Cluster structures on laminated surfaces

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Overview

- Triangulated orientable surfaces and their link to cluster algebras.

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- Generalisations of the construction.

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- Generalisations of the construction.
- Work in progress and future directions.

Triangulated Surfaces

- A **bordered surface** is a pair (S, M) where:
  - S is a compact orientable surface,
  - $M \subseteq S$ is a finite set of **marked points** such that every ∂-component of S has at least one marked point.

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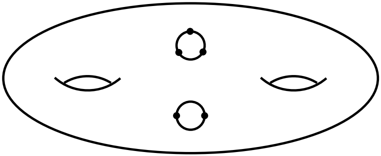
Definition

An **arc** is a curve γ in S whose endpoints are marked points and $\text{int}(\gamma) \subseteq \text{int}(S)$.



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- **Flips:**

Proposition (Harer('85) + Fomin, Shapiro, Thurston ('08))

Let T be a triangulation of an unpunctured (S, M) and let $\gamma \in T$. Then there exists a unique $\gamma' \neq \gamma$, such that $\mu_\gamma(T) := T \setminus \{\gamma\} \cup \{\gamma'\}$ is a triangulation.

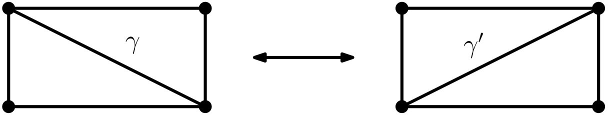

Figure: Local configuration of a flip.

Flip graph

Definition

The flip graph has vertices corresponding to triangulations and edges corresponding to flips.

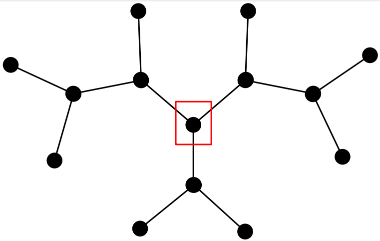

Figure: The shape of the flip graph.

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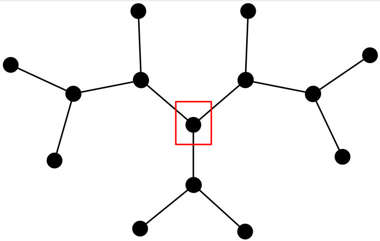

Figure: The shape of the flip graph.

Proposition (Harer)

Any two triangulations are related by a sequence of flips.

Cluster algebras (Fomin and Zelevinsky)

- Define an **initial seed** $\Sigma := (\mathbf{x}, Q)$ where:
  - $\mathbf{x} = \{x_1, ..., x_n\}$ algebraically independent variables.
  - Q is a quiver (an oriented multi-graph on n vertices without loops or 2-cycles).

Figure: non-example and example of quivers.

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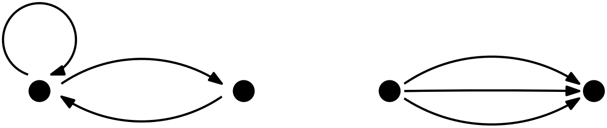
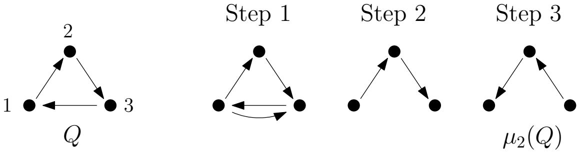


Figure: non-example and example of quivers.

- $\tilde{x} = \{x_1, ..., x_n, x_{n+1}, ..., x_m\}$ is the **extended cluster** if we add extra **frozen vertices** to the quiver Q.

Mutations

- (Quiver mutation) Let $k \in [1, n]$. We get a new quiver $\mu_k(Q)$ by:
  - For each path $i \to k \to j$ in Q add an arrow $i \to j$.
  - Remove any 2-cycles.
  - Reverse arrows incident to k.



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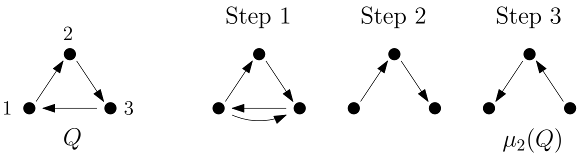

- (Variable mutation)

$$\mu_k(\mathbf{x}) := \{x'_1, \ldots, x'_n\} \text{ where } x'_i = x_i \text{ when } i \neq k \text{ and}$$

$$x'_k = \frac{1}{x_k}\Big(\prod_{i \to k \text{ in } Q} x_i + \prod_{i \leftarrow k \text{ in } Q} x_i \Big).$$

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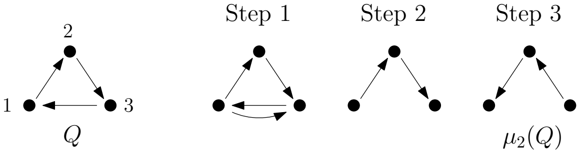

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$$\mu_k(\mathbf{x}) := \{x_1', ..., x_n'\} \text{ where } x_i' = x_i \text{ when } i \neq k \text{ and}$$

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- **Note**: $\mu_k^2(\mathbf{x}, Q) = (\mathbf{x}, Q)$.

Linking the two constructions

Given a triangulation T we define a quiver Q_T where:

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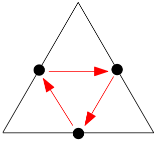

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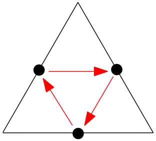
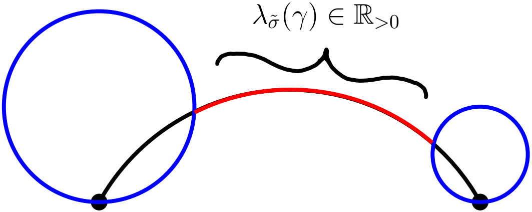


- **Note**: $\mu_\gamma(Q_T) = Q_{\mu_\gamma(T)}$.

How do the variables enter into the picture?

For each arc γ define the **lambda length** $\lambda(\gamma)$:

- Choose a metric σ and a collection of horocycles at each marked point.



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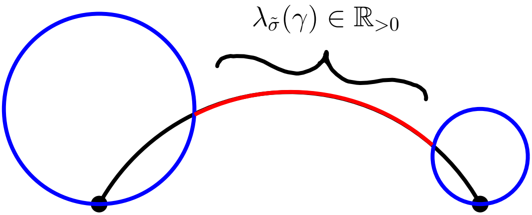

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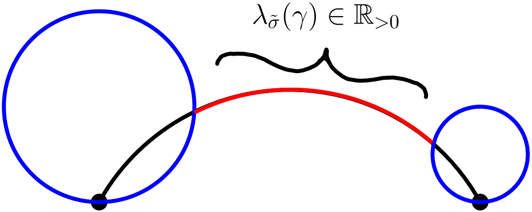

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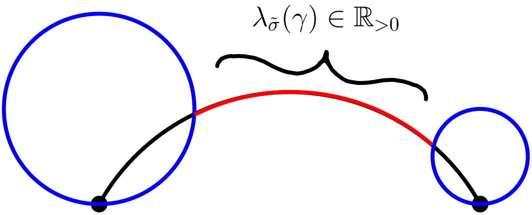

Figure: Definition of the lambda length.

- $x_\gamma := \lambda(\gamma)$

- Associate the seed $\Sigma_T := (\mathbf{x} = \{x_\gamma | \gamma \in T\}, Q_T)$ to T.

Cluster algebras from orientable surfaces.

Proposition [Penner ('87)]

The lambda lengths are related by the Ptolemy relation.

Cluster algebras from orientable surfaces.

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Theorem [Fomin-Shapiro-Thurston]

In the cluster algebra $\mathcal{A}(\Sigma_T)$ generated by the seed Σ_T we have the following:

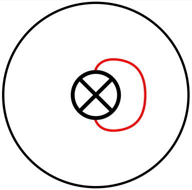
$\mathcal{A}(\boldsymbol{\Sigma}_\mathbf{T})$		$(\mathbf{S}, \mathbf{M})$
Cluster variables	$\longleftrightarrow$	Arcs
Clusters	$\longleftrightarrow$	Triangulations
Mutation	$\longleftrightarrow$	Flips

Extending to non-orientable surfaces (Dupont and Palesi)

For S non-orientable we define the bordered surafce (S, M) in the same way.

Definition

A simple closed curve is said to be **one-sided** if it has no orientable neighbourhood.



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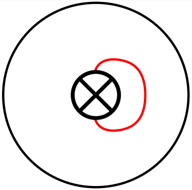


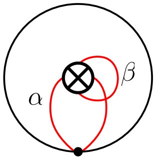
Definition

A **quasi-arc** of (S, M) is an arc (excluding arcs bounding $\mathcal{M}_1$) or a one-sided closed curve.

Extending to non-orientable surfaces (Dupont and Palesi)

Definition

Two quasi-arcs α and β of (S, M) are **compatible** if they don't intersect or are both contained in a Möbius strip with one marked point on the boundary.



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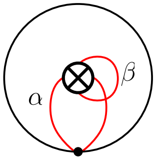


Figure: Example of compatibility.

Definition

A **quasi-triangulation** of (S, M) is a maximal collection of pairwise compatible quasi-arcs.

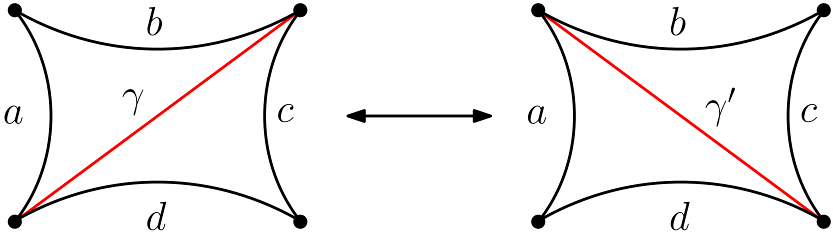
Flips

Proposition [Dupont, Palesi ('15)]

Let T be a quasi-triangulation of (S, M). Then for any $\gamma \in T$ there exists a unique quasi-arc γ' such that $\gamma' \neq \gamma$ and $\mu_\gamma(T) := T \setminus \{\gamma\} \cup \gamma'$ is a quasi-triangulation.

Exchange relations.

We describe the new exchange relations in order to obtain the cluster structure.



$$x_\gamma x_{\gamma'} = x_a x_c + x_b x_d$$

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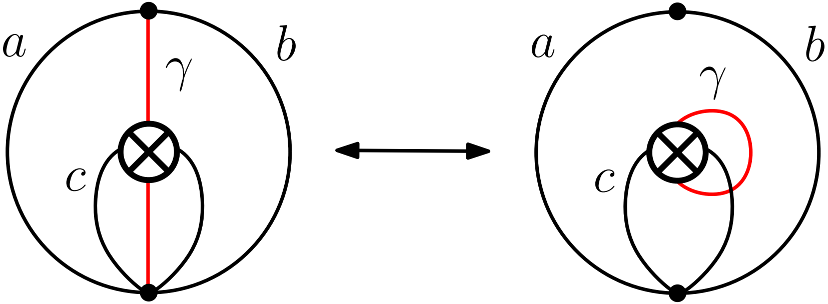

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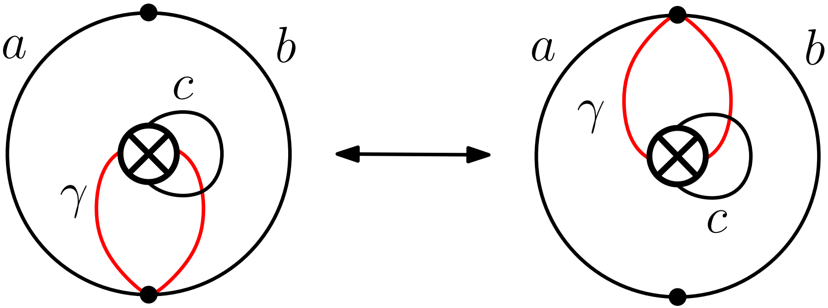
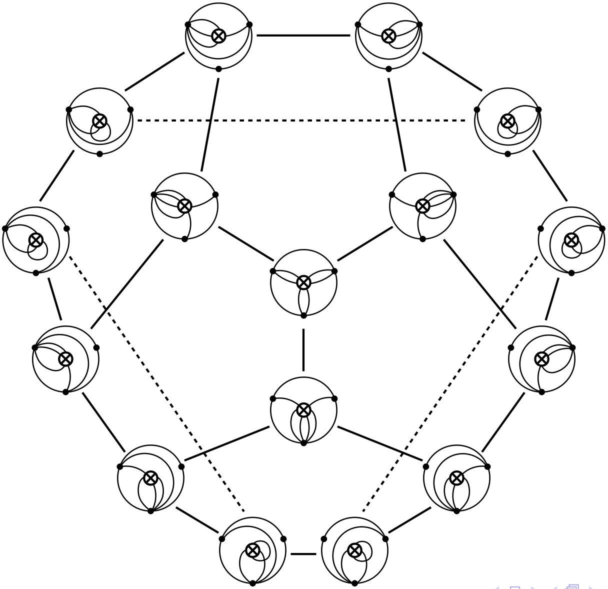


$$x_\gamma x_{\gamma'} = \frac{(x_a+x_b)^2+x_c^2x_ax_b}{x_c^2}$$

Example: $(S, M) = \mathcal{M}_3$.



Back to cluster algebras

Theorem [Fomin-Zelevinsky ('02)]

For a cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ then every cluster variable is a laurent polynomial in the initial cluster variables.

Laurent Phenomenon algebras

- Lam & Pylyavskyy ('12) cooked up a new type of cluster structure designed to produce the Laurent Phenomenon.

Cluster Algebras	**LP Algebras**

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$$(\mathbf{x}, Q) = (\mathbf{x}, \{F_1, \ldots, F_n\})$$

where F_i is binomial.

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where F_i is irreducible in $\mathbb{Z}[\mathbf{x}]$.

$$x_i x_i' = \frac{F_i}{M}$$

M a monomial in the variables $\mathbf{x} \setminus \{x_i\}$.

Linking quasi-cluster algebras and Laurent phenomenon algebras

- For a quasi-triangulation T define a (candidate) LP seed $\Sigma_T := (\mathbf{x}, \{F_1, \ldots, F_n\})$ such that the $F_i's$ are the numerators of the exchange relations of the quasi-arcs in T.

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  - Normalisation recovers the denominator.

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- Hence, for the two constructions to match we need that:
  - Σ_T is a valid LP seed.
  - Normalisation recovers the denominator.
  - LP mutation of these polynomials coincides with flips.

The **LP** structure of (unpunctured) quasi-cluster algebras.

Theorem [W]

Let T be a quasi-triangulation of an **unpunctured(!)** (S, M) and Σ_T its associated LP seed. Then in the LP algebra $\mathcal{A}_{LP}(\Sigma_T)$ generated by this seed we have the following:

$\mathcal{A}_{\mathbf{LP}}(\boldsymbol{\Sigma}_{\mathbf{Q}})$		$(\mathbf{S}, \mathbf{M})$
Cluster variables	$\longleftrightarrow$	Lambda lengths of quasi-arcs
Clusters	$\longleftrightarrow$	Quasi-triangulations
LP mutation	$\longleftrightarrow$	Flips

Adding coefficients via laminations

Definition

A **lamination** L is a curve in (S, M) whose endpoints are on $\partial S \setminus M$ or spiral infinitely around punctures (with some other mild conditions).

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Definition

A **multi-lamination** $\mathbf{L} = \{L_1, \ldots, L_k\}$ is a finite collection of laminations of (S, M).

- Analogous to Fomin and Thurston, for any quasi-arc γ we can define the **laminated lambda length** $x_{\mathbf{L}}(\gamma) := \frac{\lambda(\gamma)}{c_{\mathbf{L}}(\gamma)}$; a notion of length depending on the laminations as well as the underlying geometry.

The **LP** structure of quasi-cluster algebras.

Theorem [W]

Let T be a quasi-triangulation of (S, M). Then there exists a multi-lamination $\mathbf{L} = \{L_1, \ldots, L_k\}$, with associated LP seed $\Sigma_{T,\mathbf{L}}$, such that in the LP algebra $\mathcal{A}_{LP}(\Sigma_{T,\mathbf{L}})$ we have the following:

$\mathcal{A}_{\mathbf{LP}}(\mathbf{\Sigma_{T,L}})$		$(\mathbf{S}, \mathbf{M})$
Cluster variables	$\longleftrightarrow$	**Laminated** lambda lengths of quasi-arcs
Clusters	$\longleftrightarrow$	Quasi-triangulations
LP mutation	$\longleftrightarrow$	Flips

Summary

$$\begin{array}{ccc} \{\text{Surface cluster algebras}\} & \xhookrightarrow{\text{Fomin, Shapiro, Thurston}} & \{\text{Cluster algebras}\} \\ \downarrow \text{Dupont, Palesi} & & \downarrow \text{Lam, Pylyavskyy} \\ \{\text{Quasi-cluster algebras}\} & \xhookrightarrow{\text{[W]}} & \{\text{Laurent phenomenon algebras}\} \end{array}$$

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- The finite type arc complexes are spherical [W].
- The denominator vectors can be read off using intersection numbers [W].
- There exists a $\mathbb{Z}^n$-grading of $\mathcal{A}_{(S,M,\mathbf{L})}$ in which the quasi-cluster variables are homogenous [W].
- For finite type quasi-cluster algebras the cluster monomials form a (linear) basis for $\mathcal{A}_{(S,M)}$ [Dupont and Palesi].

Future directions

- Finding 'good' bases for $\mathcal{A}_{(S,M,\mathbf{L})}$.

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- Studying the connection between indecomposible quiver representations of ’double’ quivers (S, M) and the corresponding quasi-cluster variables.