Kostka-Foulkes polynomials in type C_n

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From weight multiplicities...

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Let $\mathfrak{g}$ be a complex, simple Lie algebra and $\mathrm{V}(\lambda)$ be a finite-dimensional, irreducible representation. Then

$$\mathrm{V}(\lambda) \cong \bigoplus_{\mu \leq \lambda} \mathrm{V}(\lambda)_\mu, \text{ and if } K_{\lambda,\mu} = \dim(\mathrm{V}(\lambda)_\mu),$$

$$K_{\lambda,\mu} = \sum_{w \in \mathrm{W}} (-1)^{l(w)} \mathcal{P}(w(\lambda + \rho) - (\mu + \rho)),$$

where:

$$\prod_{\alpha \text{ positive root}} \frac{1}{1 - x^\alpha} = \sum_\beta \mathcal{P}(\beta) x^\beta,$$

and the sum runs over all positive integral combinations of positive roots.

... to their q-analogues

Lusztig's q-analogues of weight multiplicities are given by:

$$K_{\lambda,\mu}(q) = \sum_{w \in \mathrm{W}} (-1)^{l(w)} \mathcal{P}_q(w(\lambda+\rho) - (\mu+\rho))$$

where $\mathcal{P}_q$ is a q-analoque satisfying:

$$\prod_{\alpha \text{ positive root}} \frac{1}{1 - qx^{\alpha}} = \sum_{\beta} \mathcal{P}_q(\beta) x^{\beta}.$$

In particular, $K_{\lambda,\mu}(1) = K_{\lambda,\mu}$.

Properties

- It was shown by Kato that the polynomials $K_{\lambda,\mu}(q)$ are Kostka-Foulkes polynomials defined by:

$$s_\lambda = \sum_{\mu \text{ dominant}} K_{\lambda,\mu}(q) P_\mu(x;q)$$

  where s_λ is a Weyl character and p_μ a Hall-Littlewood polynomial (a.k.a. Macdonald spherical function).
- They are affine Kazhdan-Lusztig polynomials, so their coefficients are positive, but in general, there is no positive, closed, combinatorial formula to describe them...

...except in type A_{n-1}, that is when $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$:

In this case Lascoux-Schutzenberger found a statistic

$$\mathsf{ch} : \mathsf{SSYT}_n \to \mathbb{Z}_{\geq 0}$$

on semistandard Young tableaux called **charge** which gives the following formula:

$$K_{\lambda,\mu}(q) = \sum_{\mathrm{T} \in \mathsf{SSYT}_n(\lambda,\mu)} q^{\mathsf{ch}(\mathrm{T})}.$$

The charge of a semistandard Young tableau

- Define a graph structure on SSYT_n by setting $\mathrm{T} \to \mathrm{T}'$ whenever there exists a word u and a letter $x \neq 1$ such that

$$\mathsf{word}(\mathrm{T}) \equiv xu \text{ and } \mathsf{word}(\mathrm{T}') \equiv ux.$$

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  where $\equiv$ denotes plactic equivalence on words.
- If the shape of T' is a row, it cannot be obtained from some T in this way.
- Fix a weight μ, and let T_μ be the unique tableau with row shape and content/weight μ. Then, all paths joining a tableau T of weight μ to T_μ have the same length (and there exists at least one) n_T. Then

$$\mathsf{ch}(\mathrm{T}) := \sum_i (i-1)\mu_i - n_\mathrm{T}.$$

Type C_n

From now on, $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$. Semistandard Young tableaux are replaced by Kashiwara-Nakashima tableaux KN_n, which are semistandard Young tableaux on the ordered alphabet

$$\mathcal{C}_n = \left\{ \bar{n} < \cdots < \bar{1} < 1 < ... < n \right\}$$

satisfying certain conditions.

- Lecouvey has defined a cyclage algorithm and with it a directed graph structure on the set

$$\mathsf{KN} = \bigcup_{n>0} \mathsf{KN}_n$$

in such a way that all sinks are columns of weight zero, and such that, for every $\mathrm{T} \in \mathsf{KN}$, there always exists a finite path to a unique sink C_{T}, and all paths from T to C_{T} have the same length n_{T}.

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- Let C be a column of weight zero. Define

$$\mathrm{ch}_n(\mathrm{C}) := 2 \sum_{i \in \mathrm{E}_C} (n - i),$$

where

$$\mathrm{E_C} = \{ i \geq 1 | i \in \mathrm{C}, i + 1 \notin \mathrm{C} \}.$$

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- Let $\mathrm{T} \in \mathsf{KN}_n$. Then

$$\mathsf{ch}_n(\mathrm{T}) := \mathsf{ch}_n(\mathrm{C}_\mathrm{T}) + n_\mathrm{T}.$$

Conjecture (Lecouvey)

Let $\mathsf{KN}_n(\lambda, \mu)$ denote the set of Kashiwara-Nakashima tableaux of shape λ and weight μ. The following formula holds:

$$K_{\lambda,\mu}(q) = \sum_{\mathrm{T} \in \mathsf{KN}_n(\lambda,\mu)} q^{\mathsf{ch}_n(\mathrm{T})}.$$

In recent work, Lecouvey-Lenart have proven this conjecture for columns of weight zero.

Theorem (Gerber-T.)

Lecouvey's conjecture is true for rows of weight zero. Moreover, for $\mathrm{T} \in \mathsf{KN}_n((2r), 0)$, given by

$$\mathrm{T} = \boxed{\bar{i}_r} \boxed{\ldots} \boxed{\bar{i}_1} \boxed{i_1} \boxed{\ldots} \boxed{i_r}$$

for positive integers $i_1 \leq \cdots \leq i_r$, we have

$$\mathrm{ch}_n(\mathrm{T}) = r + 2\sum_{k=1}^{r}(n - i_k).$$

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Thank you for your attention!