^{\mathrm{f}}} \boldsymbol{\nabla} u^{\mathrm{f}} \cdot \boldsymbol{\nabla} v$$

- Since $\mathrm{div}(\boldsymbol{\nabla} u^{\mathrm{f}}) = -f^{\mathrm{f}} \in L^2(\Omega^{\mathrm{f}})$, the normal trace of $\boldsymbol{\nabla} u$ is well-defined:

$$\langle \frac{\partial u^{\mathrm{f}}}{\partial \boldsymbol{n}^{\mathrm{f}}}, v \rangle \stackrel{\mathrm{def}}{=} -\int_{\Omega^{\mathrm{f}}} f^{\mathrm{f}} v + \int_{\Omega^{\mathrm{f}}} \boldsymbol{\nabla} u^{\mathrm{f}} \cdot \boldsymbol{\nabla} v$$

for all $v \in H^1(\Omega^{\mathrm{f}})$ with $v_{|\partial\Omega^{\mathrm{f}} \setminus \Sigma} = 0$.

Load computation: a toy example

- $$\begin{cases} -\Delta u^{\mathrm{f}} = f^{\mathrm{f}}, & \text{on} \quad \Omega^{\mathrm{f}}, \\ u^{\mathrm{f}} = u_{\mathrm{D}}, & \text{on} \quad \Sigma \end{cases} \longrightarrow \begin{cases} \text{Find } u^{\mathrm{f}} \in H^1(\Omega^{\mathrm{f}}) \text{ s.t. } u^{\mathrm{f}}_{|\Sigma} = u_{\mathrm{D}} \text{ and} \\ \displaystyle\int_{\Omega^{\mathrm{f}}} \boldsymbol{\nabla} u^{\mathrm{f}} \cdot \boldsymbol{\nabla} v^{\mathrm{f}} = \int_{\Omega^{\mathrm{f}}} f^{\mathrm{f}} v^{\mathrm{f}}, \quad \forall v^{\mathrm{f}} \in H^1_{\Sigma}(\Omega^{\mathrm{f}}) \end{cases}$$

$$v^{\mathrm{f}}|_{\Sigma} = 0$$

- $$\begin{cases} -\Delta u^{\mathrm{s}} = f^{\mathrm{s}}, & \text{in} \quad \Omega^{\mathrm{s}}, \\ \dfrac{\partial u^{\mathrm{s}}}{\partial \boldsymbol{n}^{\mathrm{s}}} = -\dfrac{\partial u^{\mathrm{f}}}{\partial \boldsymbol{n}^{\mathrm{f}}}, & \text{on} \quad \Sigma \end{cases} \longrightarrow \begin{cases} \text{Find } u^{\mathrm{s}} \in H^1(\Omega^{\mathrm{s}}) \text{ s.t.} \\ \displaystyle\int_{\Omega^{\mathrm{s}}} \boldsymbol{\nabla} u^{\mathrm{s}} \cdot \boldsymbol{\nabla} v^{\mathrm{s}} = \int_{\Omega^{\mathrm{s}}} f^{\mathrm{s}} v^{\mathrm{s}} - \int_{\Sigma} \frac{\partial u^{\mathrm{f}}}{\partial \boldsymbol{n}^{\mathrm{f}}} v^{\mathrm{s}}, \quad \forall v^{\mathrm{s}} \in H^1(\Omega^{\mathrm{s}}) \end{cases}$$

$$-\left\langle \frac{\partial u^{\mathrm{f}}}{\partial \boldsymbol{n}^{\mathrm{f}}}, v \right\rangle \stackrel{\text{def}}{=} \underbrace{\int_{\Omega^{\mathrm{f}}} f^{\mathrm{f}} v - \int_{\Omega^{\mathrm{f}}} \boldsymbol{\nabla} u^{\mathrm{f}} \cdot \boldsymbol{\nabla} v}_{\text{variational residual } \mathcal{R}(u^{\mathrm{f}}; v)} \quad \underbrace{\text{with } v \in H^1(\Omega^{\mathrm{f}}) \text{ s.t. } \begin{cases} v = v^{\mathrm{s}}, & \text{on} \quad \Sigma, \\ v = 0, & \text{on} \quad \partial\Omega^{\mathrm{f}} \backslash \Sigma \end{cases}}_{\text{lifting operator : } \begin{cases} \mathcal{L} : H^{\frac{1}{2}}(\Sigma) \rightarrow H^1(\Omega^{\mathrm{f}}) \\ v^{\mathrm{s}} \mapsto v = \mathcal{L} v^{\mathrm{s}} \end{cases}}$$

$$\begin{cases} \text{Find } u^{\mathrm{s}} \in H^1(\Omega^{\mathrm{s}}) \text{ s.t.} \\ \displaystyle\int_{\Omega^{\mathrm{s}}} \boldsymbol{\nabla} u^{\mathrm{s}} \cdot \boldsymbol{\nabla} v^{\mathrm{s}} = \int_{\Omega^{\mathrm{s}}} f^{\mathrm{s}} v^{\mathrm{s}} + \mathcal{R}(u^{\mathrm{f}}; \mathcal{L} v^{\mathrm{s}}), \quad \forall v^{\mathrm{s}} \in H^1(\Omega^{\mathrm{s}}) \end{cases}$$

Load computation: a toy example

Algebraic counterpart

- n^{f} nodes in Ω^{f}, n^{f}_{Σ} nodes on Σ, $n^{\mathrm{f}}_{I} = n^{\mathrm{f}} - n^{\mathrm{f}}_{\Sigma}$ "internal" nodes.
- Matrix "without boundary condition" : $\mathrm{A}^{\mathrm{f}} = \left[a_f(v^{\mathrm{f}}_j, v^{\mathrm{f}}_i)\right]_{i,j=1..n^{\mathrm{f}}}$

$$\mathrm{A}^{\mathrm{f}} = \begin{bmatrix} \mathrm{A}^{\mathrm{f}}_{\mathrm{II}} & \mathrm{A}^{\mathrm{f}}_{\mathrm{I}\Sigma} \\ \mathrm{A}^{\mathrm{f}}_{\Sigma\mathrm{I}} & \mathrm{A}^{\mathrm{f}}_{\Sigma\Sigma} \end{bmatrix}, \quad \mathrm{U}^{\mathrm{f}} = \begin{bmatrix} \mathrm{U}^{\mathrm{f}}_{\mathrm{I}} \\ \mathrm{U}^{\mathrm{f}}_{\Sigma} \end{bmatrix} \quad \mathrm{F}^{\mathrm{f}} = \begin{bmatrix} \mathrm{F}^{\mathrm{f}}_{\mathrm{I}} \\ \mathrm{F}^{\mathrm{f}}_{\Sigma} \end{bmatrix}$$

- Dirichlet problem $(\mathrm{U}^{\mathrm{f,d}}_{\Sigma}$ given) : $\boxed{\mathrm{A}^{\mathrm{f}}_{\mathrm{II}}\mathrm{U}^{\mathrm{f}}_{\mathrm{I}} = \mathrm{F}^{\mathrm{f}}_{\mathrm{I}} - \mathrm{A}^{\mathrm{f}}_{\mathrm{I}\Sigma}\mathrm{U}^{\mathrm{f,d}}_{\Sigma}}$
- Residual : $\boxed{\mathrm{R}^{\mathrm{f}}_{\Sigma} = \mathrm{F}^{\mathrm{f}}_{\Sigma} - \mathrm{A}^{\mathrm{f}}_{\Sigma\mathrm{I}}\mathrm{U}^{\mathrm{f}}_{\mathrm{I}} - \mathrm{A}^{\mathrm{f}}_{\Sigma\Sigma}\mathrm{U}^{\mathrm{f}}_{\Sigma}}$ $\qquad \mathrm{R}^{\mathrm{f}} = \mathrm{F}^{\mathrm{f}} - \mathrm{A}^{\mathrm{f}}\,\mathrm{U}^{\mathrm{f}} = \begin{bmatrix} 0 \\ \mathrm{R}^{\mathrm{f}}_{\Sigma} \end{bmatrix}$

$$\longrightarrow \text{ approximation of } -\left(\int_{\Sigma} \frac{\partial u^{\mathrm{f}}}{\partial \boldsymbol{n}^{\mathrm{f}}} v^{\mathrm{f}}_i\right)_{i=1..n^{\mathrm{f}}}$$

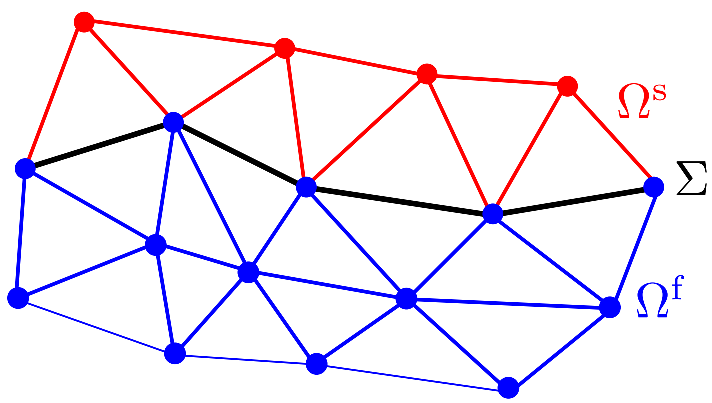
- ... but we need $\left(\int_{\Sigma} \frac{\partial u^{\mathrm{f}}}{\partial \boldsymbol{n}^{\mathrm{s}}} v^{\mathrm{s}}_i\right)_{i=1..n^{\mathrm{s}}}$

Load computation: a toy example

$$\int_\Sigma \frac{\partial u^{\mathrm{f}}}{\partial n^{\mathrm{s}}} v^{\mathrm{s}} = -\int_\Sigma \frac{\partial u^{\mathrm{f}}}{\partial n^{\mathrm{f}}} v^{\mathrm{f}} \longleftrightarrow (\mathrm{R}^{\mathrm{s}}, \mathrm{V}^{\mathrm{s}}) = (\mathrm{R}^{\mathrm{f}}, \mathrm{V}^{\mathrm{f}})$$

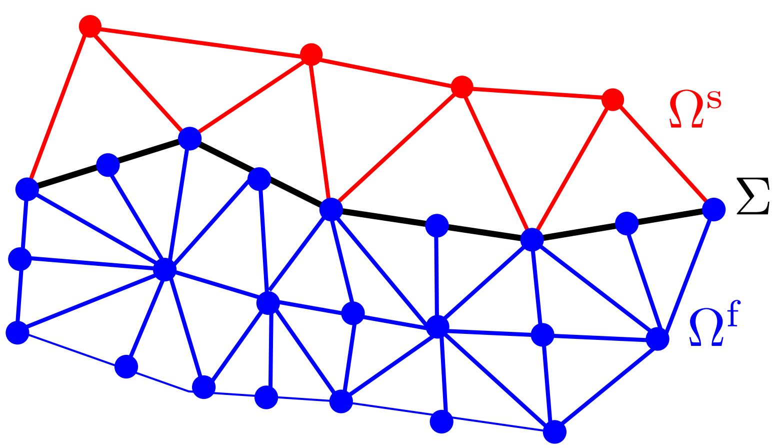
to be defined

- Conforming meshes :



$$\mathrm{V}^{\mathrm{f}}_\Sigma = \mathrm{V}^{\mathrm{s}}_\Sigma \text{ hence } \boxed{\mathrm{R}^{\mathrm{s}} = \mathrm{R}^{\mathrm{f}}}$$

- Non-conforming meshes: let K be the interpolation matrix on Σ

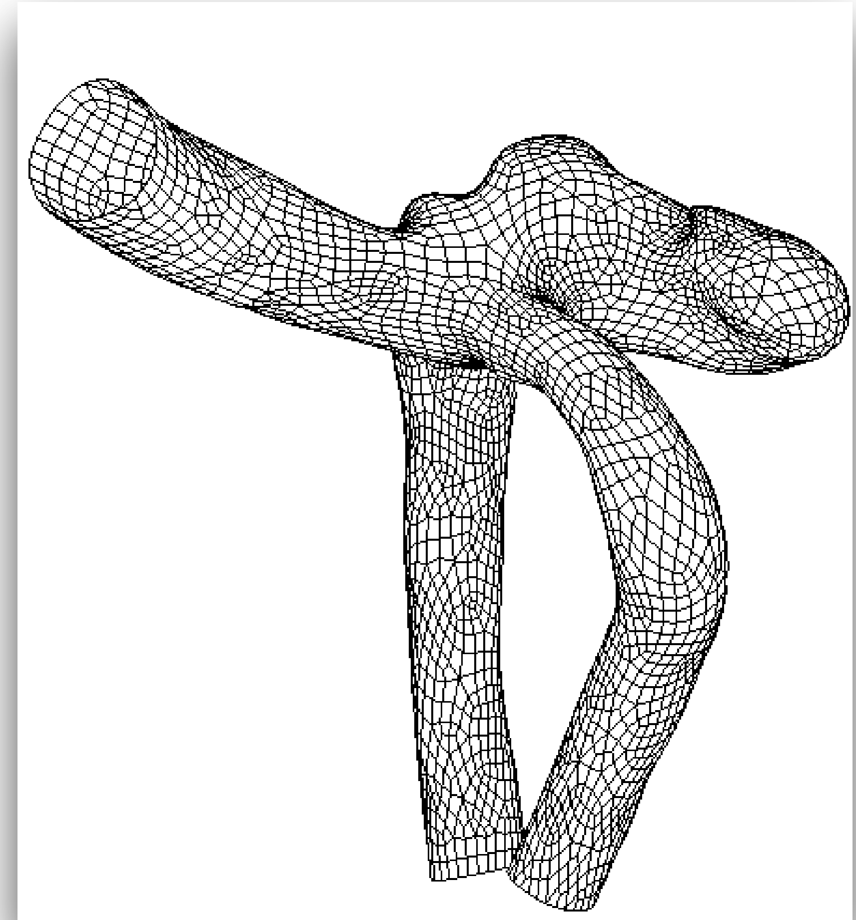


$$\mathrm{V}^{\mathrm{f}}_\Sigma = \mathrm{K}\,\mathrm{V}^{\mathrm{s}}_\Sigma \implies (\mathrm{R}^{\mathrm{s}}, \mathrm{V}^{\mathrm{s}}) = (\mathrm{R}^{\mathrm{f}}, \mathrm{K}\,\mathrm{V}^{\mathrm{s}}) = (\mathrm{K}^T\mathrm{R}^{\mathrm{f}}, \mathrm{V}^{\mathrm{s}})$$

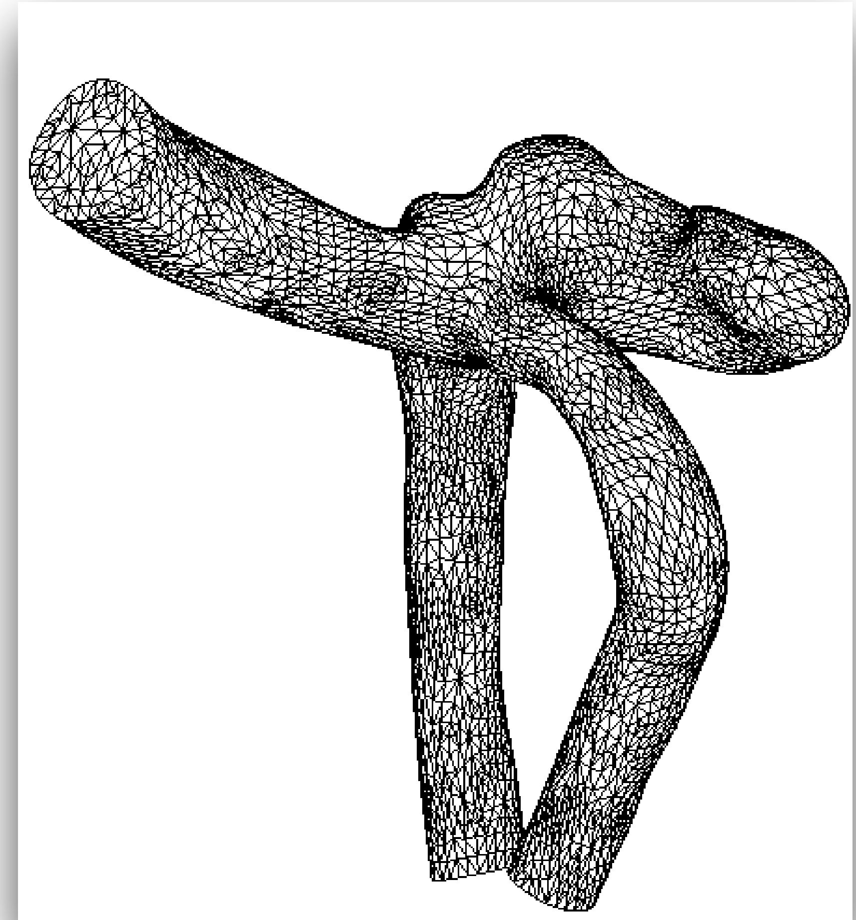
$$\text{hence } \boxed{\mathrm{R}^{\mathrm{s}} = \mathrm{K}^T\mathrm{R}^{\mathrm{f}}}$$

- Neumann problem (R^{s} given) : $\boxed{\mathrm{A}^{\mathrm{s}}\mathrm{U}^{\mathrm{s}} = \mathrm{F}^{\mathrm{s}} + \mathrm{R}^{\mathrm{s}}}$

Load computation: an example



Solid: MITC4 shell elements



Fluid: P1 tetrahedra

Same nodes, but different elements… how to transfer the load ?

$$\mathrm{K} = \mathrm{Id} \quad \text{hence} \quad \boxed{\mathrm{R}^{\mathrm{s}} = \mathrm{R}^{\mathrm{f}}}$$

Load computation: a toy example

Discrete energy balance

- Let $(\mathrm{U}^{\mathrm{f}}, \mathrm{U}^{\mathrm{s}})$ be a solution of the coupled problem (thus $\mathrm{U}_{\Sigma}^{\mathrm{f,d}} = \mathrm{K}\mathrm{U}_{\Sigma}^{\mathrm{s}}$).

- Multiplying the first equation by U^{f}:

$$\begin{aligned}(\mathrm{A}^{\mathrm{f}}\mathrm{U}^{\mathrm{f}}, \mathrm{U}^{\mathrm{f}}) &= ((\mathrm{A}^{\mathrm{f}}\mathrm{U}^{\mathrm{f}})_{\mathrm{I}}, \mathrm{U}_{\mathrm{I}}^{\mathrm{f}}) + ((\mathrm{A}^{\mathrm{f}}\mathrm{U}^{\mathrm{f}})_{\Sigma}, \mathrm{U}_{\Sigma}^{\mathrm{f}}) = (\mathrm{F}_{\mathrm{I}}^{\mathrm{f}}, \mathrm{U}_{\mathrm{I}}^{\mathrm{f}}) + (\mathrm{F}_{\Sigma}^{\mathrm{f}} - \mathrm{R}_{\Sigma}^{\mathrm{f}}, \mathrm{U}_{\Sigma}^{\mathrm{f}}) \\ &= (\mathrm{F}^{\mathrm{f}}, \mathrm{U}^{\mathrm{f}}) - (\mathrm{R}_{\Sigma}^{\mathrm{f}}, \mathrm{U}_{\Sigma}^{\mathrm{f}}) = (\mathrm{F}^{\mathrm{f}}, \mathrm{U}^{\mathrm{f}}) - (\mathrm{R}_{\Sigma}^{\mathrm{f}}, \mathrm{K}\mathrm{U}_{\Sigma}^{\mathrm{s}})\end{aligned}$$

- and the second by U^{s}:

$$(\mathrm{A}^{\mathrm{s}}\mathrm{U}^{\mathrm{s}}, \mathrm{U}^{\mathrm{s}}) = (\mathrm{F}^{\mathrm{s}}, \mathrm{U}^{\mathrm{s}}) + (\mathrm{R}^{\mathrm{s}}, \mathrm{U}^{\mathrm{s}}) = (\mathrm{F}^{\mathrm{s}}, \mathrm{U}^{\mathrm{s}}) + (\mathrm{K}^{\mathrm{T}}\mathrm{R}_{\Sigma}^{\mathrm{f}}, \mathrm{U}_{\Sigma}^{\mathrm{s}})$$

- Adding the two relations, we get the energy equation:

$$\boxed{(\mathrm{A}^{\mathrm{f}}\mathrm{U}^{\mathrm{f}}, \mathrm{U}^{\mathrm{f}}) + (\mathrm{A}^{\mathrm{s}}\mathrm{U}^{\mathrm{s}}, \mathrm{U}^{\mathrm{s}}) = (\mathrm{F}^{\mathrm{f}}, \mathrm{U}^{\mathrm{f}}) + (\mathrm{F}^{\mathrm{s}}, \mathrm{U}^{\mathrm{s}})}$$

The boundary terms cancel, as in the continuous case.

Load computation in FSI

- Let $(\boldsymbol{u}, p)$ be solution of the fluid sub-problem
- Let $\mathcal{L} : H^{1/2}(\Sigma) \to H^1(\Omega^{\mathrm{f}})$ be a lifting (extension) in the fluid domain.
- The variational residual is:

$$\mathcal{R}^{\mathrm{f}}(\boldsymbol{u}, p; \mathcal{L}\boldsymbol{v}^{\mathrm{s}}) \stackrel{\mathrm{def}}{=} \int_{\Gamma_{\mathrm{N}}^{\mathrm{f}}} \boldsymbol{g} \cdot \mathcal{L}\boldsymbol{v}^{\mathrm{s}} - \int_{\Omega^{\mathrm{f}}} \rho^{\mathrm{f}} \partial_t \boldsymbol{u} \cdot \mathcal{L}\boldsymbol{v}^{\mathrm{s}} - \int_{\Omega^{\mathrm{f}}} 2\mu \boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\mathcal{L}\boldsymbol{v}^{\mathrm{s}}) + \int_{\Omega^{\mathrm{f}}} p \operatorname{div} \mathcal{L}\boldsymbol{v}^{\mathrm{s}}$$

$$= - \int_{\Sigma} \boldsymbol{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n}^{\mathrm{f}} \cdot \boldsymbol{v}^{\mathrm{s}}$$

- Solid sub-problem:

$$\int_{\widehat{\Omega}^{\mathrm{s}}} \rho^{\mathrm{s}} \frac{\partial^2 \boldsymbol{d}}{\partial t^2} \cdot \boldsymbol{v}^{\mathrm{s}} + \int_{\widehat{\Omega}^{\mathrm{s}}} \boldsymbol{\sigma}(\boldsymbol{d}) : \boldsymbol{\nabla} \boldsymbol{v}^{\mathrm{s}} = \mathcal{R}^{\mathrm{f}}(\boldsymbol{u}, p; \mathcal{L}\boldsymbol{v}^{\mathrm{s}}).$$

- Benefits:
  - very simple algebraic operation "$\mathcal{R} = b - Au$" !
  - ensures energy conservation at the discrete level

C Farhat, M Lesoinne, P Le Tallec, "Load and motion transfer algorithms for fluid/structure interaction problems with non-matching discrete interfaces" *Comp. Meth. Applied Mech. and Engng* (1998) 157(1), 95-114.

Dirichlet-Neumann iterations

- **Fluid sub-problem**

Find $\boldsymbol{u}^{n+1}$ with $\boldsymbol{u}^{n+1}_{|\Sigma} = \frac{\boldsymbol{d}^{n+1}_{|\Sigma} - \boldsymbol{d}^{n}_{|\Sigma}}{\delta t}$ and p^{n+1} such that:

$$\begin{aligned} \int_{\Omega^{\mathrm{f}}} \rho^{\mathrm{f}} \frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n}}{\delta t} \cdot \boldsymbol{v}^{\mathrm{f}} + 2\mu \int_{\Omega^{\mathrm{f}}} \boldsymbol{\epsilon}(\boldsymbol{u}^{n+1}) : \boldsymbol{\epsilon}(\boldsymbol{v}^{\mathrm{f}}) - \int_{\Omega^{\mathrm{f}}} p^{n+1} \operatorname{div} \boldsymbol{v}^{\mathrm{f}} &= \int_{\Gamma^{\mathrm{f}}_{\mathrm{N}}} \boldsymbol{g}^{n+1} \cdot \boldsymbol{v}^{\mathrm{f}} \\ \int_{\Omega^{\mathrm{f}}} q \operatorname{div} \boldsymbol{u}^{n+1} &= 0 \end{aligned}$$

Shorthand: $(\boldsymbol{u}^{n+1}, p^{n+1}) = \mathcal{F}(\boldsymbol{d}^{n+1}_{|\Sigma})$

- **Solid sub-problem**

Find $\boldsymbol{d}^{n+1}$ such that:

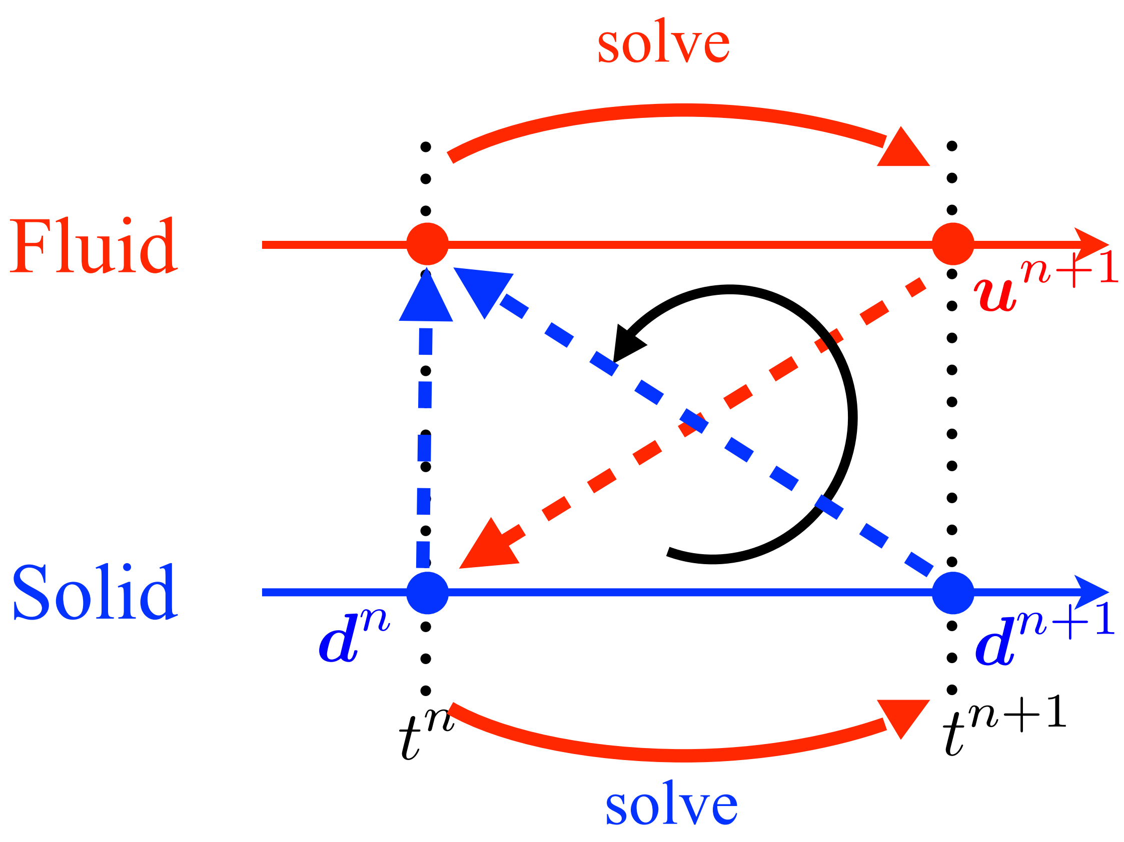
$$\int_{\widehat{\Omega}^{\mathrm{s}}} \rho^{\mathrm{s}} \frac{\boldsymbol{d}^{n+1} - 2\boldsymbol{d}^{n} + \boldsymbol{d}^{n-1}}{(\delta t)^2} \cdot \boldsymbol{v}^{\mathrm{s}} + \int_{\widehat{\Omega}^{\mathrm{s}}} \boldsymbol{\sigma}(\boldsymbol{d}^{n+1}) : \boldsymbol{\nabla} \boldsymbol{v}^{\mathrm{s}} = \mathcal{R}^{\mathrm{f}}(\boldsymbol{u}^{n+1}, p^{n+1}; \mathcal{L}\boldsymbol{v}^{\mathrm{s}})$$

Shorthand: $\boldsymbol{d}^{n+1}_{|\Sigma} = \mathcal{S}(\boldsymbol{u}^{n+1}, p^{n+1})$

Dirichlet-Neumann iterations

- **Implicit coupling schemes**
  Find a fixed point:

$$\boldsymbol{d}_{|\Sigma}^{n+1} = \mathcal{S} \circ \mathcal{F}\big(\boldsymbol{d}_{|\Sigma}^{n+1}\big)$$



- Several approches: accelerated fixed point, Newton, Inexact Newton,...

- In general:

$$\text{FSI cost} \gg \text{FLUID cost} + \text{SOLID cost}$$

Energy estimate

Energy estimate for the implicit coupling scheme

If the system is isolated, then:

$$\frac{1}{\delta t}\left[\int_{\Omega^{\mathrm{f}}} \frac{\rho^{\mathrm{f}}}{2}|\boldsymbol{u}^{n+1}|^2 - \int_{\Omega^{\mathrm{f}}} \frac{\rho^{\mathrm{f}}}{2}|\boldsymbol{u}^{n}|^2 + \int_{\widehat{\Omega}^{\mathrm{s}}} \frac{\rho^{\mathrm{s}}}{2}\left|\frac{\boldsymbol{d}^{n+1}-\boldsymbol{d}^{n}}{\delta t}\right|^2 - \int_{\widehat{\Omega}^{\mathrm{s}}} \frac{\rho^{\mathrm{s}}}{2}\left|\frac{\boldsymbol{d}^{n}-\boldsymbol{d}^{n-1}}{\delta t}\right|^2\right]$$
$$+\frac{1}{\delta t}\left[\int_{\widehat{\Omega}^{\mathrm{s}}} W\big(\boldsymbol{\epsilon}(\boldsymbol{d}^{n+1})\big) - \int_{\widehat{\Omega}^{\mathrm{s}}} W\big(\boldsymbol{\epsilon}(\boldsymbol{d}^{n})\big)\right] + \int_{\Omega^{\mathrm{f}}} 2\mu|\boldsymbol{\epsilon}(\boldsymbol{u}^{n+1})|^2 \leq 0$$

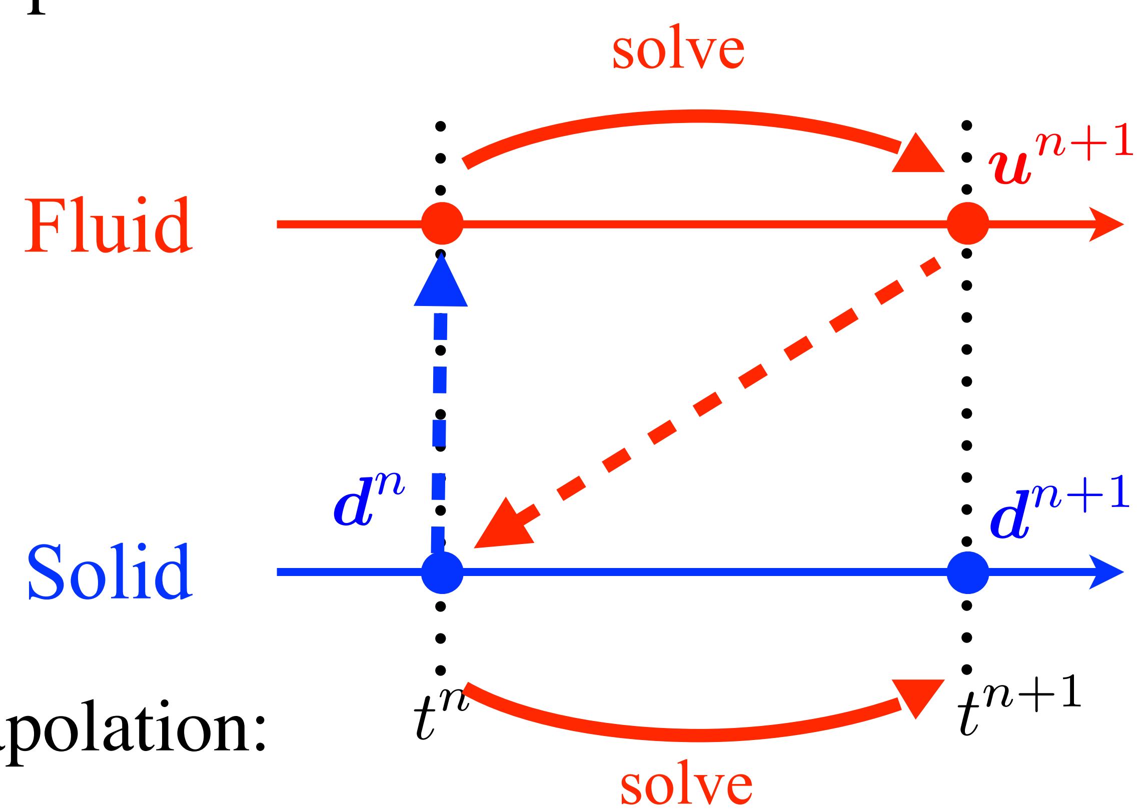
- The interface terms cancel: stability in the energy norm
- Proof
  - In the fluid, take $\boldsymbol{v}^{\mathrm{f}} = \boldsymbol{u}^{n+1} - \mathcal{L}\left(\frac{\boldsymbol{d}^{n+1}-\boldsymbol{d}^{n}}{\delta t}\right)$.
  - In the solid, take $\boldsymbol{v}^{\mathrm{s}} = \frac{\boldsymbol{d}^{n+1}-\boldsymbol{d}^{n}}{\delta t}$.

Dirichlet-Neumann iterations

- **Explicit coupling schemes**
  Solve only once the fluid and the solid sub-problems:

$$(\boldsymbol{u}^{n+1}, p^{n+1}) = \mathcal{F}(\boldsymbol{d}^n_{|\Sigma})$$

$$\boldsymbol{d}^{n+1}_{|\Sigma} = \mathcal{S}(\boldsymbol{u}^{n+1}, p^{n+1})$$



Many possible variants, for example extrapolation:

$$\tilde{\boldsymbol{d}}^n{}_{|\widehat{\Sigma}} = \boldsymbol{d}^n_{|\widehat{\Sigma}} + \delta t \dot{\boldsymbol{d}}^n_{|\widehat{\Sigma}}$$

- Very much used in aeroelasticity

$$\text{FSI cost} \approx \text{FLUID cost} + \text{SOLID cost}$$

Energy estimate

Energy estimate for the explicit coupling scheme

If the system is isolated, then:

$$\frac{1}{\delta t}\left[\int_{\Omega^{\mathrm{f}}}\frac{\rho^{\mathrm{f}}}{2}|\boldsymbol{u}^{n+1}|^2-\int_{\Omega^{\mathrm{f}}}\frac{\rho^{\mathrm{f}}}{2}|\boldsymbol{u}^{n}|^2+\int_{\widehat{\Omega}^{\mathrm{s}}}\frac{\rho^{\mathrm{s}}}{2}\left|\frac{\boldsymbol{d}^{n+1}-\boldsymbol{d}^{n}}{\delta t}\right|^2-\int_{\widehat{\Omega}^{\mathrm{s}}}\frac{\rho^{\mathrm{s}}}{2}\left|\frac{\boldsymbol{d}^{n}-\boldsymbol{d}^{n-1}}{\delta t}\right|^2\right]$$

$$+\frac{1}{\delta t}\left[\int_{\widehat{\Omega}^{\mathrm{s}}}W\big(\boldsymbol{\epsilon}(\boldsymbol{d}^{n+1})\big)-\int_{\widehat{\Omega}^{\mathrm{s}}}W\big(\boldsymbol{\epsilon}(\boldsymbol{d}^{n})\big)\right]+\int_{\Omega^{\mathrm{f}}}2\mu|\boldsymbol{\epsilon}(\boldsymbol{u}^{n+1})|^2$$

$$\underbrace{-\,\mathcal{R}^{\mathrm{f}}\Big(\boldsymbol{u}^{n+1},p^{n+1};\mathcal{L}\Big(\boldsymbol{u}^{n+1}-\frac{\boldsymbol{d}^{n+1}-\boldsymbol{d}^{n}}{\delta t}\Big)\Big)}_{\approx\int_{\Sigma}\boldsymbol{\sigma}(\boldsymbol{u}^{n+1},p^{n+1})\boldsymbol{n}\cdot\left(\boldsymbol{u}^{n+1}-\frac{\boldsymbol{d}^{n+1}-\boldsymbol{d}^{n}}{\delta t}\right)}\leq 0$$

artificial interface numerical power

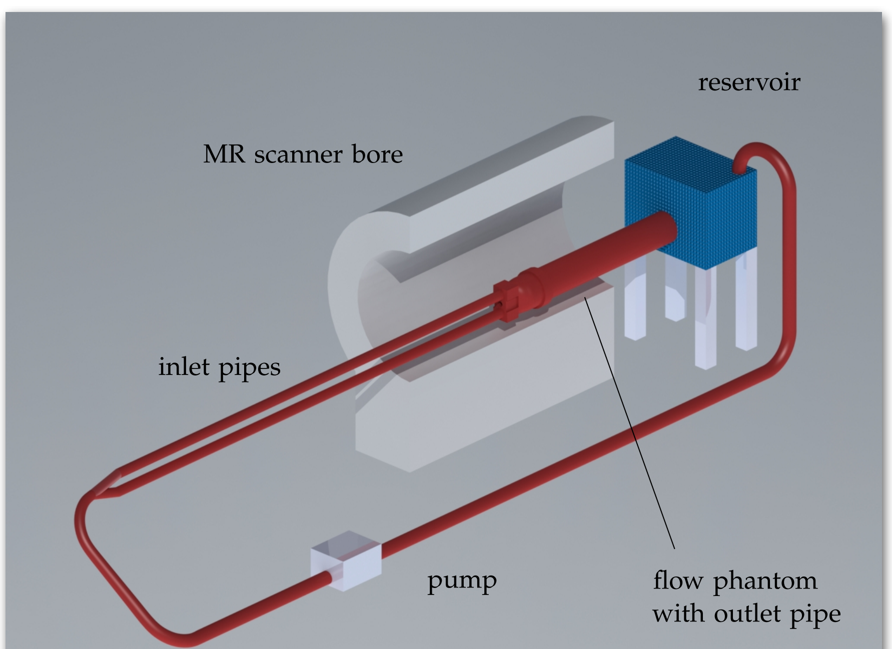
- The interface terms do not cancel: **stability is not granted**
- Proof
  - In the fluid, take $\boldsymbol{v}^{\mathrm{f}}=\boldsymbol{u}-\mathcal{L}\left(\frac{\boldsymbol{d}^{n}-\boldsymbol{d}^{n-1}}{\delta t}\right)$.
  - In the solid, take $\boldsymbol{v}^{\mathrm{s}}=\frac{\boldsymbol{d}^{n+1}-\boldsymbol{d}^{n}}{\delta t}$.

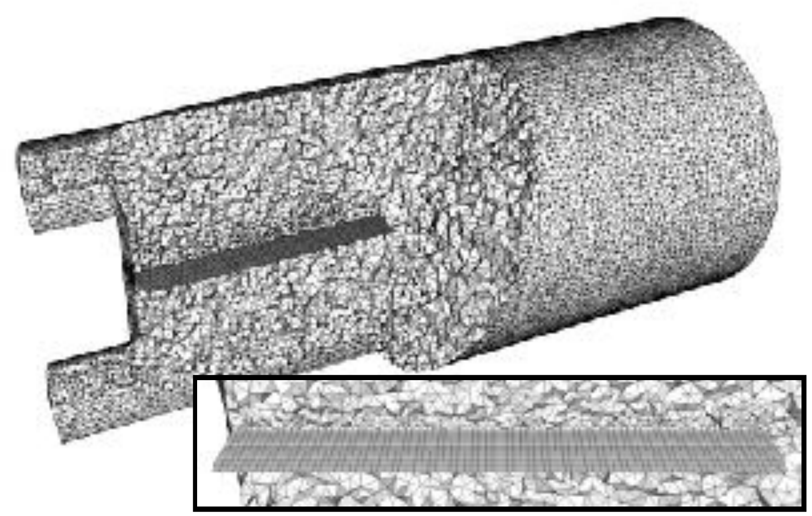
1997-2007 2007-2009 2010-2015

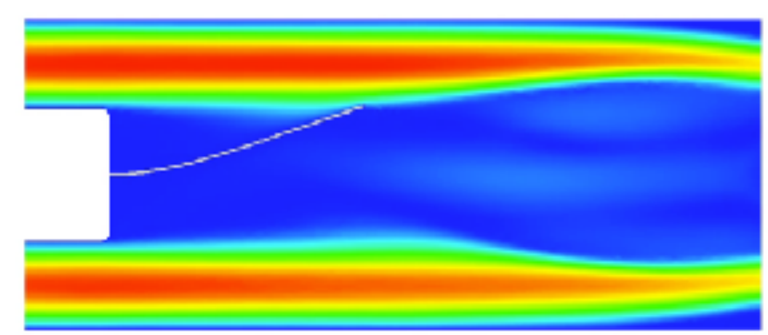
Implicit	Semi-implicit	Explicit monolithic	Explicit fractional step
18	2.5	2	1

Normalized CPU time (on the same computer, in 2015)

Benchmark with validation against experimental measurements:







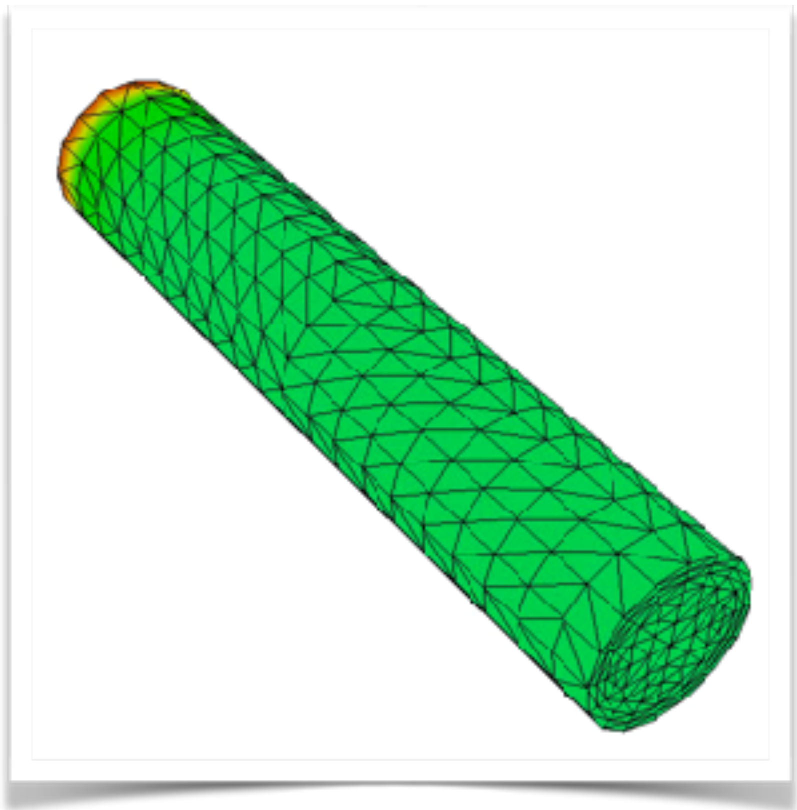
Landajuela, Vidrascu, Chapelle, Fernández, *Int. J. Numerical Methods Biomed Engng. 2016*

Outline

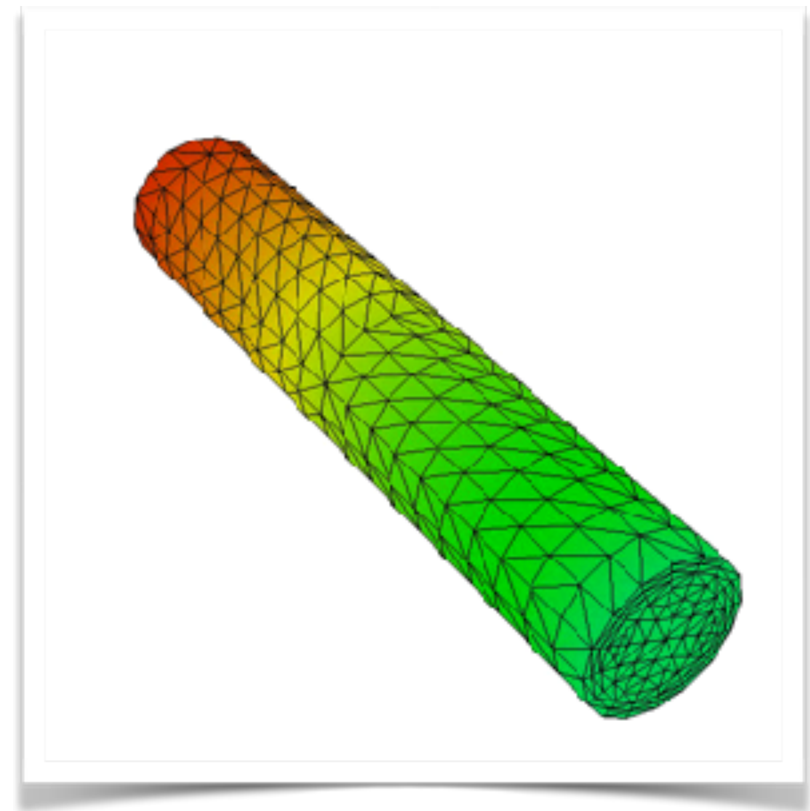
- Generality: energy balance, load computation, coupling
- Added mass effect and incompressibility
- Semi-implicit coupling *via* projection schemes
- Explicit coupling *via* Nitsche and Robin formulations
- Explicit coupling for thin structures

P Causin, JF Gerbeau, F Nobile. "Added-mass effect in the design of partitioned algorithms for fluid–structure problems." *Computer Methods in Applied Mechanics and Engineering* 194, no. 42 (2005): 4506-4527.

Explicit coupling: some observations



implicit coupling



explicit coupling

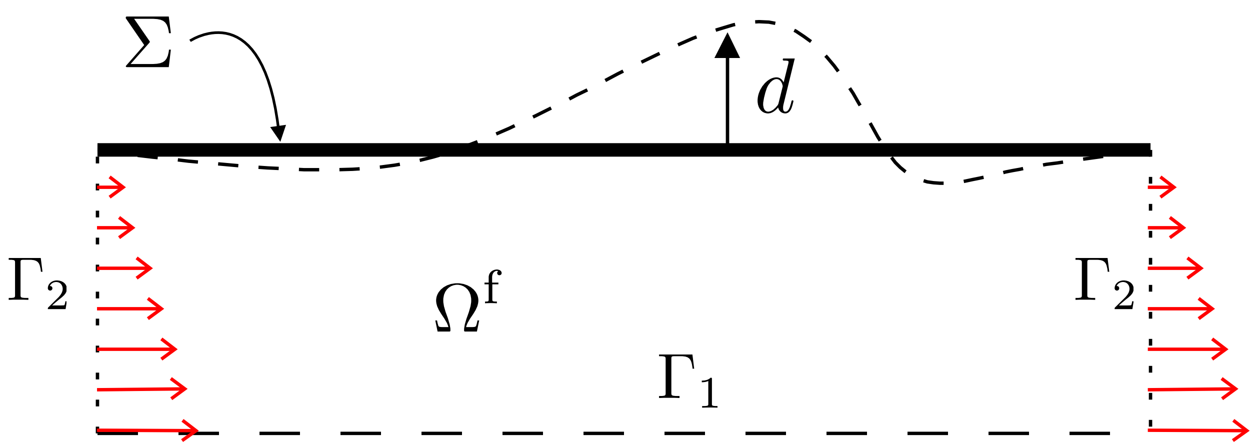
- Empirical observations for explicit coupling in blood flows:

  ➡ Instabilities disappear when the solid density is (artificially) increased

  ➡ Instabilities are independent of the time step

  ➡ The instability is sensitive to the **length** of the domain

Analysis of a toy model



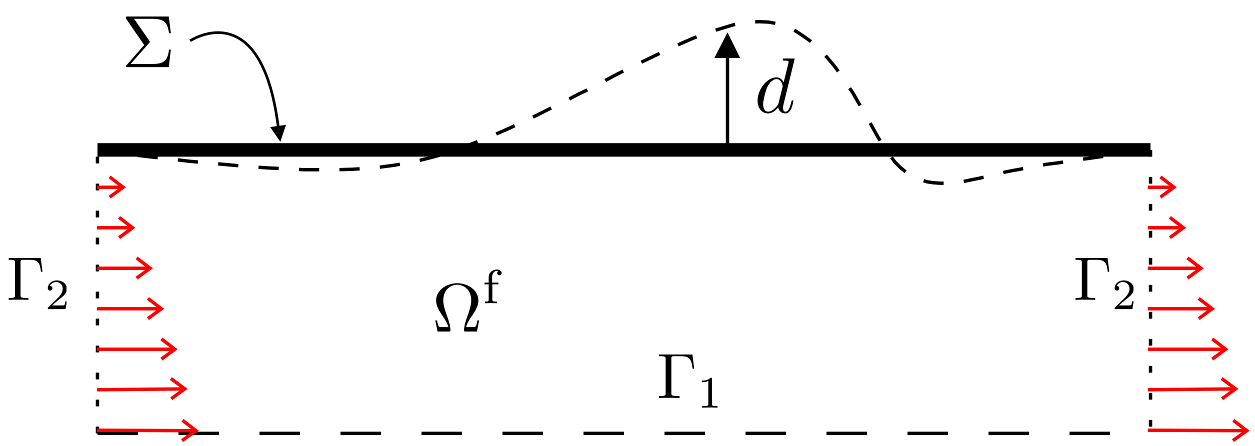
- Solid: string model (infinitesimal displacements)

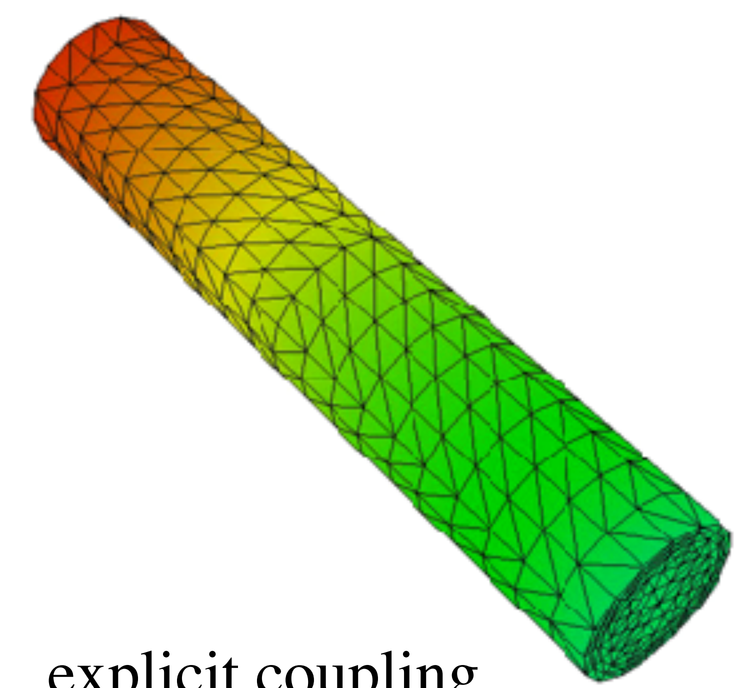
$$\rho^{\mathrm{s}}\varepsilon\ddot{d} + Ld = p_{|\Sigma}, \quad \text{in} \quad \Sigma,$$

with

- d: vertical displacement
- ε: vessel thickness
- L: linear operator (for instance $L\eta = a\eta - b\frac{\partial^2\eta}{\partial x^2}$)

Analysis of a toy model





explicit coupling

- **Solid**: string model (infinitesimal displacements)

$$\rho^{\mathrm{s}}\varepsilon\ddot{d} + Ld = p_{|\Sigma}, \quad \text{in} \quad \Sigma,$$

- **Fluid**: fixed fluid domain, no viscous/convective terms

$$\left\{\begin{aligned} \rho^{\mathrm{f}}\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} p &= 0, && \text{in} \quad \Omega^{\mathrm{f}} \\ \operatorname{div}\boldsymbol{u} &= 0, && \text{in} \quad \Omega^{\mathrm{f}} \\ \boldsymbol{u}\cdot\boldsymbol{n} &= \dot{d}, && \text{on} \quad \Sigma \\ \boldsymbol{u}\cdot\boldsymbol{n} &= 0, && \text{on} \quad \Gamma_1 \\ p &= 0, && \text{on} \quad \Gamma_2 \end{aligned}\right. \quad \underset{\text{div}}{\Longrightarrow} \quad \left\{\begin{aligned} &-\Delta p = 0, && \text{in} \quad \Omega^{\mathrm{f}} \\ &\frac{\partial p}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}}\frac{\partial \boldsymbol{u}}{\partial t}\cdot\boldsymbol{n} = -\rho^{\mathrm{f}}\ddot{d}, && \text{on} \quad \Sigma \\ &\frac{\partial p}{\partial \boldsymbol{n}} = 0, && \text{on} \quad \Gamma_1 \\ &p = 0 && \text{on} \quad \Gamma_2 \end{aligned}\right.$$

- **Physics**: reproduces propagation phenomena
- **Numerics**: explicit coupling unstable

The added-mass operator

Fluid: $$\begin{cases} -\Delta p = 0, & \text{in} \quad \Omega^{\mathrm{f}} \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}} \ddot{d}, & \text{on} \quad \Sigma \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = 0, & \text{on} \quad \Gamma_1 \\ p = 0 & \text{on} \quad \Gamma_2 \end{cases}$$

Solid: $$\rho^{\mathrm{s}} \varepsilon \ddot{d} + Ld = p_{|\Sigma}, \quad \text{in} \quad \Sigma,$$

Steklov-Poincaré operator

The operator $\mathcal{M}_{\mathrm{A}} : H^{-\frac{1}{2}}(\Sigma) \to H^{\frac{1}{2}}(\Sigma)$ defined as: for each $g \in H^{-\frac{1}{2}}(\Sigma)$ we set $\mathcal{M}_{\mathrm{A}}(g) \stackrel{\text{def}}{=} q_{|\Gamma^{\mathrm{w}}}$, where $q \in H^1(\Omega^{\mathrm{f}})$ solves

$$\begin{cases} -\Delta q = 0, & \text{in} \quad \Omega^{\mathrm{f}} \\ \dfrac{\partial q}{\partial \boldsymbol{n}} = g, & \text{on} \quad \Sigma \\ \dfrac{\partial q}{\partial \boldsymbol{n}} = 0, & \text{on} \quad \Gamma_1 \\ q = 0, & \text{on} \quad \Gamma_2 \end{cases}$$

is a linear, compact, positive and self-adjoint operator in $L^2(\Sigma)$.

From this definition, we have $$\boxed{p_{|\Sigma} = \mathcal{M}_{\mathrm{A}}(-\rho^{\mathrm{f}} \ddot{d}) = -\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \ddot{d}}$$

The added-mass operator

Fluid: $$\begin{cases} -\Delta p = 0, & \text{in} \quad \Omega^{\mathrm{f}} \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}} \ddot{d}, & \text{on} \quad \Sigma \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = 0, & \text{on} \quad \Gamma_1 \\ p = 0 & \text{on} \quad \Gamma_2 \end{cases}$$

Solid: $$\rho^{\mathrm{s}} \varepsilon \ddot{d} + L d = p_{|\Sigma}, \quad \text{in} \quad \Sigma, \qquad (1)$$

$$\boxed{p_{|\Sigma} = -\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \ddot{d}}$$

$$\big(\rho^{\mathrm{s}} \varepsilon + \rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}}\big) \ddot{d} + L d = 0, \quad \text{in} \quad \Sigma \qquad (2)$$

What kind of time integration scheme of (2) arises from the explicit coupling of (1) ?

Monolithic scheme

Fluid:
$$\begin{cases} \rho^{\mathrm{f}} \dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\delta t} + \nabla p^{n+1} = 0 \\ \operatorname{div} \boldsymbol{u}^{n+1} = 0 \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \dfrac{d^{n+1} - d^n}{\delta t} \end{cases} \xRightarrow{\operatorname{div}} \begin{cases} -\Delta p^{n+1} = 0 \\ \dfrac{\partial p^{n+1}}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}} \dfrac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} \end{cases}$$

Solid:
$$\rho^{\mathrm{s}} \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} + L d^{n+1} = p^{n+1}_{|\Sigma}$$

$$p^{n+1}_{|\Sigma} = -\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2}$$

Condensed solid problem:

$$\left(\rho^{\mathrm{s}} \varepsilon + \rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}}\right) \underbrace{\frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2}}_{\textbf{implicit}} + L d^{n+1} = 0$$

No surprise: no stability problem…

Explicit coupling and added-mass

Fluid:
$$\begin{cases} \rho^{\mathrm{f}} \dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\delta t} + \nabla p^{n+1} = 0 \\ \operatorname{div} \boldsymbol{u}^{n+1} = 0 \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \dfrac{d^n - d^{n-1}}{\delta t} \end{cases} \xRightarrow[\text{div}]{} \begin{cases} -\Delta p^{n+1} = 0 \\ \dfrac{\partial p^{n+1}}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}} \dfrac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2} \end{cases}$$

Solid:
$$\rho^{\mathrm{s}} \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} + L d^n = p^{n+1}_{|\Sigma}$$

$$p^{n+1}_{|\Sigma} = -\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2}$$

Condensed solid problem:

$$\underbrace{\rho^{\mathrm{s}} \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2}}_{\textbf{implicit}} + \rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \underbrace{\frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2}}_{\textbf{explicit}} + L d^n = 0$$

Explicit coupling yields an explicit discretization of the added mass

Stability analysis

$$\rho^{\mathrm{s}}\varepsilon\frac{d^{n+1}-2d^{n}+d^{n-1}}{\delta t^2}+\rho^{\mathrm{f}}\mathcal{M}_{\mathrm{A}}\frac{d^{n}-2d^{n-1}+d^{n-2}}{\delta t^2}+a\eta^{n+1}=0$$

- Note that $\mathcal{M}_{\mathrm{A}}$ is a compact, positive and self-adjoint operator on $L^2(\Sigma)$

- Expand d^{n+1}, d^{n}, d^{n-1} and d^{n-2} on a orthonormal basis made of eigen-vectors of $\mathcal{M}_{\mathrm{A}}$

- Diagonalization of $\mathcal{M}_{\mathrm{A}}$ leads to the linear homogeneous recurrence:

$$\rho^{\mathrm{s}}\varepsilon\frac{d_i^{n+1}-2d_i^{n}+d_i^{n-1}}{\delta t^2}+\rho^{\mathrm{f}}\mu_i\frac{d_i^{n}-2d_i^{n-1}+d_i^{n-2}}{\delta t^2}+a\eta_i^{n+1}=0,\quad i\in\mathbb{N}$$

- Whose characteristic polynomial is:

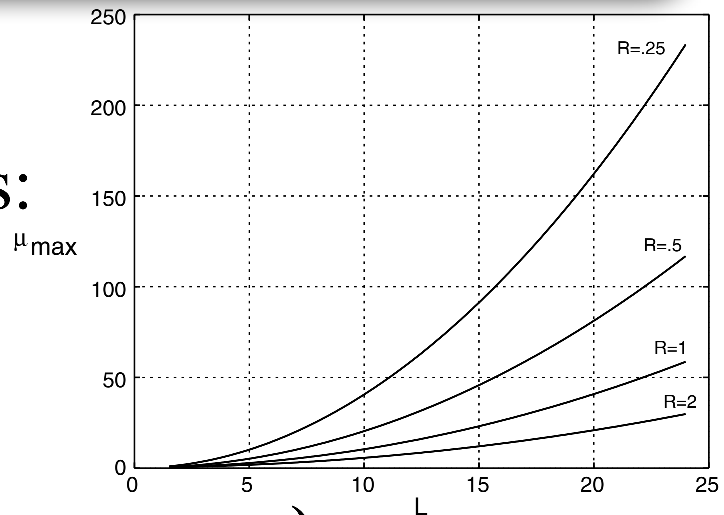
$$P_i(r)\stackrel{\mathrm{def}}{=}(\rho^{\mathrm{s}}\varepsilon+a\delta t^2)r^3+(\rho^{\mathrm{f}}\mu_i-2\rho^{\mathrm{s}}\varepsilon)r^2+(\rho^{\mathrm{s}}\varepsilon-2\rho^{\mathrm{f}}\mu_i)r+\rho^{\mathrm{f}}\mu_i$$

An unconditional instability result

Proposition

Let $\mu_{\max}$ be the largest eigenvalue of $\mathcal{M}_{\mathrm{A}}$ and assume that $L\eta = a\eta$. Then, the previous explicit coupling scheme is unconditionally unstable whenever

$$\frac{\rho^{\mathrm{f}}\mu_{\max}}{\rho^{\mathrm{s}}\varepsilon} \geq 1. \quad (1)$$



- The instability condition confirms the empirical observations:
  - Instabilities depend on the density ratio
  - The instability condition does not depend on the time step
  - Instabilities occur when the structure is **thin** and **slender (larger** $\mu_{\max}$ **)**
- Other time schemes have been considered by *Förster-Wall-Ramm 07* with analogous conclusions
- Do not forget that the first assumption was **incompressiblity**

Why fixed-point iterations can be inefficient ?

Fluid:
$$\begin{cases} \rho^{\mathrm{f}} \dfrac{\boldsymbol{u}_{k+1} - \boldsymbol{u}^n}{\delta t} + \nabla p_{k+1} = 0 \\ \operatorname{div} \boldsymbol{u}_{k+1} = 0 \\ \boldsymbol{u}_{k+1} \cdot \boldsymbol{n} = \dfrac{d_k - d^n}{\delta t} \end{cases} \xRightarrow[\text{div}]{} \begin{cases} -\Delta p_{k+1} = 0 \\ \dfrac{\partial p_{k+1}}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}} \dfrac{d_k - 2d^n + d^{n-1}}{\delta t^2} \end{cases}$$

Solid:
$$\rho^{\mathrm{s}} \varepsilon \frac{\widetilde{d}_{k+1} - 2d^n + d^{n-1}}{\delta t^2} + L\widetilde{d}_{k+1} = p_{k+1|\Sigma}$$

$$p_{k+1|\Sigma} = -\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \frac{d_k - 2d^n + d^{n-1}}{\delta t^2}$$

Relaxation:
$$d_{k+1} = \omega \tilde{d}_{k+1} + (1 - \omega) d_k$$

Proposition

Let $\mu_{\max}$ be the largest eigenvalue of $\mathcal{M}_{\mathrm{A}}$ and assume that $L\eta = a\eta$. The above relaxed fixed point iterations converge if and only if

$$0 < \omega < \frac{2(\rho^{\mathrm{s}}\varepsilon + a\delta t^2)}{\rho^{\mathrm{s}}\varepsilon + \rho^{\mathrm{f}}\mu_{\max} + a\delta t^2}$$

P Causin, JF Gerbeau, F Nobile. “Added-mass effect in the design of partitioned algorithms for fluid–structure problems.” *Computer Methods in Applied Mechanics and Engineering* 194, no. 42 (2005): 4506-4527.

Implicit / Explicit coupling

Two families of solution:

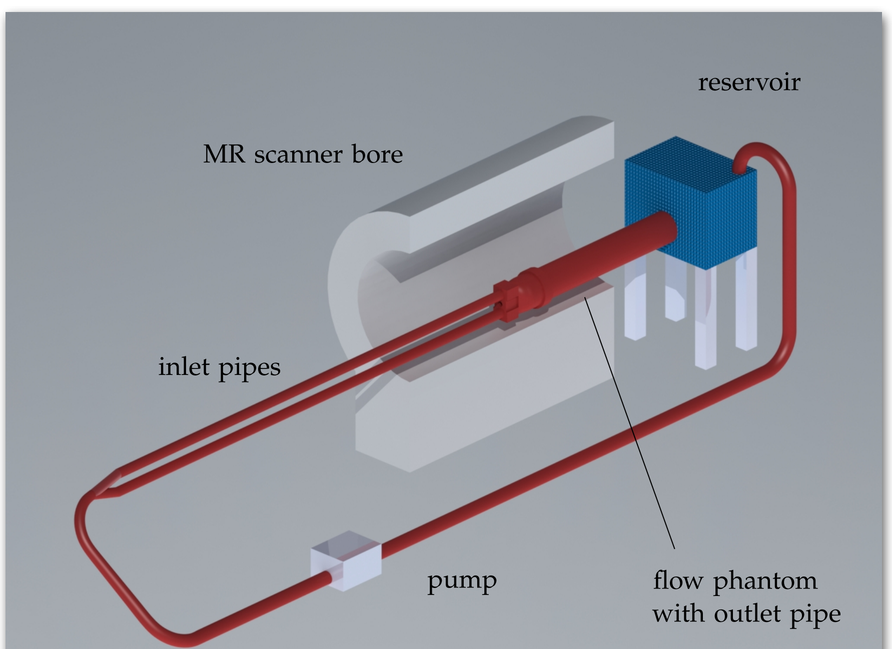
- Improve **implicit** iterations (Fixed point, Newton, quasi-Newton,…)
  - Le Tallec-Mouro (1999) Wall-Ramm (2001), Fernández-Moubachir (2003), Matthies-Steindorf (2003), JFG-Vidrascu (2003), Mischler-van Brummelen-de Borst (2005), Deparis-Discacciati-Quarteroni (2005), Badia-Nobile-Vergara (2007), Vierendeels (2006), Vierendeels-Lanoye-Degroote-Verdonck (2007), Degroote-Annerel-Vierendeels (2010), and **many others**...

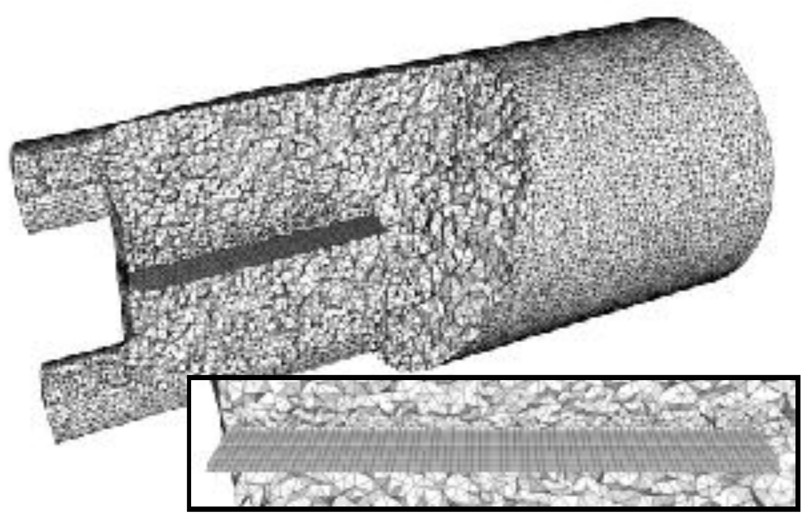
- Devise **explicit** coupling algorithms:
  - Projection semi-implicit coupling: Fernández-JFG-Grandmont (2007), Badia-Quaini-Quarteroni (2008)
  - Robin-Neuman : Burman-Fernández (2008)
  - Kinematically coupled time-splitting: Guidoboni-Glowinski-Cavallini-Canic (2009), Fernández (2012)

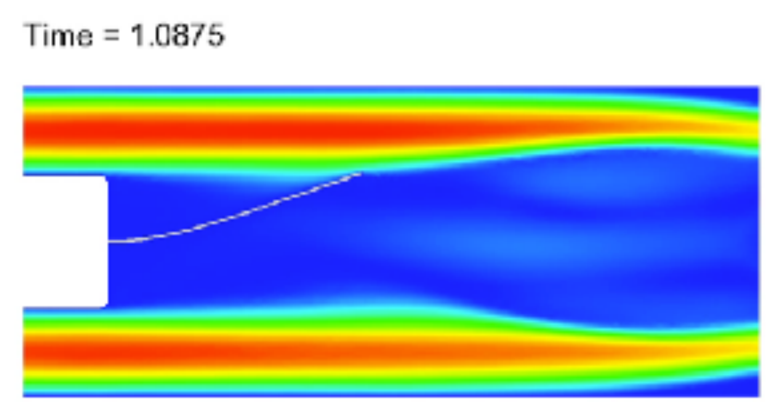
1997-2007	2007-2009	2010-2015	
Implicit	**Semi-implicit**	**Explicit monolithic**	**Explicit fractional step**
18	2.5	2	1

Normalized CPU time (on the same computer, in 2015)

Benchmark with validation against experimental measurements:







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Outline

- Generality: energy balance, load computation, coupling
- Added mass effect and incompressibility
- Semi-implicit coupling *via* projection schemes
- Explicit coupling *via* Nitsche and Robin formulations
- Explicit coupling for thin structures

MA Fernández, JF Gerbeau, C Grandmont. "A projection semi−implicit scheme for the coupling of an elastic structure with an incompressible fluid." *International Journal for Numerical Methods in Engineering* (2007) 69(4): 794-821.

Semi-implicit coupling

Basic ideas:

- Couple implicitly the added-mass effect (incompressibility, pressure stress)
- Couple explicitly the fluid domain motion, convective and viscous effects
- Use the splitting provided by fractional time stepping in the fluid (Chorin-Temam projection scheme)

The Chorin-Teman projection scheme

R Temam, *Arch Rat Mech Anal (1969)*
AJ Chorin, *Math Comp (1968)*

- Transient Stokes equations:

$$\rho^{\mathrm{f}} \frac{\partial \boldsymbol{u}}{\partial t} - \mathbf{div}(2\mu \boldsymbol{\epsilon}(\boldsymbol{u})) + \boldsymbol{\nabla} p = \mathbf{0}, \quad \text{in} \quad \Omega^{\mathrm{f}}$$

$$\operatorname{div} \boldsymbol{u} = 0, \quad \text{in} \quad \Omega^{\mathrm{f}}$$

- **Viscous** step:

$$\begin{cases} \rho^{\mathrm{f}} \dfrac{\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^n}{\delta t} - \operatorname{div}\left(2\mu\epsilon(\widetilde{\boldsymbol{u}}^{n+1})\right) = 0, \quad \text{in} \quad \Omega^{\mathrm{f}} \\ \widetilde{\boldsymbol{u}}^{n+1} = 0, \quad \text{on} \quad \partial\Omega^{\mathrm{f}} \end{cases}$$

- **Projection** step:

$$\begin{cases} \rho^{\mathrm{f}} \dfrac{\boldsymbol{u}^{n+1} - \widetilde{\boldsymbol{u}}^{n+1}}{\delta t} + \boldsymbol{\nabla} p^{n+1} = 0, \text{ in } \Omega^{\mathrm{f}} \\ \operatorname{div} \boldsymbol{u}^{n+1} = 0, \text{ in } \Omega^{\mathrm{f}} \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = 0, \text{ on } \partial\Omega^{\mathrm{f}} \end{cases} \xrightarrow[\text{div}]{} \begin{cases} -\Delta p^{n+1} = -\dfrac{\rho^{\mathrm{f}}}{\delta t} \operatorname{div} \widetilde{\boldsymbol{u}}^{n+1}, \text{ in } \Omega^{\mathrm{f}} \\ \dfrac{\partial p^{n+1}}{\partial \boldsymbol{n}} = 0, \text{ on } \partial\Omega^{\mathrm{f}} \end{cases}$$

Accuracy: velocity: $\mathcal{O}(\delta t)$ in L^2, $\mathcal{O}(\delta t^{1/2})$ in H^1, pressure $\mathcal{O}(\delta t^{1/2})$ in L^2
but can be improved with pressure correction ...

JL Guermond, P Minev, J Shen, *CMAME, 2006*

Semi-implicit coupling: explicit part

- Viscous sub-step:

$$\begin{aligned}
\rho^{\mathrm{f}} \frac{\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^n}{\delta t} - \mathbf{div}\,(2\mu\boldsymbol{\epsilon}(\widetilde{\boldsymbol{u}}^{n+1})) = 0, \quad &\text{in} \quad \Omega^{\mathrm{f}} \\
\widetilde{\boldsymbol{u}}^{n+1} = \dot{\boldsymbol{d}}^n, \quad &\text{on} \quad \Sigma
\end{aligned}$$

- **N.B.:** in the more realistic cases of Navier-Stokes equations on moving domains, the mesh is moved during this step, only *once* (explicit).

Semi-implicit coupling: implicit part

- Fluid projection sub-step (in a known domain):

$$\left\{\begin{aligned} \rho^{\mathrm{f}} \frac{\boldsymbol{u}^{n+1} - \widetilde{\boldsymbol{u}}^{n+1}}{\delta t} + \boldsymbol{\nabla} p^{n+1} &= 0, \text{ in } \Omega^{\mathrm{f}} \\ \operatorname{div} \boldsymbol{u}^{n+1} &= 0, \text{ in } \Omega^{\mathrm{f}} \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \frac{\boldsymbol{d}^{n+1} - \boldsymbol{d}^{n}}{\delta t} \cdot \boldsymbol{n}&, \text{ on } \Sigma \end{aligned}\right. \quad \overset{\Longrightarrow}{\text{div}} \quad \left\{\begin{aligned} -\Delta p^{n+1} &= -\frac{\rho^{\mathrm{f}}}{\delta t} \operatorname{div} \widetilde{\boldsymbol{u}}^{n+1}, \text{ in } \Omega^{\mathrm{f}} \\ \frac{\partial p^{n+1}}{\partial \boldsymbol{n}} &= -\rho^{\mathrm{f}} \frac{\boldsymbol{d}^{n+1} - 2\boldsymbol{d}^{n} + \boldsymbol{d}^{n-1}}{\delta t^2}, \text{ on } \Sigma \end{aligned}\right.$$

- Solid equation:

$$\left\{\begin{aligned} \rho^{\mathrm{s}} \frac{\boldsymbol{d}^{n+1} - 2\boldsymbol{d}^{n} + \boldsymbol{d}^{n-1}}{\delta t^2} - \mathbf{div}\left(\boldsymbol{\sigma}(\boldsymbol{d}^{n+1})\right) = \mathbf{0}, &\quad \text{in} \quad \widehat{\Omega}^{\mathrm{s}} \\ \boldsymbol{\sigma}(\boldsymbol{d}^{n+1})\widehat{\boldsymbol{n}} = \boldsymbol{\sigma}(\widetilde{\boldsymbol{u}}^{n+1}, p^{n+1})\widehat{\boldsymbol{n}}, &\quad \text{on} \quad \widehat{\Sigma} \end{aligned}\right.$$

  - Projection sub-step in a fixed fluid domain (even in ALE formulation)
  - Explicit viscous load, implicit pressure load
  - Implicit part solved with cheaper inner iterations compared to previous schemes

Semi-implicit coupling: the nonlinear case

1) Explicit sub-step:

- Fluid domain motion

$$\boldsymbol{d}^{\mathrm{f},n+1} = \mathrm{Ext}(\boldsymbol{d}^{n}_{|\widehat{\Sigma}}), \quad \boldsymbol{w}^{n+1} = \frac{\boldsymbol{d}^{\mathrm{f},n+1} - \boldsymbol{d}^{n}}{\delta t}, \quad \Omega^{\mathrm{f},n+1} = (I + \boldsymbol{d}^{\mathrm{f},n+1})(\widehat{\Omega}^{\mathrm{f}}),$$

- Fluid viscous sub-step

$$\rho^{\mathrm{f}} \left(\left. \frac{\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^{n}}{\delta t} \right|_{\widehat{\boldsymbol{x}}} + (\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{w}^{n+1}) \cdot \boldsymbol{\nabla} \widetilde{\boldsymbol{u}}^{n+1} \right) - 2\mu \,\mathbf{div}\, \boldsymbol{\epsilon}(\widetilde{\boldsymbol{u}}^{n+1}) = 0, \quad \text{in} \quad \Omega^{\mathrm{f},n+1}$$

$$\widetilde{\boldsymbol{u}}^{n+1} = \boldsymbol{w}^{n+1}, \quad \text{on} \quad \Sigma^{n+1}$$

2) Implicit sub-step: (fixed fluid domain)

- Fluid projection sub-step

$$\begin{cases} \rho^{\mathrm{f}} \dfrac{\boldsymbol{u}^{n+1} - \widetilde{\boldsymbol{u}}^{n+1}}{\delta t} + \boldsymbol{\nabla} p^{n+1} = 0, & \text{in} \quad \Omega^{\mathrm{f},n+1} \\ \mathrm{div} \boldsymbol{u}^{n+1} = 0, & \text{in} \quad \Omega^{\mathrm{f},n+1} \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \dfrac{\boldsymbol{d}^{n+1} - \boldsymbol{d}^{n}}{\delta t} \cdot \boldsymbol{n}, & \text{on} \quad \Sigma^{n+1} \end{cases}$$

- Solid equation:

$$\begin{cases} \rho^{\mathrm{s}} \dfrac{\dot{\boldsymbol{d}}^{n+1} - \dot{\boldsymbol{d}}^{n}}{\delta t} - \dfrac{1}{2} \mathbf{div} \left[\boldsymbol{F}(\boldsymbol{d}^{n+1}) \boldsymbol{S}(\boldsymbol{d}^{n+1}) + \boldsymbol{F}(\boldsymbol{d}^{n}) \boldsymbol{S}(\boldsymbol{d}^{n}) \right] = \boldsymbol{0}, & \text{in} \quad \widehat{\Omega}^{\mathrm{s}} \\ \dfrac{1}{2} \left[\boldsymbol{F}(\boldsymbol{d}^{n+1}) \boldsymbol{S}(\boldsymbol{d}^{n+1}) + \boldsymbol{F}(\boldsymbol{d}^{n}) \boldsymbol{S}(\boldsymbol{d}^{n}) \right] \widehat{\boldsymbol{n}} = J(\boldsymbol{d}^{\mathrm{f},n+1}) \boldsymbol{\sigma}(\widetilde{\boldsymbol{u}}^{n+1}, p^{n+1}) \boldsymbol{F}(\boldsymbol{d}^{\mathrm{f},n+1})^{-\mathrm{T}} \widehat{\boldsymbol{n}}, & \text{on} \quad \widehat{\Sigma} \end{cases}$$

Variational formulation

1) Explicit sub-step:

$$\begin{cases} \tilde{\boldsymbol{u}}^{n+1} = \boldsymbol{w}^{n+1}, \quad \text{on} \quad \Sigma^{n+1} \\ \dfrac{1}{\delta t}\left[\displaystyle\int_{\Omega^{\mathrm{f},n+1}} \rho^{\mathrm{f}} \tilde{\boldsymbol{u}}^{n+1} \cdot \boldsymbol{v}_1^{\mathrm{f}} - \int_{\Omega^{\mathrm{f},n}} \rho^{\mathrm{f}} \boldsymbol{u}^{n} \cdot \boldsymbol{v}_1^{\mathrm{f}}\right] - \displaystyle\int_{\Omega^{\mathrm{f},n+1}} \rho^{\mathrm{f}} (\operatorname{div} \boldsymbol{w}^{n+1}) \tilde{\boldsymbol{u}}^{n+1} \cdot \boldsymbol{v}_1^{\mathrm{f}} \\ + \displaystyle\int_{\Omega^{\mathrm{f},n+1}} \rho^{\mathrm{f}} (\tilde{\boldsymbol{u}}^{n+1} - \boldsymbol{w}^{n+1}) \cdot \nabla \tilde{\boldsymbol{u}}^{n+1} \cdot \boldsymbol{v}_1^{\mathrm{f}} + 2\mu \int_{\Omega^{\mathrm{f},n+1}} \boldsymbol{\epsilon}(\tilde{\boldsymbol{u}}^{n+1}) : \boldsymbol{\epsilon}(\boldsymbol{v}_1^{\mathrm{f}}) = 0, \quad \forall \widehat{\boldsymbol{v}_1^{\mathrm{f}}} \in V^{\mathrm{f}} \end{cases}$$

2) Implicit sub-step:

$$\begin{cases} \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \dfrac{\boldsymbol{d}^{n+1} - \boldsymbol{d}^{n}}{\delta t} \cdot \boldsymbol{n}, \quad \text{on} \quad \Sigma^{n+1} \\ \dfrac{1}{\delta t}\left[\displaystyle\int_{\Omega^{\mathrm{f},n+1}} \rho^{\mathrm{f}} \boldsymbol{u}^{n+1} \cdot \boldsymbol{v}_2^{\mathrm{f}} - \int_{\Omega^{\mathrm{f},n+1}} \rho^{\mathrm{f}} \tilde{\boldsymbol{u}}^{n+1} \cdot \boldsymbol{v}_2^{\mathrm{f}}\right] - \displaystyle\int_{\Omega^{\mathrm{f},n+1}} p^{n+1} \operatorname{div} \boldsymbol{v}_2^{\mathrm{f}} \\ \qquad + \displaystyle\int_{\Omega^{\mathrm{f},n+1}} q \operatorname{div} \boldsymbol{u}^{n+1} = 0, \quad \forall \widehat{\boldsymbol{v}_2^{\mathrm{f}}} \in H_{\Sigma}(\operatorname{div}; \widehat{\Omega}^{\mathrm{f}}), \quad \forall \widehat{q} \in L^2(\widehat{\Omega}^{\mathrm{f}}) \end{cases}$$

$$\int_{\widehat{\Omega}^{\mathrm{s}}} \rho^{\mathrm{s}} \frac{\dot{\boldsymbol{d}}^{n+1} - \dot{\boldsymbol{d}}^{n}}{\delta t} \cdot \boldsymbol{v}^{\mathrm{s}} + \frac{1}{2} \int_{\widehat{\Omega}^{\mathrm{s}}} \left[\boldsymbol{F}(\boldsymbol{d}^{n+1}) \boldsymbol{S}(\boldsymbol{d}^{n+1}) + \boldsymbol{F}(\boldsymbol{d}^{n}) \boldsymbol{S}(\boldsymbol{d}^{n})\right] : \boldsymbol{\nabla} \boldsymbol{v}^{\mathrm{s}}$$

$$= \underbrace{-\int_{\Sigma^{n+1}} 2\mu \boldsymbol{\epsilon}(\tilde{\boldsymbol{u}}^{n+1}) \boldsymbol{n} \cdot \boldsymbol{v}^{\mathrm{s}}}_{-\mathcal{R}_{\mu}^{\mathrm{f}}(\tilde{\boldsymbol{u}}^{n+1}; \mathcal{L}(\boldsymbol{v}^{\mathrm{s}}))} + \underbrace{\int_{\Sigma^{n+1}} p^{n+1} \boldsymbol{n} \cdot \boldsymbol{v}^{\mathrm{s}}}_{-\mathcal{R}_{p}^{\mathrm{f}}(p^{n+1}; \mathcal{L}(\boldsymbol{v}^{\mathrm{s}}))}, \quad \forall \boldsymbol{v}^{\mathrm{s}} \in V^{\mathrm{s}}.$$

viscous load

pressure load

Energy estimate

Energy estimate for the semi-implicit coupling scheme

If the system is isolated, then:

$$\frac{1}{\delta t}\left[\int_{\Omega^{\mathrm{f}}} \frac{\rho^{\mathrm{f}}}{2}|\boldsymbol{u}^{n+1}|^2 - \int_{\Omega^{\mathrm{f}}} \frac{\rho^{\mathrm{f}}}{2}|\boldsymbol{u}^{n}|^2 + \int_{\widehat{\Omega}^{\mathrm{s}}} \frac{\rho^{\mathrm{s}}}{2}\left|\frac{\boldsymbol{d}^{n+1}-\boldsymbol{d}^{n}}{\delta t}\right|^2 - \int_{\widehat{\Omega}^{\mathrm{s}}} \frac{\rho^{\mathrm{s}}}{2}\left|\frac{\boldsymbol{d}^{n}-\boldsymbol{d}^{n-1}}{\delta t}\right|^2\right]$$

$$+\frac{1}{\delta t}\left[\int_{\widehat{\Omega}^{\mathrm{s}}} W\big(\boldsymbol{\epsilon}(\boldsymbol{d}^{n+1})\big) - \int_{\widehat{\Omega}^{\mathrm{s}}} W\big(\boldsymbol{\epsilon}(\boldsymbol{d}^{n})\big)\right] + \int_{\Omega^{\mathrm{f}}} 2\mu|\boldsymbol{\epsilon}(\boldsymbol{u}^{n+1})|^2$$

$$\underbrace{-\,\mathcal{R}^{\mathrm{f}}_{\mu}\Big(\tilde{\boldsymbol{u}}^{n+1}, p^{n+1}; \mathcal{L}\big(\tilde{\boldsymbol{u}}^{n+1} - \frac{\boldsymbol{d}^{n+1}-\boldsymbol{d}^{n}}{\delta t}\big)\Big)}_{\approx \int_{\Sigma} 2\mu\boldsymbol{\epsilon}(\tilde{\boldsymbol{u}}^{n+1})\boldsymbol{n}\cdot\left(\tilde{\boldsymbol{u}}^{n+1} - \frac{\boldsymbol{d}^{n+1}-\boldsymbol{d}^{n}}{\delta t}\right)} \leq 0$$

artificial interface numerical power

- The interface terms do not cancel: **stability is not granted**
- But compare with the spurious power of the explicit scheme:

$$\approx \int_{\Sigma} \boldsymbol{\sigma}(\boldsymbol{u}^{n+1}, p^{n+1})\boldsymbol{n}\cdot\left(\boldsymbol{u}^{n+1} - \frac{\boldsymbol{d}^{n+1}-\boldsymbol{d}^{n}}{\delta t}\right)$$

A stability result (linear case)

Proposition

Assume the interface matching operator to be L^2-stable. Then, under condition

$$\rho^{\mathrm{s}} \geq C\left(\rho^{\mathrm{f}}\frac{h}{H^{\alpha}} + 2\frac{\mu\delta t}{hH^{\alpha}}\right), \quad \text{with} \quad \alpha \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if} \quad \overline{\Omega^{\mathrm{s}}} = \Sigma, \\ 1, & \text{if} \quad \overline{\Omega^{\mathrm{s}}} \neq \Sigma, \end{cases}$$

the following discrete energy inequality holds:

$$\frac{1}{\delta t}\left[\frac{\rho^{\mathrm{f}}}{2}\|\boldsymbol{u}_h^{n+1}\|^2_{0,\Omega^{\mathrm{f}}} - \frac{\rho^{\mathrm{f}}}{2}\|\boldsymbol{u}_h^{n}\|^2_{0,\Omega^{\mathrm{f}}} + \frac{\rho^{\mathrm{s}}}{2}\left\|\frac{\boldsymbol{d}_H^{n+1}-\boldsymbol{d}_H^{n}}{\delta t}\right\|^2_{0,\Omega^{\mathrm{f}}} - \frac{\rho^{\mathrm{s}}}{2}\left\|\frac{\boldsymbol{d}_H^{n}-\boldsymbol{d}_H^{n-1}}{\delta t}\right\|^2_{0,\Omega^{\mathrm{f}}}\right]$$
$$+\frac{1}{2\delta t}\left[a^{\mathrm{s}}(\boldsymbol{d}_H^{n+1},\boldsymbol{d}_H^{n+1}) - a^{\mathrm{s}}(\boldsymbol{d}_H^{n},\boldsymbol{d}_H^{n})\right] + \mu\|\epsilon(\widetilde{\boldsymbol{u}}_h^{n+1})\|^2_{0,\Omega^{\mathrm{f}}} \leq 0$$

Therefore, the semi-implicit coupling scheme is conditionally stable in the energy norm.

- If $\Omega^{\mathrm{s}} = \Sigma$ the scheme can be stabilized by decreasing δt
- If $\Omega^{\mathrm{s}} = \Sigma$ and $H = h$ the conditions is not too restrictive (numerics)
- Stable in practice...

MA Fernández, JF Gerbeau, C Grandmont. "A projection semi−implicit scheme for the coupling of an elastic structure with an incompressible fluid." *International Journal for Numerical Methods in Engineering* (2007) 69(4): 794-821.

Idea of the proof

1) Explicit sub-step:

$$
\begin{cases}
\widetilde{\boldsymbol{u}}^{n+1} = \dfrac{\boldsymbol{d}^n - \boldsymbol{d}^{n-1}}{\delta t}, \quad \text{on} \quad \Sigma^{n+1} \\[2ex]
\dfrac{1}{\delta t}\left[\displaystyle\int_{\Omega^{\mathrm{f}}} \rho^{\mathrm{f}} \tilde{\boldsymbol{u}}^{n+1} \cdot \boldsymbol{v}_1^{\mathrm{f}} - \int_{\Omega^{\mathrm{f}}} \rho^{\mathrm{f}} \boldsymbol{u}^n \cdot \boldsymbol{v}_1^{\mathrm{f}}\right] + \displaystyle\int_{\Omega^{\mathrm{f}}} 2\mu \boldsymbol{\epsilon}(\tilde{\boldsymbol{u}}^{n+1}) : \boldsymbol{\epsilon}(\boldsymbol{v}_1^{\mathrm{f}}) = 0, \quad \forall \boldsymbol{v}_1^{\mathrm{f}} \in V^{\mathrm{f}}
\end{cases}
$$

take
$$\boldsymbol{v}_1^{\mathrm{f}} = \widetilde{\boldsymbol{u}}^{n+1} - \mathcal{L}(\boldsymbol{v}^{\mathrm{s}}) - \mathcal{L}\left(\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{v}^{\mathrm{s}}\right) \in V^{\mathrm{f}}$$

2) Implicit sub-step:

$$
\begin{cases}
\displaystyle\int_{\Omega^{\mathrm{f}}} \rho^{\mathrm{f}} \frac{\boldsymbol{u}^{n+1} - \tilde{\boldsymbol{u}}^{n+1}}{\delta t} \cdot \boldsymbol{v}_2^{\mathrm{f}} - \int_{\Omega^{\mathrm{f}}} p^{n+1} \operatorname{div} \boldsymbol{v}_2^{\mathrm{f}} + \int_{\Omega^{\mathrm{f}}} q \operatorname{div} \boldsymbol{u}^{n+1} = 0, \\[2ex]
\boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \dfrac{\boldsymbol{d}^{n+1} - \boldsymbol{d}^n}{\delta t} \cdot \boldsymbol{n}, \quad \text{on} \quad \Sigma
\end{cases}
$$

take
$$\boldsymbol{v}_2^{\mathrm{f}} = \boldsymbol{u}^{n+1} - \mathcal{L}(\boldsymbol{v}^{\mathrm{s}}) \in H_\Sigma(\mathrm{div}; \Omega^{\mathrm{f}})$$

$$
\int_{\widehat{\Omega}^{\mathrm{s}}} \rho^{\mathrm{s}} \frac{\boldsymbol{d}^{n+1} - 2\boldsymbol{d}^n + \boldsymbol{d}^{n-1}}{(\delta t)^2} \cdot \boldsymbol{v}^{\mathrm{s}} + \int_{\widehat{\Omega}^{\mathrm{s}}} \boldsymbol{\epsilon}(\boldsymbol{d}^{n+1}) : \boldsymbol{\nabla} \boldsymbol{v}^{\mathrm{s}}
$$
$$
= -\mathcal{R}_\mu^{\mathrm{f}}\big(\tilde{\boldsymbol{u}}^{n+1}; \mathcal{L}(\boldsymbol{v}^{\mathrm{s}})\big) - \mathcal{R}_p^{\mathrm{f}}\big(p^{n+1}; \mathcal{L}(\boldsymbol{v}^{\mathrm{s}})\big), \quad \forall \boldsymbol{v}^{\mathrm{s}} \in V^{\mathrm{s}}
$$

take
$$\boldsymbol{v}^{\mathrm{s}} = \frac{\boldsymbol{d}^{n+1} - \boldsymbol{d}^n}{\delta t} \in V^{\mathrm{s}}$$

Idea of the proof

1) Under reasonable assumptions

$$\|\mathcal{L}_h \boldsymbol{v}_H^{\mathrm{s}}\|^2_{L^2(\Omega^{\mathrm{f}})} \leq \frac{Ch}{H^\alpha}\|\boldsymbol{v}_H^{\mathrm{s}}\|^2_{L^2(\Omega^{\mathrm{s}})}$$

$$\|\boldsymbol{\nabla}(\mathcal{L}_h \boldsymbol{v}_H^{\mathrm{s}})\|^2_{L^2(\Omega^{\mathrm{f}})} \leq \frac{C}{hH^\alpha}\|\boldsymbol{v}_H^{\mathrm{s}}\|^2_{L^2(\Omega^{\mathrm{s}})}$$

2) This allows to control the unbalanced terms in the fluid,
with the diffusion of the structural time scheme (hence the solid density…)

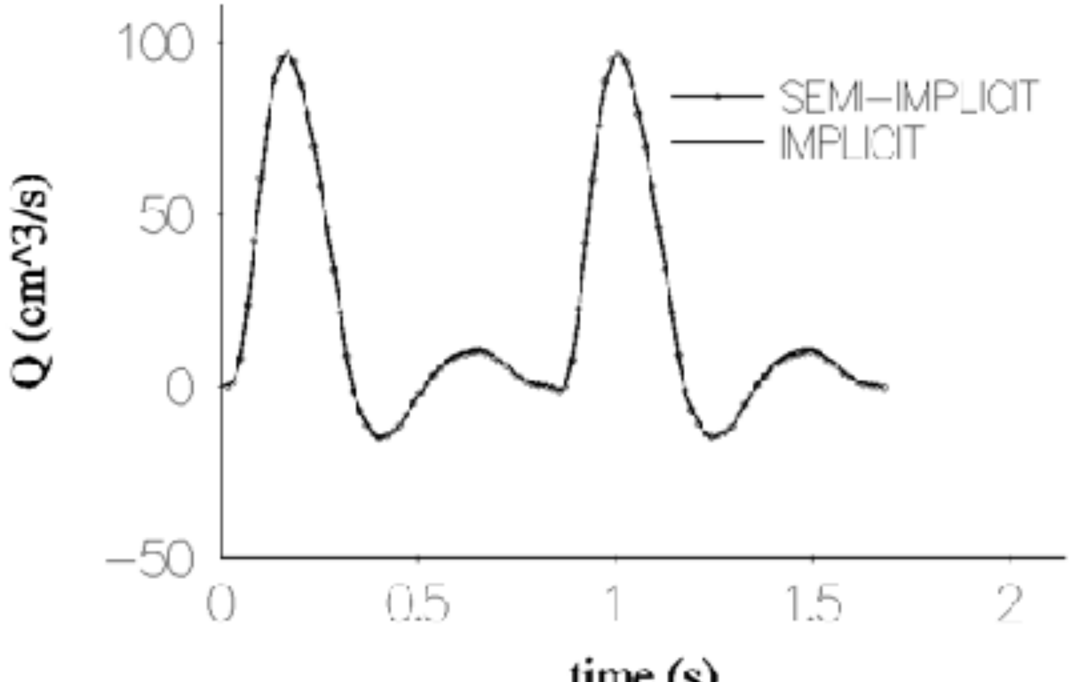
$$\left\|\mathcal{L}_h\left(\frac{\boldsymbol{d}^{n+1}-2\boldsymbol{d}^n+\boldsymbol{d}^{n-1}}{\delta t}\right)\right\|^2_{L^2(\Omega^{\mathrm{f}})} \leq \frac{Ch}{H^\alpha}\left\|\frac{\boldsymbol{d}^{n+1}-2\boldsymbol{d}^n+\boldsymbol{d}^{n-1}}{\delta t}\right\|^2_{L^2(\Omega^{\mathrm{s}})}$$

$$\left\|\boldsymbol{\epsilon}\left(\mathcal{L}_h\left(\frac{\boldsymbol{d}^{n+1}-2\boldsymbol{d}^n+\boldsymbol{d}^{n-1}}{\delta t}\right)\right)\right\|^2_{L^2(\Omega^{\mathrm{f}})} \leq \frac{C}{hH^\alpha}\left\|\frac{\boldsymbol{d}^{n+1}-2\boldsymbol{d}^n+\boldsymbol{d}^{n-1}}{\delta t}\right\|^2_{L^2(\Omega^{\mathrm{s}})}$$

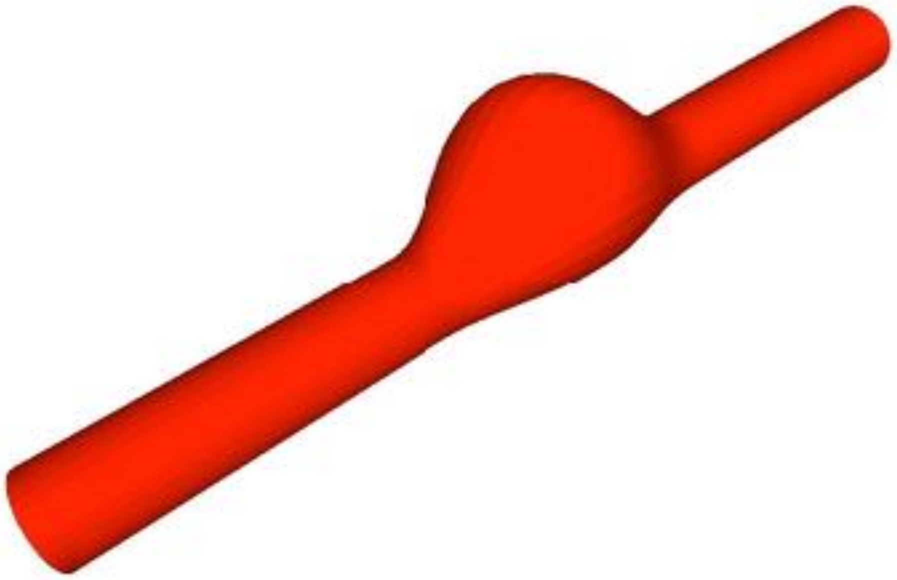
3) Remark: fortunately, numerical diffusion in the solid does not seem necessary in practical simulations in hemodynamics

Navier-Stokes / Nonlinear shell coupling

- Abdominal aortic aneurysm (in-vitro model): 2 cardiac cycles, 1000 times steps
  - $\delta t = 1.68 \times 10^{-3} s$
  - Fluid: 26950 Hexahedra ($\mathbb{Q}_1/\mathbb{Q}_1$ FE)
  - Solid: 2240 Quadrilaterals (MITC4 FE)
  - Parameters: $\mu = 0.035\, poise$, $\rho^f = 1\, g/cm^3$, $\rho^s = 1.2\, g/cm^3$, $E = 610^6\, dynes/cm^2$, $\nu = 0.3$



Out-flow

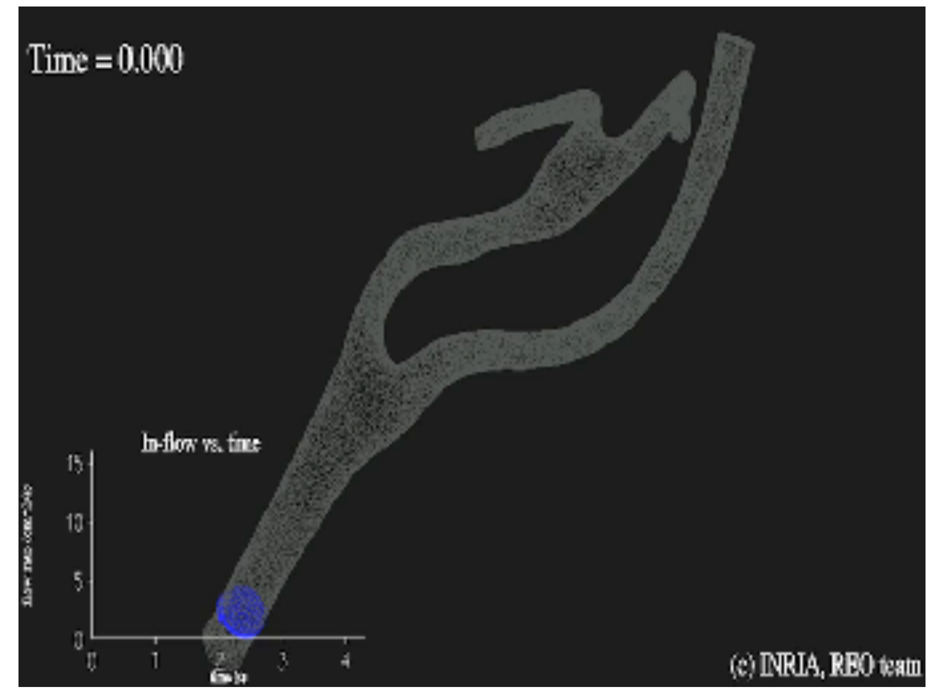


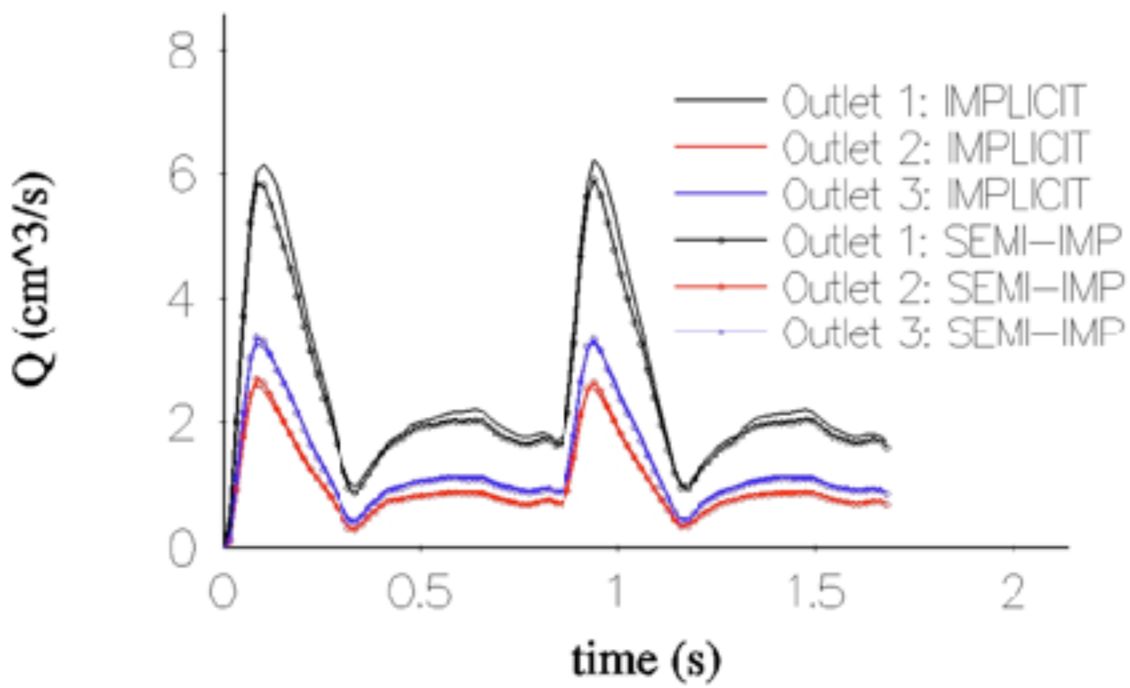
COUPLING	CPU time
Implicit	**9.3**
Semi-Implicit	**1.0**

Dimensionless CPU time

Carotid artery

- Carotid artery (in-vivo model): 9 cardiac cycles, 4500 times steps
  - $\delta t = 1.68 \times 10^{-3} s$
  - Fluid: 70047 Tetrahedra ($\mathbb{P}_1/\mathbb{P}_1$ FE)
  - Solid: 8103 Quadrilaterals (MITC4 FE)
  - Parameters: $\mu = 0.035\, poise$, $\rho^f = 1\, g/cm^3$, $\rho^s = 1.2\, g/cm^3$, $E = 6 \times 10^6\, dynes/cm^2$, $\nu = 0.3$.





Out-flow

COUPLING	CPU time
Implicit	**6.7**
Semi-Implicit	**1.0**

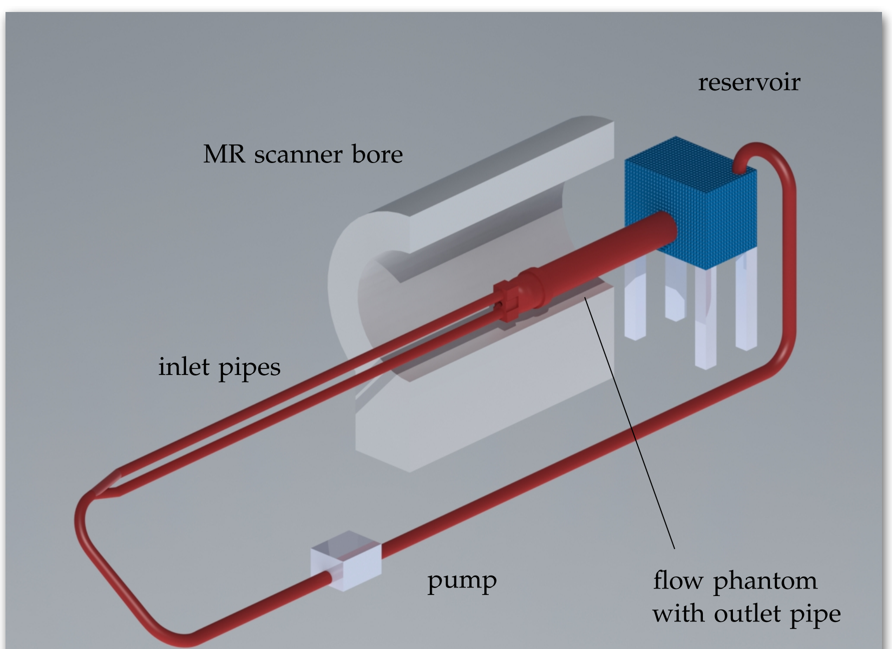
Dimensionless CPU time

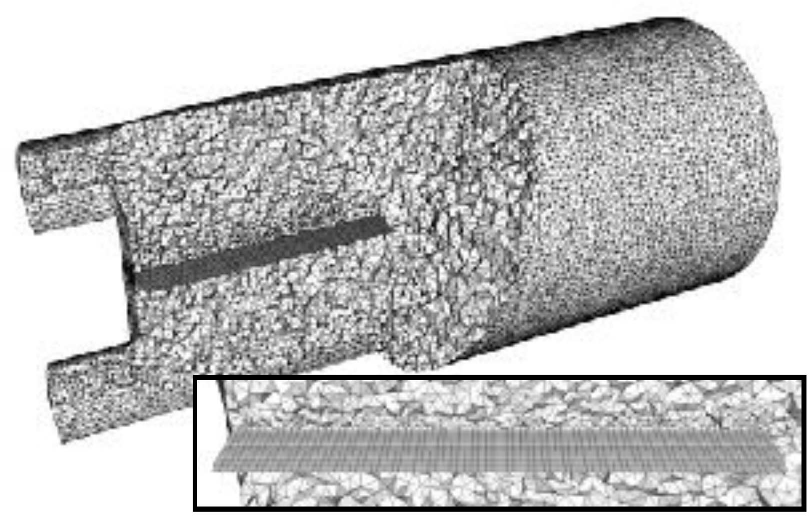
1997-2007 2007-2009 2010-2015

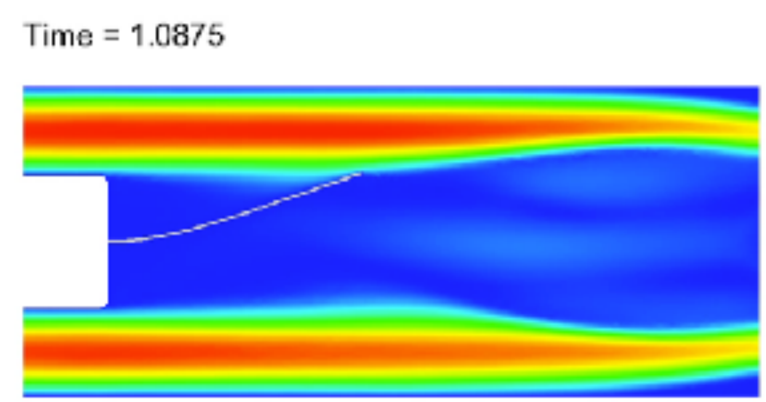
Implicit	Semi-implicit	Explicit monolithic	Explicit fractional step
18	2.5	2	1

Normalized CPU time (on the same computer, in 2015)

Benchmark with validation against experimental measurements:







Landajuela, Vidrascu, Chapelle, Fernández, *Int. J. Numerical Methods Biomed Engng. 2016*

Outline

- Generality: energy balance, load computation, coupling
- Added mass effect and incompressibility
- Semi-implicit coupling *via* projection schemes
- Explicit coupling *via* Nitsche and Robin formulations
- Explicit coupling for thin structures

E. Burman, M Fernández, *"Stabilization of explicit coupling in fluid-structure interaction involving fluid incompressibility"*, Comput Methods Appl. Mech. Engrg. (2009), 198, 766–784

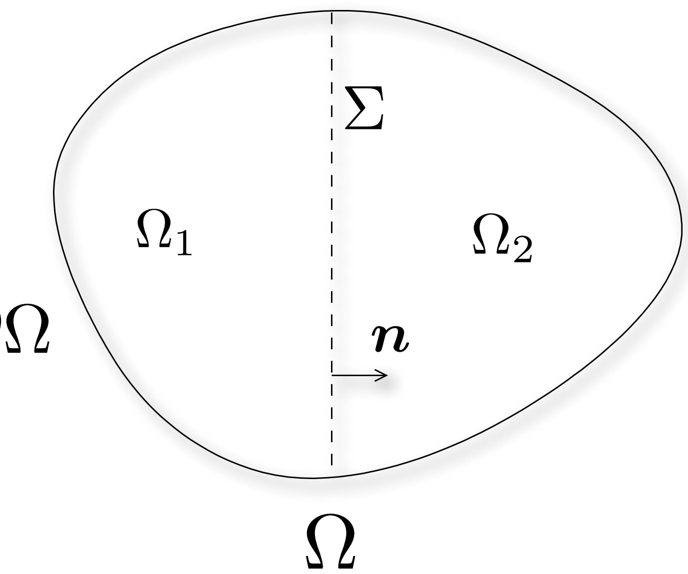
E. Burman, M Fernández, *"Explicit strategies for incompressible fluid-structure interaction problems: Nitsche type mortaring versus Robin-Robin coupling"*, Int. J. Numer. Meth. Engng (2014), 7, 739–758

D-N formulation for interface problems

- A model problem:

$$i = 1,2, \quad \left.\begin{aligned} -\Delta u_i &= f, && \text{in} \quad \Omega_i \\ u_i &= 0, && \text{on} \quad \partial\Omega_i \backslash \Sigma \end{aligned}\right\}$$

$$\left.\begin{aligned} u_1 &= u_2, && \text{on} \quad \Sigma \\ \frac{\partial u_1}{\partial \boldsymbol{n}} &= \frac{\partial u_2}{\partial \boldsymbol{n}}, && \text{on} \quad \Sigma \end{aligned}\right\} \iff \left\{\begin{aligned} -\Delta u &= f, && \text{in} \quad \Omega \\ u &= 0, && \text{on} \quad \partial\Omega \end{aligned}\right.$$



- The conforming Dirichlet-Neumann (D-N) formulation:

Find $u_1 \in H^1(\Omega_1)$ s.t. $u_{1|\Sigma} = u_{2|\Sigma}$ and

$$\int_{\Omega_1} \boldsymbol{\nabla} u_1 \cdot \boldsymbol{\nabla} v_1 = \int_{\Omega_1} f v_1, \quad \forall v_1 \in H^1_\Sigma(\Omega_1)$$

Find $u_2 \in H^1(\Omega_2)$ s.t.

$$\int_{\Omega_2} \boldsymbol{\nabla} u_2 \cdot \boldsymbol{\nabla} v_2 = \int_{\Omega_2} f v_2 - \left[\int_{\Omega_1} \boldsymbol{\nabla} u_1 \cdot \boldsymbol{\nabla} \mathcal{L} v_2 - \int_{\Omega_1} f \mathcal{L} v_2\right], \quad \forall v_2 \in H^1(\Omega_2)$$

- By adding these expressions we recover the global weak formulation:

$$\int_{\Omega_1} \boldsymbol{\nabla} u_1 \cdot \boldsymbol{\nabla}(v_1 + \mathcal{L} v_2) + \int_{\Omega_2} \boldsymbol{\nabla} u_2 \cdot \boldsymbol{\nabla} v_2 = \int_{\Omega_1} f(v_1 + \mathcal{L} v_2) + \int_{\Omega_2} f v_2$$

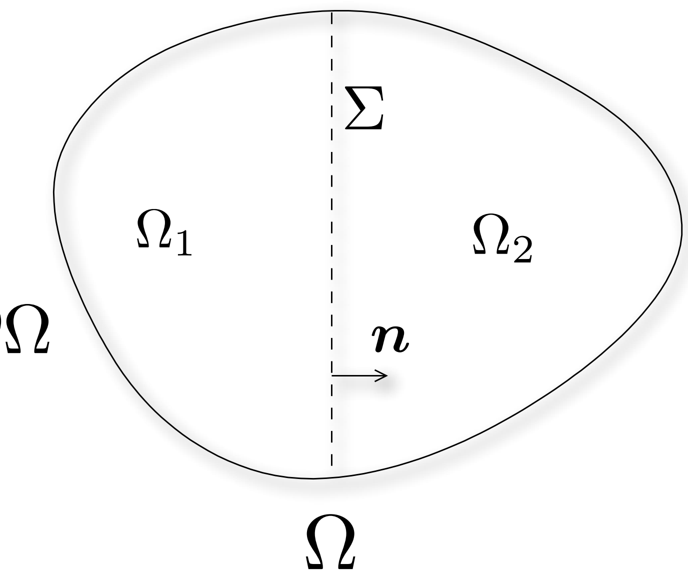
- Remark: $H^1(\Omega) = \left(H^1_\Sigma(\Omega_1), 0\right) \oplus \left(\mathcal{L} H^1(\Omega_2), H^1(\Omega_2)\right)$

Nitsche's formulation for interface problems

- A model problem:

$$i = 1,2, \quad \left.\begin{aligned} -\Delta u_i &= f, \quad \text{in} \quad \Omega_i \\ u_i &= 0, \quad \text{on} \quad \partial\Omega_i \backslash \Sigma \end{aligned}\right\}$$

$$\left.\begin{aligned} u_1 &= u_2, \quad \text{on} \quad \Sigma \\ \frac{\partial u_1}{\partial \boldsymbol{n}} &= \frac{\partial u_2}{\partial \boldsymbol{n}}, \quad \text{on} \quad \Sigma \end{aligned}\right\} \iff \left\{\begin{aligned} -\Delta u &= f, \quad \text{in} \quad \Omega \\ u &= 0, \quad \text{on} \quad \partial\Omega \end{aligned}\right.$$



- The Nitsche's formulation (*Nitsche 78, Becker et al. 03, Hansbo 05*):

$$\int_{\Omega_1} \nabla u_1 \cdot \nabla v_1 - \int_\Sigma \frac{\partial u_1}{\partial \boldsymbol{n}} v_1 + \frac{\gamma}{h} \int_\Sigma (u_1 - u_2) v_1 = \int_{\Omega_1} f v_1$$

$$\int_{\Omega_2} \nabla u_2 \cdot \nabla v_2 + \int_\Sigma \frac{\partial u_1}{\partial \boldsymbol{n}} v_2 + \frac{\gamma}{h} \int_\Sigma (u_2 - u_1) v_2 = \int_{\Omega_2} f v_2$$

by adding these expressions we get:

$$\sum_{i=1}^{2} \int_{\Omega_i} \nabla u_i \cdot \nabla v_i - \int_\Sigma \frac{\partial u_1}{\partial \boldsymbol{n}} (v_1 - v_2) - \int_\Sigma (u_1 - u_2) \frac{\partial v_1}{\partial \boldsymbol{n}} + \frac{\gamma}{h} \int_\Sigma (u_1 - u_2)(v_1 - v_2) = \sum_{i=1}^{2} \int_{\Omega_i} f v_i$$

Nitsche's method for interface problems

- Let $X_{1,h} \subset H^1(\Omega_1)$ and $X_{2,h} \subset H^1(\Omega_2)$ be given conforming finite element approximation spaces

- Consider the discrete problem: Find $(u_{1,h}, u_{2,h}) \in X_{1,h} \times X_{2,h}$ such that

$$\sum_{i=1}^{2} \int_{\Omega_i} \boldsymbol{\nabla} u_{i,h} \cdot \boldsymbol{\nabla} v_{i,h} - \int_{\Sigma} \frac{\partial u_{1,h}}{\partial \boldsymbol{n}} (v_{1,h} - v_{2,h}) - \int_{\Sigma} (u_{1,h} - u_{2,h}) \frac{\partial v_{1,h}}{\partial \boldsymbol{n}}$$

$$+ \frac{\gamma}{h} \int_{\Sigma} (u_{1,h} - u_{2,h})(v_{1,h} - v_{2,h}) = \sum_{i=1}^{2} \int_{\Omega_i} f v_{i,h} \quad \forall (v_{1,h}, v_{2,h}) \in X_{1,h} \times X_{2,h}$$

$$a\big((u_{1,h}, u_{2,h}), (v_{1,h}, v_{2,h})\big)$$

- The formulation is (strongly) consistent
- The parameter $\gamma > 0$ is chosen (irrespectively of h) to ensure coercivity
- Optimal and handles non-matching discretizations
- This is a Discontinuous Galerkin (DG) over the interface Σ

Coercivity

Proposition

For each $(u_{1,h}, u_{2,h}) \in X_{1,h} \times X_{2,h}$ we have

$$a\big((u_{1,h}, u_{2,h}), (u_{1,h}, u_{2,h})\big) \geq \left(1 - \frac{2C_{\mathrm{T}}}{\gamma}\right) \|\boldsymbol{\nabla} u_{1,h}\|^2_{0,\Omega_1} + \|\boldsymbol{\nabla} u_{2,h}\|^2_{0,\Omega_2} + \frac{\gamma}{2h}\|u_{1,h} - u_{2,h}\|^2_{0,\Sigma}$$

Therefore coercivity holds for $\gamma > 2C_{\mathrm{T}}$, where C_{T} is a constant independent of the mesh size h, but might depend on the polynomial approximation order.

Remark:

$\|u_{1,h} - u_{2,h}\|_{0,\Sigma} \to 0$ as $h \to 0$

Sketch of the proof

- Find $(u_{1,h}, u_{2,h}) \in X_{1,h} \times X_{2,h}$ such that

$$\sum_{i=1}^{2} \int_{\Omega_i} \boldsymbol{\nabla} u_{i,h} \cdot \boldsymbol{\nabla} v_{i,h} - \int_{\Sigma} \frac{\partial u_{1,h}}{\partial \boldsymbol{n}} (v_{1,h} - v_{2,h}) - \int_{\Sigma} (u_{1,h} - u_{2,h}) \frac{\partial v_{1,h}}{\partial \boldsymbol{n}}$$

$$+ \frac{\gamma}{h} \int_{\Sigma} (u_{1,h} - u_{2,h})(v_{1,h} - v_{2,h}) = \sum_{i=1}^{2} \int_{\Omega_i} f v_{i,h} \quad \forall (v_{1,h}, v_{2,h}) \in X_{1,h} \times X_{2,h}$$

$$a\big((u_{1,h}, u_{2,h}), (u_{1,h}, u_{2,h})\big) = \sum_{i=1}^{2} \|\boldsymbol{\nabla} u_{i,h}\|_{0,\Omega_i}^2 + \frac{\gamma}{h} \|u_{1,h} - u_{2,h}\|_{0,\Sigma}^2 - 2 \int_{\Sigma} \frac{\partial u_{1,h}}{\partial \boldsymbol{n}} (u_{1,h} - u_{2,h})$$

$ab \leq \frac{1}{2\epsilon} a^2 + \frac{\epsilon}{2} b^2, \quad \epsilon > 0$

$$\geq \sum_{i=1}^{2} \|\boldsymbol{\nabla} u_{i,h}\|_{0,\Omega_i}^2 + \frac{\gamma}{h} \|u_{1,h} - u_{2,h}\|_{0,\Sigma}^2 - \frac{1}{\epsilon} \left\| \frac{\partial u_{1,h}}{\partial \boldsymbol{n}} \right\|_{0,\Sigma}^2 - \epsilon \|u_{1,h} - u_{2,h}\|_{0,\Sigma}^2$$

discrete trace inequality

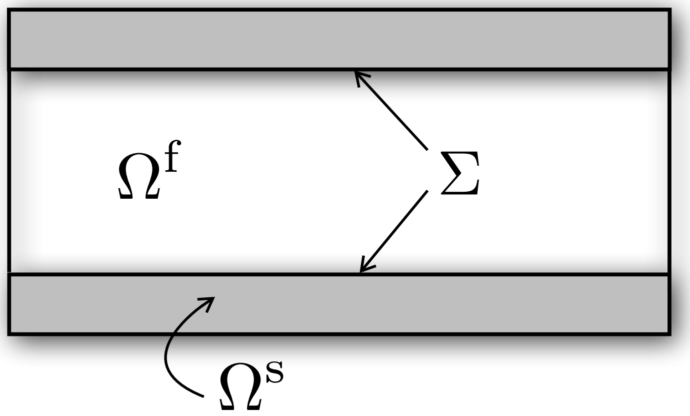
$$\geq \sum_{i=1}^{2} \|\boldsymbol{\nabla} u_{i,h}\|_{0,\Omega_i}^2 + \frac{\gamma}{h} \|u_{1,h} - u_{2,h}\|_{0,\Sigma}^2 - \frac{C_{\mathrm{T}}}{h\epsilon} \|\boldsymbol{\nabla} u_{1,h}\|_{0,\Omega_1}^2 - \epsilon \|u_{1,h} - u_{2,h}\|_{0,\Sigma}^2$$

take $\begin{cases} \epsilon = \dfrac{\gamma}{2h} \\ \gamma > 2C_{\mathrm{T}} \end{cases}$

A linear FSI model problem

- Fluid: transient Stokes

$$\begin{cases} \rho^{\mathrm{f}} \dfrac{\partial \boldsymbol{u}}{\partial t} - \mathbf{div} \boldsymbol{\sigma}(\boldsymbol{u}, p) = \mathbf{0}, & \text{in} \quad \Omega^{\mathrm{f}} \\ \mathrm{div} \boldsymbol{u} = 0, & \text{in} \quad \Omega^{\mathrm{f}} \end{cases}$$



- Structure: linear elasticity

$$\rho^{s} \ddot{\boldsymbol{d}} - \mathbf{div} \boldsymbol{\sigma}(\boldsymbol{d}) = \mathbf{0}, \quad \text{in} \quad \Omega^{\mathrm{s}}$$

- Coupling conditions:

$$\begin{cases} \boldsymbol{u} = \dot{\boldsymbol{d}}, & \text{on} \quad \Sigma \\ \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} = \boldsymbol{\sigma}(\boldsymbol{d})\boldsymbol{n}, & \text{on} \quad \Sigma \end{cases}$$

The FSI Nitsche's semi-discrete formulation

Find $(\boldsymbol{u}_h, p_h, \boldsymbol{d}_h) \in V_h \times Q_h \times X_h$ such that:

$$\rho^{\mathrm{f}} \int_{\Omega^{\mathrm{f}}} \frac{\partial \boldsymbol{u}_h}{\partial t} \cdot \boldsymbol{v}_h + \int_{\Omega^{\mathrm{f}}} \boldsymbol{\sigma}(\boldsymbol{u}_h, p_h) : \boldsymbol{\epsilon}(\boldsymbol{v}_h) + \int_{\Omega^{\mathrm{f}}} q_h \boldsymbol{\nabla} \cdot \boldsymbol{u}_h$$

$$+\rho^{\mathrm{s}} \int_{\Omega^{\mathrm{s}}} \ddot{\boldsymbol{d}}_h \cdot \boldsymbol{w}_h + a^{\mathrm{s}}(\boldsymbol{d}_h, \boldsymbol{w}_h)$$

consistency

$$- \int_{\Sigma} \boldsymbol{\sigma}(\boldsymbol{u}_h, p_h)\boldsymbol{n} \cdot (\boldsymbol{v}_h - \boldsymbol{w}_h) - \int_{\Sigma} (\boldsymbol{u}_h - \dot{\boldsymbol{d}}_h) \cdot \boldsymbol{\sigma}(\boldsymbol{v}_h, -q_h)\boldsymbol{n}$$

symmetry

$$+\gamma \frac{\mu}{h} \int_{\Sigma} (\boldsymbol{u}_h - \dot{\boldsymbol{d}}_h) \cdot (\boldsymbol{v}_h - \boldsymbol{w}_h) = 0$$

"penalty"

for all $(\boldsymbol{v}_h, q_h, \boldsymbol{w}_h) \in V_h \times Q_h \times X_h$

Partitioned Nitsche's formulation

- Solid subproblem: Find $\boldsymbol{d}_h \in X_h$

$$\rho^{\mathrm{s}} \int_{\Omega^{\mathrm{s}}} \ddot{\boldsymbol{d}}_h \cdot \boldsymbol{w}_h + a^{\mathrm{s}}(\boldsymbol{d}_h, \boldsymbol{w}_h) + \gamma \frac{\mu}{h} \int_{\Sigma} (\partial_t \boldsymbol{d}_h - \boldsymbol{u}_h) \cdot \boldsymbol{w}_h + \int_{\Sigma} \boldsymbol{\sigma}(\boldsymbol{u}_h, p_h)\boldsymbol{n} \cdot \boldsymbol{w}_h = 0$$

for all $\boldsymbol{w}_h \in X_h$

- Fluid subproblem: Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$

$$\begin{aligned}\rho^{\mathrm{f}} \int_{\Omega^{\mathrm{f}}} \frac{\partial \boldsymbol{u}_h}{\partial t} \cdot \boldsymbol{v}_h + \int_{\Omega^{\mathrm{f}}} \boldsymbol{\sigma}(\boldsymbol{u}_h, p_h) : \boldsymbol{\epsilon}(\boldsymbol{v}_h) + \int_{\Omega^{\mathrm{f}}} q_h \boldsymbol{\nabla} \cdot \boldsymbol{u}_h - \int_{\Sigma} \boldsymbol{\sigma}(\boldsymbol{u}_h, p_h)\boldsymbol{n} \cdot \boldsymbol{v}_h \\ - \int_{\Sigma} \left(\boldsymbol{u}_h - \dot{\boldsymbol{d}}_h\right) \cdot \boldsymbol{\sigma}(\boldsymbol{v}_h, -q_h)\boldsymbol{n} + \gamma \frac{\mu}{h} \int_{\Sigma} \left(\boldsymbol{u}_h - \dot{\boldsymbol{d}}_h\right) \cdot \boldsymbol{v}_h = 0\end{aligned}$$

for all $(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$

- Robin coupling in the solid
- Dirichlet-Nitsche coupling in the fluid

Time discretization: coupling procedures

- Solid:

$$\partial_{\delta t} X^{n+1} \stackrel{\text{def}}{=} \frac{X^{n+1} - X^n}{\delta t}$$

$$\rho^{\mathrm{s}} \int_{\Omega^{\mathrm{s}}} \frac{\boldsymbol{d}_h^{n+1} - 2\boldsymbol{d}_h^{n} + \boldsymbol{d}_h^{n-1}}{\delta t^2} \cdot \boldsymbol{w}_h + a^{\mathrm{s}}(\boldsymbol{d}_h^{n+1}, \boldsymbol{w}_h) + \gamma \frac{\mu}{h} \int_{\Sigma} (\partial_{\delta t} \boldsymbol{d}_h^{n+1} - \boldsymbol{u}_h^{*}) \cdot \boldsymbol{w}_h + \int_{\Sigma} \boldsymbol{\sigma}(\boldsymbol{u}_h^{*}, p_h^{*}) \boldsymbol{n} \cdot \boldsymbol{w}_h = 0$$

- Fluid:

$$\rho^{\mathrm{f}} \int_{\Omega^{\mathrm{f}}} \partial_{\delta t} \boldsymbol{u}_h^{n+1} \cdot \boldsymbol{v}_h + \int_{\Omega^{\mathrm{f}}} \boldsymbol{\sigma}(\boldsymbol{u}_h^{n+1}, p_h^{n+1}) : \boldsymbol{\epsilon}(\boldsymbol{v}_h) + \int_{\Omega^{\mathrm{f}}} q_h \boldsymbol{\nabla} \cdot \boldsymbol{u}_h^{n+1} - \int_{\Sigma} \boldsymbol{\sigma}(\boldsymbol{u}_h^{*}, p_h^{*}) \boldsymbol{n} \cdot \boldsymbol{v}_h - \int_{\Sigma} (\boldsymbol{u}_h^{n+1} - \partial_{\delta t} \boldsymbol{d}_h^{n+1}) \cdot \boldsymbol{\sigma}(\boldsymbol{v}_h, -q_h) \boldsymbol{n} + \gamma \frac{\mu}{h} \int_{\Sigma} (\boldsymbol{u}_h^{n+1} - \partial_{\delta t} \boldsymbol{d}_h^{n+1}) \cdot \boldsymbol{v}_h = 0$$

Coupling strategies:

- Implicit: $\boldsymbol{u}_h^{*} = \boldsymbol{u}_h^{n+1}, \ p_h^{*} = p_h^{n+1}$
- Explicit: $\boldsymbol{u}_h^{*} = \boldsymbol{u}_h^{n}, \ p_h^{*} = p_h^{n}$

Implicit Nitsche's coupling

Proposition

Define the discrete energy by:

$$E^n \stackrel{\mathrm{def}}{=} \frac{\rho^{\mathrm{f}}}{2}\|\boldsymbol{u}_h^n\|_{0,\Omega^{\mathrm{f}}}^2 + \frac{\rho^{\mathrm{s}}}{2}\|\partial_{\delta t}\boldsymbol{d}_h^n\|_{0,\Omega^{\mathrm{s}}}^2 + a^{\mathrm{s}}(\boldsymbol{d}_h^n, \boldsymbol{d}_h^n) + \delta t \mu \sum_{m=0}^{n-1} \|\boldsymbol{\epsilon}(\boldsymbol{u}_h^{m+1})\|_{0,\Omega^{\mathrm{f}}}^2 + \delta t \frac{\gamma\mu}{h} \sum_{m=0}^{n-1} \|\boldsymbol{u}_h^{m+1} - \partial_{\delta t}\boldsymbol{d}_h^{m+1}\|_{0,\Sigma}^2$$

Under the conditions $\gamma > 2C_{\mathrm{T}}$,

We have: $E^n \lesssim E^0$

Explicit Nitsche's coupling

Proposition

Under the conditions

$$\gamma > 256 C_{\mathrm{T}}, \quad \gamma \delta t = O(h),$$

we have

$$E^n \leq 6E^0 + 3C_\Sigma \mu \|\boldsymbol{u}_h^0\|_{0,\Sigma}^2 + \frac{3\mu}{4}\|\boldsymbol{\epsilon}(\boldsymbol{u}_h^0)\|_{0,\Omega_{\mathrm{f}}}^2$$

$$+ 24\frac{h}{\gamma\mu}\delta t \sum_{m=0}^{n-1} \|p_h^{m+1} - p_h^m\|_{0,\Sigma}^2.$$

- **Idea to stabilize the scheme:** add the temporal weakly consistent penalty term

$$\frac{\gamma_0 h}{\gamma \mu}\int_\Sigma \left(p_h^{n+1} - p_h^n\right) q_h$$

E. Burman, M Fernández, *"Stabilization of explicit coupling in fluid-structure interaction involving fluid incompressibility"*, Comput Methods Appl. Mech. Engrg. (2009), 198, 766–784

The stabilized explicit coupling scheme

1) Solid substep:

$$\rho^{\mathrm{s}} \int_{\Omega^{\mathrm{s}}} \frac{\boldsymbol{d}_h^{n+1} - 2\boldsymbol{d}_h^{n} + \boldsymbol{d}_h^{n-1}}{\delta t^2} \cdot \boldsymbol{w}_h + a^{\mathrm{s}}(\boldsymbol{d}_h^{n+1}, \boldsymbol{w}_h) + \gamma \frac{\mu}{h} \int_{\Sigma} \left(\partial_{\delta t} \boldsymbol{d}_h^{n+1} - \boldsymbol{u}_h^{n}\right) \cdot \boldsymbol{w}_h + \int_{\Sigma} \boldsymbol{\sigma}(\boldsymbol{u}_h^{n}, p_h^{n})\boldsymbol{n} \cdot \boldsymbol{w}_h = 0$$

2) Fluid substep:

$$\rho^{\mathrm{f}} \int_{\Omega^{\mathrm{f}}} \partial_{\delta t} \boldsymbol{u}_h^{n+1} \cdot \boldsymbol{v}_h + \int_{\Omega^{\mathrm{f}}} \boldsymbol{\sigma}(\boldsymbol{u}_h^{n+1}, p_h^{n+1}) : \boldsymbol{\epsilon}(\boldsymbol{v}_h) + \int_{\Omega^{\mathrm{f}}} q_h \boldsymbol{\nabla} \cdot \boldsymbol{u}_h^{n+1} - \int_{\Sigma} \boldsymbol{\sigma}(\boldsymbol{u}_h^{n}, p_h^{n})\boldsymbol{n} \cdot \boldsymbol{v}_h - \int_{\Sigma} \left(\boldsymbol{u}_h^{n+1} - \partial_{\delta t} \boldsymbol{d}_h^{n+1}\right) \cdot \boldsymbol{\sigma}(\boldsymbol{v}_h, -q_h)\boldsymbol{n} + \gamma \frac{\mu}{h} \int_{\Sigma} \left(\boldsymbol{u}_h^{n+1} - \partial_{\delta t} \boldsymbol{d}_h^{n+1}\right) \cdot \boldsymbol{v}_h + \frac{\gamma_0 h}{\gamma \mu} \int_{\Sigma} \left(p_h^{n+1} - p_h^{n}\right) q_h = 0$$

3) Next time step

E. Burman, M Fernández, *"Stabilization of explicit coupling in fluid-structure interaction involving fluid incompressibility"*, Comput Methods Appl. Mech. Engrg. (2009), 198, 766–784

Robin-Robin and quasi-compressibility

- Explicit Robin-Robin coupling:

$$\begin{cases} \boldsymbol{\sigma}(\boldsymbol{d}^{n+1})\boldsymbol{n}^{\mathrm{s}} + \dfrac{\gamma\mu}{h}\dot{\boldsymbol{d}}^{n+1} = \dfrac{\gamma\mu}{h}\boldsymbol{u}^{n} - \boldsymbol{\sigma}(\boldsymbol{u}^{n}, p^{n})\boldsymbol{n} & \text{on} \quad \Sigma \\ \boldsymbol{\sigma}(\boldsymbol{u}^{n+1}, p^{n+1})\boldsymbol{n} + \dfrac{\gamma\mu}{h}\boldsymbol{u}^{n+1} = \dfrac{\gamma\mu}{h}\dot{\boldsymbol{d}}^{n+1} + \boldsymbol{\sigma}(\boldsymbol{u}^{n}, p^{n})\boldsymbol{n} & \text{on} \quad \Sigma \end{cases}$$

- The discrete incompressibility condition is given by

$$\frac{\gamma_0 h \delta t}{\gamma\mu}\int_{\Sigma} \partial_{\delta t} p_h^{n+1} q_h + \int_{\Omega_{\mathrm{f}}} q_h \boldsymbol{\nabla}\cdot\boldsymbol{u}_h^{n+1} = \int_{\Sigma}(\boldsymbol{u}_h^{n+1} - \partial_{\delta t}\boldsymbol{\eta}_h^{n+1})\cdot\boldsymbol{n} q_h, \quad \forall q_h \in Q_h$$

- Remark: the idea of pseudo-compressibility to accelerate the convergence of the fixed-point iterations (implicit coupling) was present in

P Raback, J Ruokolainen, M Lyly, E Järvinen, *ECCOMAS CFD,* 2001

J.A. Vierendeels, K. Riemslagh, E. Dick, P.R. Verdonck, *J. Biomech. Eng. Trans. ASME* 122 (6) (2000) 667–674.

An energy based stability result

Proposition

Under the conditions

$$\gamma > 256 C_{\mathrm{T}}, \quad \gamma \delta t = O(h), \quad \gamma_0 > 8,$$

there holds

$$E^n \leq 6E^0 + 3C_\Sigma \mu \|\boldsymbol{u}_h^0\|_{0,\Sigma}^2 + \frac{3\mu}{4} \|\boldsymbol{\epsilon}(\boldsymbol{u}_h^0)\|_{0,\Omega_{\mathrm{f}}}^2 + 3\frac{\gamma_0 h}{\gamma \mu} \delta t \|p_h^0\|_{0,\Sigma}^2.$$

Thus, the stabilized explicit coupling scheme is (conditionally) stable in the energy norm.

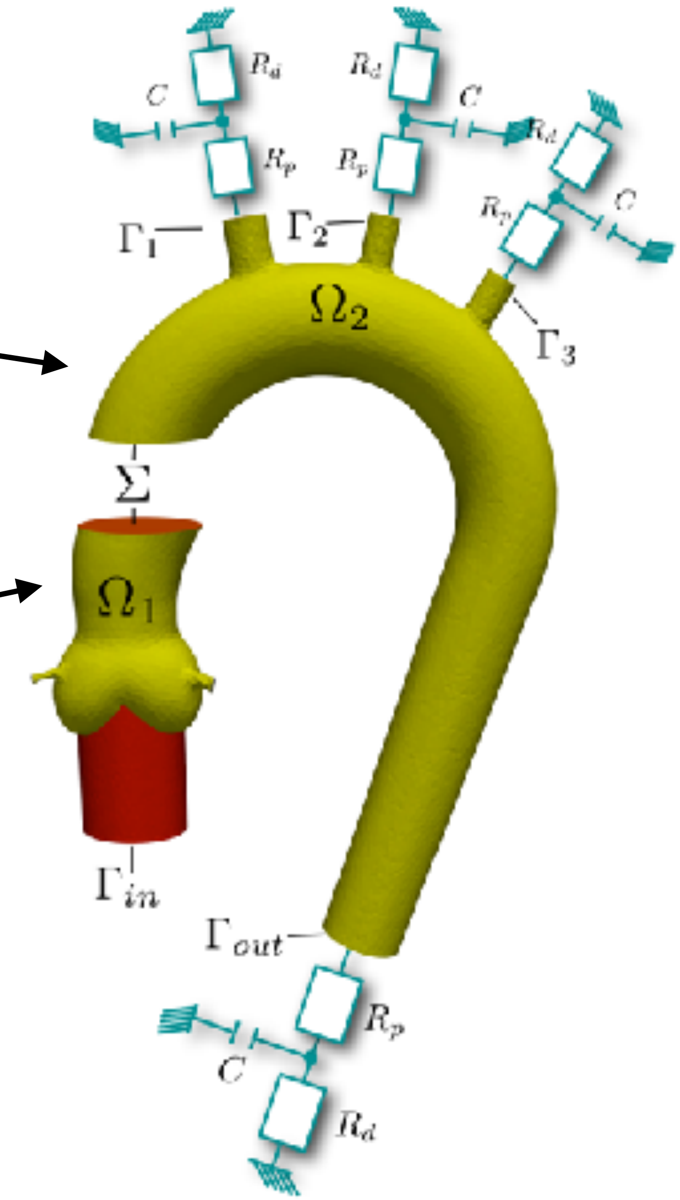
E. Burman, M Fernández, *"Stabilization of explicit coupling in fluid-structure interaction involving fluid incompressibility"*, Comput Methods Appl. Mech. Engrg. (2009), 198, 766–784

Side remark: fluid-fluid coupling

- The same idea applies to fluid-fluid explicit coupling

$$\left\{\begin{aligned} \rho\partial_\tau \mathbf{u}_1 + \rho(\mathbf{u}_1\cdot\nabla)\mathbf{u}_1 - \nabla\cdot\boldsymbol{\sigma}(\mathbf{u}_1,p_1) &= 0 \\ \nabla\cdot\mathbf{u}_1 &= 0 \end{aligned}\right. \quad \text{in } \Omega_1$$

$$\left\{\begin{aligned} \rho\partial_\tau \mathbf{u}_2 + \rho(\mathbf{u}_2\cdot\nabla)\mathbf{u}_2 - \nabla\cdot\boldsymbol{\sigma}(\mathbf{u}_2,p_2) &= 0 \\ \nabla\cdot\mathbf{u}_2 &= 0 \end{aligned}\right. \quad \text{in } \Omega_2$$



Transmission conditions on Σ

$$\left\{\begin{array}{lcl} \boldsymbol{\sigma}(\mathbf{u}_1^n,p_1^n)\cdot\mathbf{n}_1 + \alpha\mathbf{u}_1^n & = & \alpha\mathbf{u}_2^{n-1} - \boldsymbol{\sigma}(\mathbf{u}_2^{n-1},p_2^{n-1})\cdot\mathbf{n}_2 \\ \boldsymbol{\sigma}(\mathbf{u}_2^n,p_2^n)\cdot\mathbf{n}_2 + \alpha\mathbf{u}_2^n & = & \alpha\mathbf{u}_1^n \quad + \boldsymbol{\sigma}(\mathbf{u}_2^{n-1},p_2^{n-1})\cdot\mathbf{n}_2 \end{array}\right.$$

Control of pressure on Σ

$$S(p_2^n,p_2^{n-1}) = \beta\int_\Sigma (p_2^n - p_2^{n-1})q_2$$

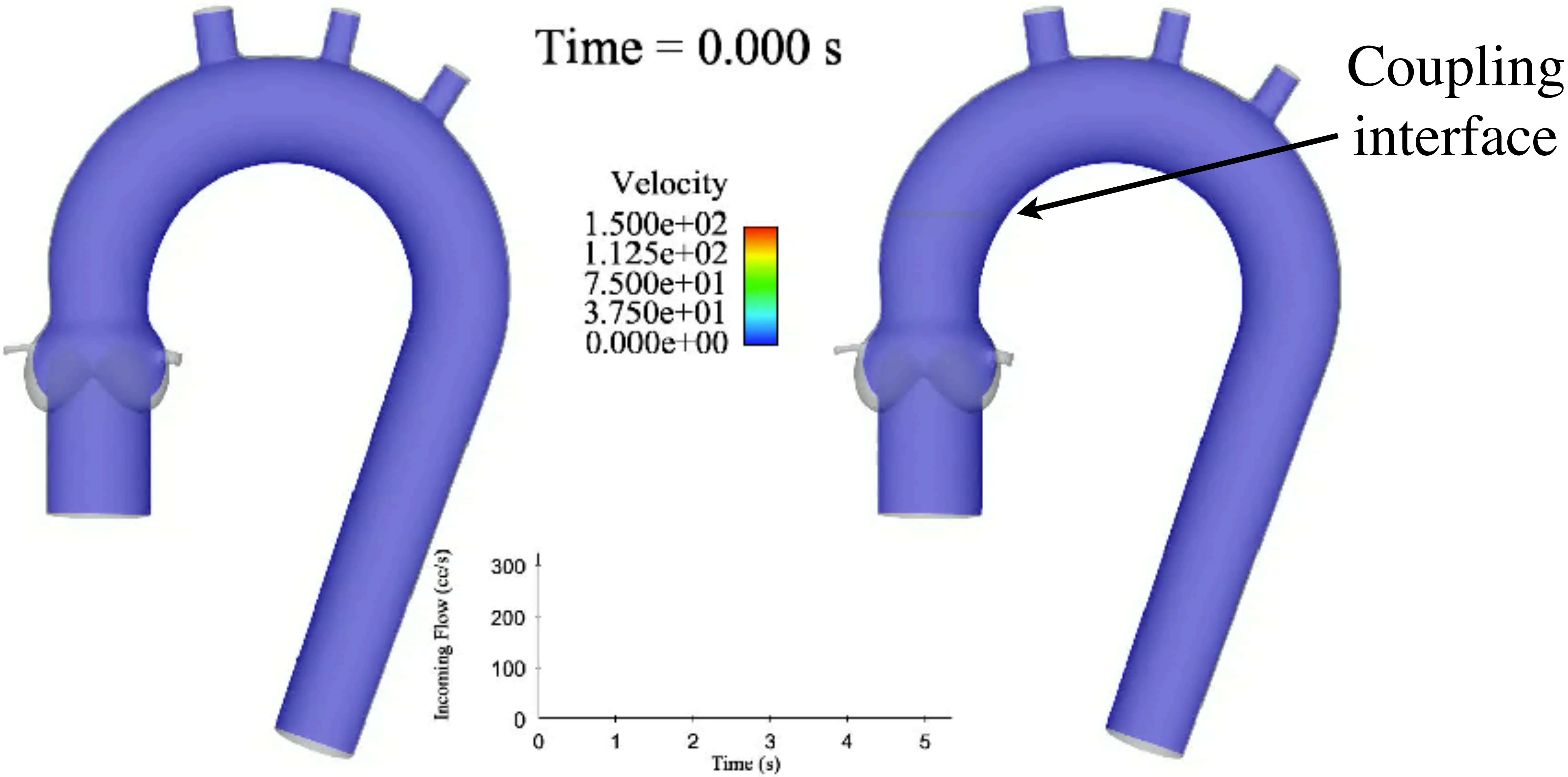
Control of kinetic energy flux

$$\begin{aligned} \rho\mathbf{u}\cdot\nabla\mathbf{u} + \boldsymbol{\nabla}p \quad &= \rho\mathbf{u}\cdot\nabla\mathbf{u} - \frac{\rho}{2}\boldsymbol{\nabla}\mathbf{u}^2 + \frac{\rho}{2}\boldsymbol{\nabla}\mathbf{u}^2 + \boldsymbol{\nabla}p \\ &= \rho\mathbf{u}\cdot\nabla\mathbf{u} - \rho\mathbf{u}\cdot(\nabla\mathbf{u})^T + \boldsymbol{\nabla}\pi \end{aligned}$$

Side remark: fluid-fluid coupling

Reference solution (monolithic)

“Total pressure” (staggered scheme)



M Fernández, JF Gerbeau, S Smaldone “Explicit coupling schemes for a fluid-fluid interaction problem arising in hemodynamics”, *SIAM Journal on Scientific Computing* (2014), 36(6), 2557-2583

Outline

- Generality: energy balance, load computation, coupling
- Added mass effect and incompressibility
- Semi-implicit coupling *via* projection schemes
- Explicit coupling *via* Nitsche and Robin formulations
- Explicit coupling for thin structures

M Fernández. "Incremental displacement-correction schemes for incompressible fluid-structure interaction: stability and convergence analysis", *Numerische Math.* (2012), 123(1), 21-65.

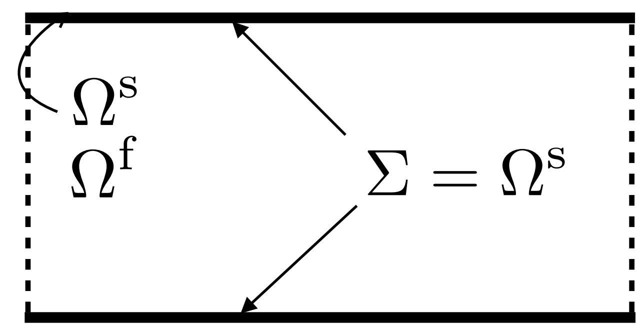
G Guidoboni, R Glowinski, N Cavallini, S Canic. "Stable loosely-coupled-type algorithm for fluid–structure interaction in blood flow" *Journal of Computational Physics* (2009), 228(18), 6916-6937.

FSI models with thin structure

- Fluid: transient Stokes

$$\left\{\begin{aligned} \rho^{\mathrm{f}} \frac{\partial \boldsymbol{u}}{\partial t} - \mathbf{div}\boldsymbol{\sigma}(\boldsymbol{u}, p) &= \mathbf{0}, \quad \text{in} \quad \Omega^{\mathrm{f}} \\ \mathrm{div}\boldsymbol{u} &= 0, \quad \text{in} \quad \Omega^{\mathrm{f}} \end{aligned}\right.$$

- Thin structure (membrane, shell,...)



$$\rho^{\mathrm{s}} \epsilon \partial_t \dot{\boldsymbol{d}} + \boldsymbol{L}^{\mathrm{e}} \boldsymbol{d} = -\boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} \quad \text{on} \quad \Sigma$$

- Coupling conditions:

$$\boldsymbol{u} = \dot{\boldsymbol{d}}, \qquad \text{on} \quad \Sigma$$

FSI models with thin structure

- Idea: only solid inertia needs to be implicitly coupled to the fluid

- Fluid

$$\left\{\begin{aligned} \rho^{\mathrm{f}}\partial_t \boldsymbol{u} - \mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u},p) &= \mathbf{0} \quad \text{in} \quad \Omega^{\mathrm{f}} \\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in} \quad \Omega^{\mathrm{f}} \\ \boldsymbol{u} &= \dot{\boldsymbol{d}} \quad \text{on} \quad \Sigma \end{aligned}\right.$$

$$\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n} + \rho^{\mathrm{s}}\epsilon\partial_t\boldsymbol{u} = -\boldsymbol{L}^{\mathrm{e}}\boldsymbol{d} \quad \text{on} \quad \Sigma$$

- Thin solid

$$\left\{\begin{aligned} \rho^{\mathrm{s}}\epsilon\partial_t\dot{\boldsymbol{d}} + \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d} &= -\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n} \quad \text{on} \quad \Sigma \\ \dot{\boldsymbol{d}} &= \partial_t\boldsymbol{d} \quad \text{on} \quad \Sigma \end{aligned}\right.$$

$$\boxed{\boldsymbol{\sigma}(\boldsymbol{u}^n,p^n)\boldsymbol{n} + \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\boldsymbol{u}^n = \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\dot{\boldsymbol{d}}^{n-1} - \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}^\star} \quad \text{on} \quad \Sigma, \quad \boldsymbol{d}^\star = \left|\begin{array}{l} 0 \\ \boldsymbol{d}^{n-1} \\ \boldsymbol{d}^{n-1} + \tau\dot{\boldsymbol{d}}^{n-1} \end{array}\right.$$

- Added-mass free *and* parameter free
- **Key issue is now the accuracy !**

Robin-"Neumann" explicit coupling schemes

1) Solve the fluid:

$$
\left\{
\begin{aligned}
\rho^{\mathrm{f}} \partial_\tau \boldsymbol{u}^n - \mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{u}^n, p^n) &= \mathbf{0} \quad \text{in} \quad \Omega^{\mathrm{f}} \\
\operatorname{div} \boldsymbol{u}^n &= 0 \quad \text{in} \quad \Omega^{\mathrm{f}} \\
\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} + \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\boldsymbol{u}^n &= \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\dot{\boldsymbol{d}}^{n-1} - \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}^\star \quad \text{on} \quad \Sigma\,,
\end{aligned}
\right.
\qquad
\boldsymbol{d}^\star = \left|
\begin{array}{l}
0 \\
\boldsymbol{d}^{n-1} \\
\boldsymbol{d}^{n-1} + \tau \dot{\boldsymbol{d}}^{n-1}
\end{array}
\right.
$$

2) Solve the solid:

$$
\left\{
\begin{aligned}
\rho^{\mathrm{s}}\epsilon \partial_\tau \dot{\boldsymbol{d}}^n + \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}^n &= -\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} \quad \text{on} \quad \Sigma \\
\dot{\boldsymbol{d}}^n &= \partial_\tau \boldsymbol{d}^n \quad \text{on} \quad \Sigma
\end{aligned}
\right.
$$

Same method, alternative formulations

- Explicit Robin-Neumann coupling:

$$\begin{cases} \boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} + \dfrac{\rho^{\mathrm{s}}\epsilon}{\tau}\boldsymbol{u}^n = \dfrac{\rho^{\mathrm{s}}\epsilon}{\tau}\dot{\boldsymbol{d}}^{n-1} - \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}^{\star} \quad \text{on} \quad \Sigma \\ \rho^{\mathrm{s}}\epsilon\partial_\tau\dot{\boldsymbol{d}}^n + \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}^n = -\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} \quad \text{on} \quad \Sigma \end{cases}$$

- Robin based kinematic relaxation *(form convenient to the implementation)*:

$$\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} + \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\boldsymbol{u}^n = \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\left(\dot{\boldsymbol{d}}^{n-1} + \tau\partial_\tau\dot{\boldsymbol{d}}^{\star}\right) + \boldsymbol{\sigma}(\boldsymbol{u}^{\star}, p^{\star})\boldsymbol{n} \quad \text{on} \quad \Sigma$$

- *Incremental* displacement-correction *(form convenient to the analysis)*:

$$\frac{\rho^{\mathrm{s}}\epsilon}{\tau}\left(\dot{\boldsymbol{d}}^n - \boldsymbol{u}^n\right) + \boldsymbol{L}^{\mathrm{e}}\left(\boldsymbol{d}^n - \boldsymbol{d}^{\star}\right) = 0 \quad \text{on} \quad \Sigma$$

Remark: $\boldsymbol{d}^{\star} = 0$ corresponds to the stable loosely-coupled scheme by

G Guidoboni, R Glowinski, N Cavallini, S Canic. "Stable loosely-coupled-type algorithm for fluid–structure interaction in blood flow" *Journal of Computational Physics* (2009), 228(18), 6916-6937.

Stability and accuracy: main principle

Proposition

- Energy-norm error:

$$e^n \stackrel{\text{def}}{=} \sqrt{\frac{\rho^{\mathrm{f}}}{2}\|\boldsymbol{u}^n - \boldsymbol{u}(t_n)\|^2_{0,\Omega^{\mathrm{f}}} + \frac{\rho^{\mathrm{s}}\epsilon}{2}\|\dot{\boldsymbol{d}}^n - \dot{\boldsymbol{d}}(t_n)\|^2_{0,\Sigma} + \frac{1}{2}\|\boldsymbol{d}^n - \boldsymbol{d}(t_n)\|^2_{\mathrm{e}}}$$

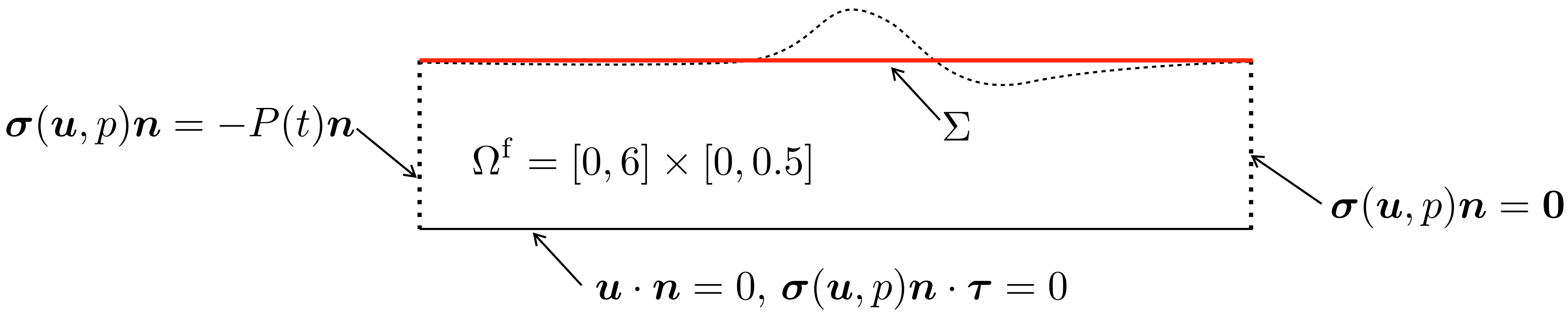
- For smooth enough solutions and $n \geq 1$, there holds:

$$e^n \lesssim h^k + \tau + \frac{\beta_{\mathrm{e}}}{\sqrt{\rho^{\mathrm{s}}\epsilon}} \cdot \begin{cases} \tau^{\frac{1}{2}} & \text{if} \quad \boldsymbol{d}^\star = \boldsymbol{0} \\ \tau & \text{if} \quad \boldsymbol{d}^\star = \boldsymbol{d}^{n-1} \\ \tau^2 & \text{if} \quad \boldsymbol{d}^\star = \boldsymbol{d}^{n-1} + \tau\dot{\boldsymbol{d}}^{n-1} \end{cases}$$

with $k \geq 1$ the convergence order of the Stokes-projection.

M Fernández. "Incremental displacement-correction schemes for incompressible fluid-structure interaction: stability and convergence analysis", *Numer. Math.*, 2012, 123(1), 21-65.

Pressure-wave propagation benchmark



$$P(t) = \begin{cases} \frac{p_{\max}}{2}\big(1 - \cos(2\pi t/0.005)\big) & t \in [0, 0.005] \\ 0 & t > 0.005 \end{cases}$$

- Thin-solid: generalized string

$$L^{\mathrm{e}} d_y \stackrel{\mathrm{def}}{=} -\frac{E\epsilon}{2(1+\nu)}\partial_{xx} d_y + \frac{E\epsilon}{0.25(1-\nu^2)} d_y$$

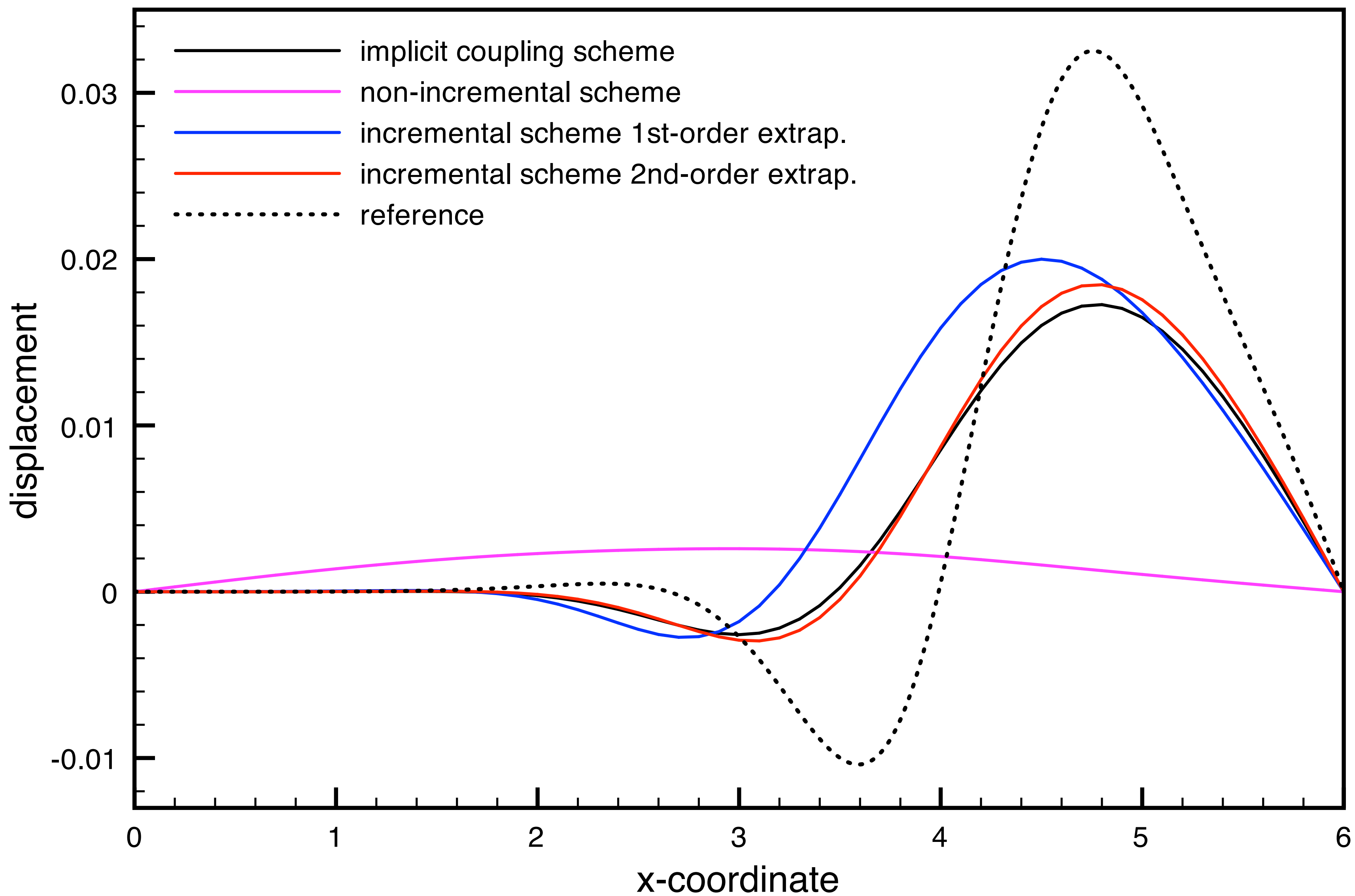
- Physical data:

$$\begin{aligned} \epsilon &= 0.1\,\mathrm{cm} & \rho^{\mathrm{f}} &= 1.0\,\mathrm{g/cm^3} \\ \rho^{\mathrm{s}} &= 1.1\,\mathrm{g/cm^3} & \mu &= 0.035\,\mathrm{P} \\ E &= 0.75 \times 10^6\,\mathrm{dynes/cm^2} & & \\ \nu &= 0.5 & & \end{aligned}$$

- Space discretization: $\mathbb{P}_1$ for the solid, $\mathbb{P}_1/\mathbb{P}_1$ stabilized for the fluid

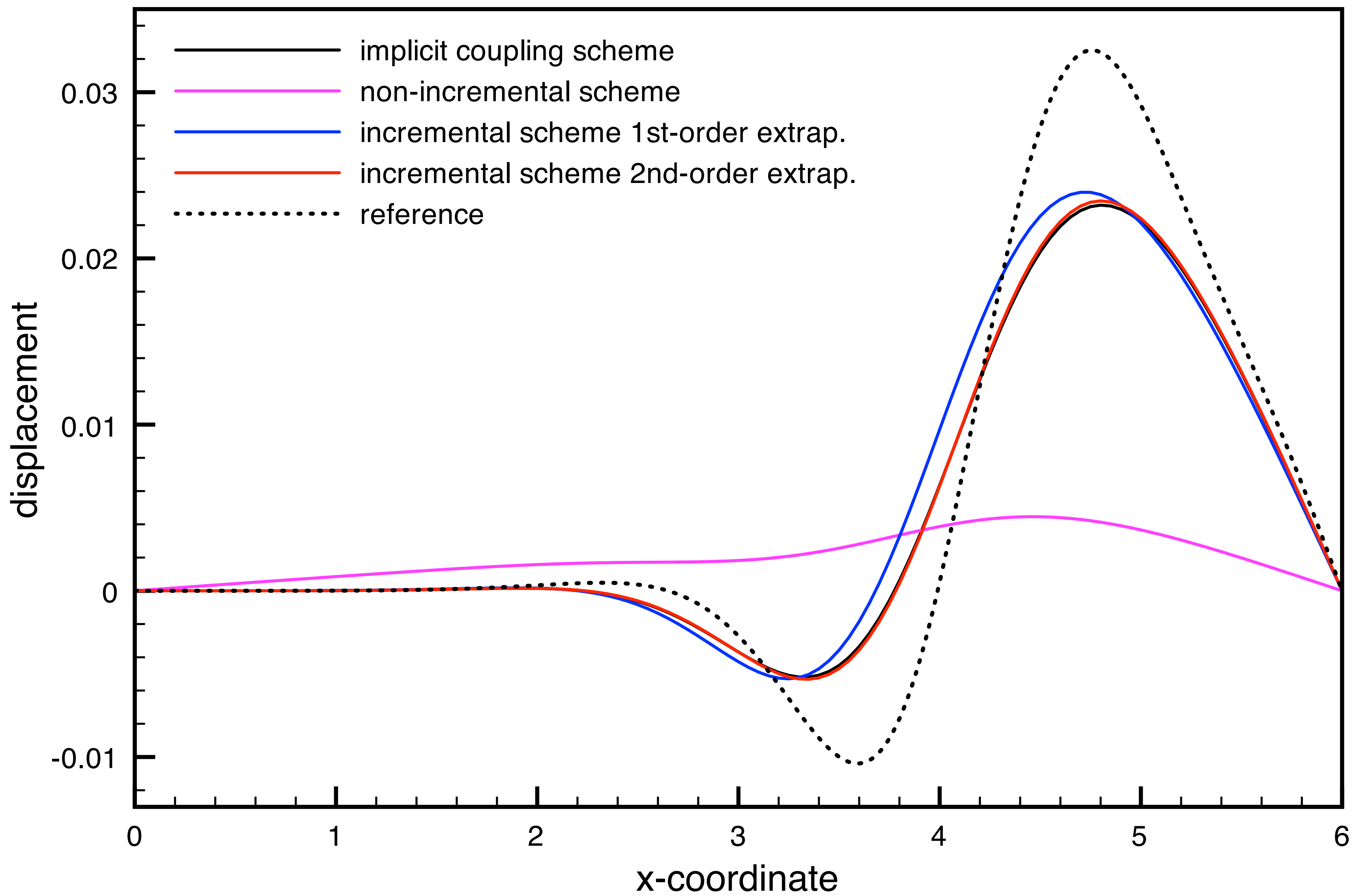
Accuracy vs. time and space refinement

$\tau = 2 \times 10^{-4}$, $h = 0.1$, $t_n = 0.015$



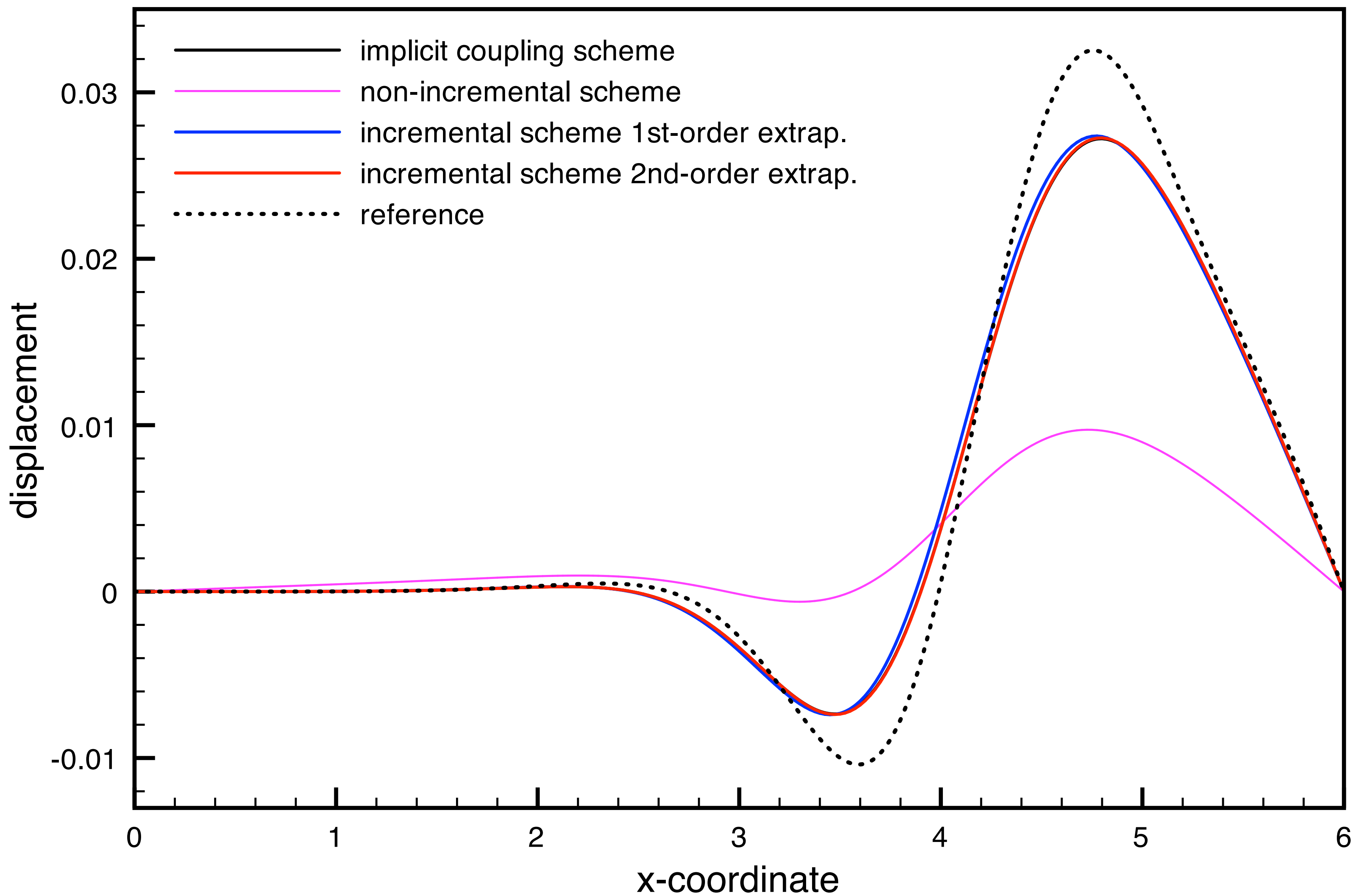
Accuracy vs. time and space refinement

$\tau = 10^{-4}$, $h = 0.05$, $t_n = 0.015$



Accuracy vs. time and space refinement

$\tau = 0.5 \times 10^{-4}$, $h = 0.025$, $t_n = 0.015$



Concluding remarks

- A wide variety of partitioned coupling methods for incompressible FSI
  - Implicit, semi-implicit, explicit coupling schemes
- Fractional time stepping
  - In the fluid: semi-implicit projection scheme
  - In the solid: explicit schemes for thin structures
- Added-mass stability issues
  - Can be circumvented via, e.g., Robin-Robin or Robin-Neumann coupling schemes
- The fundamental issue used to be stability with Dirichlet-Neumann. With Nitsche, Robin, etc. it is now accuracy!
  - Be careful with the parameters and the constants…
  - Thin-walled solids: first-order accurate Robin-Neumann schemes
  - Open problems: optimal convergence for the coupling with non-thin structure