Emmanuel LETELLIER

Exotic Fourier transforms on connected reductive groups

(Work in progress with G. Laumon)

Introduction

- Classical Fourier transforms on $\mathrm{Lie}(G)$ and relationship with representation theory of G through $G_{\mathrm{uni}} \simeq \mathrm{Lie}(G)_{\mathrm{nil}}$ (Springer, Kazhdan, Kawanaka, Lusztig, Waldspurger, L. , Achar-Henderson-Juteau-Riche).

- Classical Fourier transforms on $\mathrm{Lie}(G)$ and relationship with representation theory of G through $G_{\mathrm{uni}} \simeq \mathrm{Lie}(G)_{\mathrm{nil}}$ (Springer, Kazhdan, Kawanaka, Lusztig, Waldspurger, L. , Achar-Henderson-Juteau-Riche).
- Special case $\mathrm{GL}_n \subset \mathrm{gl}_n = \mathrm{Lie}(\mathrm{GL}_n)$ (equivariant $\mathrm{GL}_n \times \mathrm{GL}_n$ open embedding). Fourier on gl_n provides Fourier on GL_n (many applications to quiver representations for instance).

Introduction

- Classical Fourier transforms on $\mathrm{Lie}(G)$ and relationship with representation theory of G through $G_{\mathrm{uni}} \simeq \mathrm{Lie}(G)_{\mathrm{nil}}$ (Springer, Kazhdan, Kawanaka, Lusztig, Waldspurger, L. , Achar-Henderson-Juteau-Riche).
- Special case $\mathrm{GL}_n \subset \mathrm{gl}_n = \mathrm{Lie}(\mathrm{GL}_n)$ (equivariant $\mathrm{GL}_n \times \mathrm{GL}_n$ open embedding). Fourier on gl_n provides Fourier on GL_n (many applications to quiver representations for instance).
- Generalization of the case of GL_n to other reductive groups ? Given $\rho^\flat : G^\flat \to \mathrm{GL}_n$, would like to transfert $(\mathrm{GL}_n \subset \mathrm{gl}_n, \mathcal{F}^{\mathrm{gl}_n})$ to $(G \subset \mathcal{X}_\rho, \mathcal{F}^{\mathcal{X}_\rho})$.

Classical Fourier transforms

- G connected reductive group defined over $\mathbb{F}_q$, with Frobenius $F : G \to G$. Then $G^F = G(\mathbb{F}_q)$ finite group.

Classical Fourier transforms

- G connected reductive group defined over $\mathbb{F}_q$, with Frobenius $F : G \to G$. Then $G^F = G(\mathbb{F}_q)$ finite group.

  Ex : $G = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ and $F : (a_{ij})_{i,j} \mapsto (a_{ij}^q)_{i,j}$.

Classical Fourier transforms

- G connected reductive group defined over $\mathbb{F}_q$, with Frobenius $F : G \to G$. Then $G^F = G(\mathbb{F}_q)$ finite group.

  Ex : $G = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ and $F : (a_{ij})_{i,j} \mapsto (a_{ij}^q)_{i,j}$.

  $\mathfrak{g} = \mathrm{Lie}(G)$ and $F : \mathfrak{g} \to \mathfrak{g}$ Frobenius.

Classical Fourier transforms

- G connected reductive group defined over $\mathbb{F}_q$, with Frobenius $F : G \to G$. Then $G^F = G(\mathbb{F}_q)$ finite group.

  Ex : $G = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ and $F : (a_{ij})_{i,j} \mapsto (a_{ij}^q)_{i,j}$.

  $\mathfrak{g} = \mathrm{Lie}(G)$ and $F : \mathfrak{g} \to \mathfrak{g}$ Frobenius.

  $\mu : \mathfrak{g} \times \mathfrak{g} \to \overline{\mathbb{F}}_q$ non-degenerate G-invariant symmetric bilinear from which commutes with Frobenius (ex. $(x, y) \mapsto \mathrm{Tr}(xy)$ if $G = \mathrm{GL}_n$).

Classical Fourier transforms

- G connected reductive group defined over $\mathbb{F}_q$, with Frobenius $F : G \to G$. Then $G^F = G(\mathbb{F}_q)$ finite group.

  Ex : $G = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ and $F : (a_{ij})_{i,j} \mapsto (a_{ij}^q)_{i,j}$.

  $\mathfrak{g} = \mathrm{Lie}(G)$ and $F : \mathfrak{g} \to \mathfrak{g}$ Frobenius.

  $\mu : \mathfrak{g} \times \mathfrak{g} \to \overline{\mathbb{F}}_q$ non-degenerate G-invariant symmetric bilinear from which commutes with Frobenius (ex. $(x, y) \mapsto \mathrm{Tr}(xy)$ if $G = \mathrm{GL}_n$).

  $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell{}^\times$ non-trivial additive character.

Classical Fourier transforms

- G connected reductive group defined over $\mathbb{F}_q$, with Frobenius $F : G \to G$. Then $G^F = G(\mathbb{F}_q)$ finite group.

  Ex : $G = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ and $F : (a_{ij})_{i,j} \mapsto (a_{ij}^q)_{i,j}$.

  $\mathfrak{g} = \mathrm{Lie}(G)$ and $F : \mathfrak{g} \to \mathfrak{g}$ Frobenius.

  $\mu : \mathfrak{g} \times \mathfrak{g} \to \overline{\mathbb{F}}_q$ non-degenerate G-invariant symmetric bilinear from which commutes with Frobenius (ex. $(x, y) \mapsto \mathrm{Tr}(xy)$ if $G = \mathrm{GL}_n$).

  $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$ non-trivial additive character.

  Fourier transform $\mathcal{F}^{\mathfrak{g}} : \mathrm{Fun}(\mathfrak{g}^F, \overline{\mathbb{Q}}_\ell) \to \mathrm{Fun}(\mathfrak{g}^F, \overline{\mathbb{Q}}_\ell)$,

$$\mathcal{F}^{\mathfrak{g}}(f)(x) = \sum_{y \in \mathfrak{g}^F} \psi(\mu(x, y)) f(y).$$

Representations of G^F

G-equivariant isomorphism

$$\phi : G_{\mathrm{uni}} \simeq \mathfrak{g}_{\mathrm{nil}}.$$

Representations of G^F

G-equivariant isomorphism

$$\phi : G_{\text{uni}} \simeq \mathfrak{g}_{\text{nil}}.$$

We have a connection

$$\{\text{Irreducible } \overline{\mathbb{Q}}_\ell\text{-char. of } G^F\} \longleftrightarrow \{\text{Fourier trans. of } G^F\text{-orbits of } \mathfrak{g}^F\}$$

G-equivariant isomorphism

$$\phi : G_{\mathrm{uni}} \simeq \mathfrak{g}_{\mathrm{nil}}.$$

We have a connection

$$\{\text{Irreducible } \overline{\mathbb{Q}}_\ell\text{-char. of } G^F\} \longleftrightarrow \{\text{Fourier trans. of } G^F\text{-orbits of } \mathfrak{g}^F\}$$

Remark : If H finite abelian group, then

$$\{\text{Irreducible } \overline{\mathbb{Q}}_\ell\text{-char. of } H\} = \text{Fourier}(\{\text{conjugacy classes of } H\})$$

Special case $G = \mathrm{GL}_n$

- $G = \mathrm{GL}_n$ with standard Frobenius $F : (a_{ij}) \mapsto (a_{ij}^q)$.
  Then $\mathrm{GL}_n \subset \mathrm{gl}_n$ compatible with Frobenius.

Special case $G = \mathrm{GL}_n$

- $G = \mathrm{GL}_n$ with standard Frobenius $F : (a_{ij}) \mapsto (a_{ij}^q)$.
  Then $\mathrm{GL}_n \subset \mathrm{gl}_n$ compatible with Frobenius.

$$\{\text{Unipotent char. of } \mathrm{GL}_n^F\} = (\mathcal{F}^{\mathrm{gl}_n}\{\text{nilpotent orbits of } \mathrm{gl}_n^F\})|_{\mathrm{GL}_n^F}$$

Special case $G = \mathrm{GL}_n$

- $G = \mathrm{GL}_n$ with standard Frobenius $F : (a_{ij}) \mapsto (a_{ij}^q)$.
  Then $\mathrm{GL}_n \subset \mathrm{gl}_n$ compatible with Frobenius.

  $$\{\text{Unipotent char. of } \mathrm{GL}_n^F\} = (\mathcal{F}^{\mathrm{gl}_n}\{\text{nilpotent orbits of } \mathrm{gl}_n^F\})|_{\mathrm{GL}_n^F}$$

- Many applications (Hausel-L.-Villegas, L.) :

Special case $G = \mathrm{GL}_n$

- $G = \mathrm{GL}_n$ with standard Frobenius $F : (a_{ij}) \mapsto (a_{ij}^q)$.
  Then $\mathrm{GL}_n \subset \mathrm{gl}_n$ compatible with Frobenius.

$$\{\text{Unipotent char. of } \mathrm{GL}_n^F\} = (\mathcal{F}^{\mathrm{gl}_n}\{\text{nilpotent orbits of } \mathrm{gl}_n^F\})|_{\mathrm{GL}_n^F}$$

- Many applications (Hausel-L.-Villegas, L.) :

  - Computation of Poincaré polynomial of Nakajima's quiver varieties.

Special case $G = \mathrm{GL}_n$

- $G = \mathrm{GL}_n$ with standard Frobenius $F : (a_{ij}) \mapsto (a_{ij}^q)$.
  Then $\mathrm{GL}_n \subset \mathrm{gl}_n$ compatible with Frobenius.

  $\{$Unipotent char. of $\mathrm{GL}_n^F\} = (\mathcal{F}^{\mathrm{gl}_n}\{$nilpotent orbits of $\mathrm{gl}_n^F\})|_{\mathrm{GL}_n^F}$

- Many applications (Hausel-L.-Villegas, L.) :

  - Computation of Poincaré polynomial of Nakajima's quiver varieties.

  - Proof of Kac conjectures on quiver representations.

Special case $G = \mathrm{GL}_n$

- $G = \mathrm{GL}_n$ with standard Frobenius $F : (a_{ij}) \mapsto (a_{ij}^q)$.
  Then $\mathrm{GL}_n \subset \mathrm{gl}_n$ compatible with Frobenius.

$$\{\text{Unipotent char. of } \mathrm{GL}_n^F\} = (\mathcal{F}^{\mathrm{gl}_n}\{\text{nilpotent orbits of } \mathrm{gl}_n^F\})|_{\mathrm{GL}_n^F}$$

- Many applications (Hausel-L.-Villegas, L.) :

  - Computation of Poincaré polynomial of Nakajima's quiver varieties.

  - Proof of Kac conjectures on quiver representations.

  - Study of structure coefficients of the character ring of $\mathrm{GL}_n(\mathbb{F}_q)$.

General case ?

- **Example :** $G = \mathrm{PGL}_n$.

General case ?

- **Example :** $G = \mathrm{PGL}_n$.

$$\mathsf{PGL}_n \subset [\mathrm{gl}_n/\mathrm{GL}_1].$$

General case ?

- **Example :** $G = \mathrm{PGL}_n$.

$$\mathsf{PGL}_n \subset [\mathrm{gl}_n/\mathrm{GL}_1].$$

Then $\mathrm{Fun}([\mathrm{gl}_n/\mathrm{GL}_1]^F, \overline{\mathbb{Q}}_\ell) \simeq \mathrm{Fun}(\mathrm{gl}_n^F, \overline{\mathbb{Q}}_\ell)^{\mathrm{GL}_1^F}$,

General case ?

- **Example :** $G = \mathrm{PGL}_n$.

$$\mathsf{PGL}_n \subset [\mathrm{gl}_n/\mathrm{GL}_1].$$

Then $\mathrm{Fun}([\mathrm{gl}_n/\mathrm{GL}_1]^F, \overline{\mathbb{Q}}_\ell) \simeq \mathrm{Fun}(\mathrm{gl}_n^F, \overline{\mathbb{Q}}_\ell)^{\mathrm{GL}_1^F}$,

$$\mathcal{F}(f)([x]) = \sum_{[y]\in[\mathrm{gl}_n/\mathrm{GL}_1]^F} \frac{1}{|\mathrm{Stab}_{\mathrm{GL}_1^F}(y)|} K([x],[y]) f([y]),$$

where

$$K([x],[y]) = \sum_{z\in\mathrm{GL}_1^F} \psi(\mathrm{Tr}(\lambda xy)) = \begin{cases} q-1 & \text{if } \mathrm{Tr}(xy) = 0 \\ -1 & \text{if } \mathrm{Tr}(xy) \neq 0 \end{cases}.$$

General case ?

- **Starting datum :** an F-stable maximal torus T of G and a representation

$$\rho^\flat : N_{G^\flat}(T^\flat) \to \mathrm{GL}_n$$

which commutes with Frobenius.

- **Starting datum :** an F-stable maximal torus T of G and a representation

$$\rho^\flat : N_{G^\flat}(T^\flat) \to \mathrm{GL}_n$$

which commutes with Frobenius.

- In the previous example : $G = \mathrm{PGL}_n$, $\rho^\flat : \mathrm{SL}_n \subset \mathrm{GL}_n$ which dualizes into $\rho : \mathrm{GL}_n \to \mathrm{PGL}_n$,

$$\mathcal{X} = [\mathrm{gl}_n / \mathrm{Ker}(\rho)] = [\mathrm{gl}_n / \mathrm{GL}_1],$$

and $\mathcal{F}^{\mathcal{X}}$ obtained from $\mathcal{F}^{\mathrm{gl}_n}$ by descent.

Spectral aspect of Fourier transform $\mathcal{F}^{\mathrm{gl}_n}$

- If X finite set, put $\mathcal{C}(X) = \mathrm{Fun}(X, \overline{\mathbb{Q}}_\ell)$

Spectral aspect of Fourier transform $\mathcal{F}^{\mathrm{gl}_n}$

- If X finite set, put $\mathcal{C}(X) = \mathrm{Fun}(X, \overline{\mathbb{Q}}_\ell)$
- $\mathrm{GL}_n^F \times \mathrm{GL}_n^F$ acts on $\mathcal{C}(\mathrm{gl}_n^F)$ and

$$\mathcal{F}^{\mathrm{gl}_n}((g,h)\cdot f) = (h,g)\cdot \mathcal{F}^{\mathrm{gl}_n}(f).$$

I.e. $\mathcal{F}^{\mathrm{gl}_n} : \mathcal{C}(\mathrm{gl}_n^F) \to \mathcal{C}(\mathrm{gl}_n^F)^\iota$ isomorphism of $\mathrm{GL}_n^F \times \mathrm{GL}_n^F$-mod.

Spectral aspect of Fourier transform $\mathcal{F}^{\mathrm{gl}_n}$

- If X finite set, put $\mathcal{C}(X) = \mathrm{Fun}(X, \overline{\mathbb{Q}}_\ell)$
- $\mathrm{GL}_n^F \times \mathrm{GL}_n^F$ acts on $\mathcal{C}(\mathrm{gl}_n^F)$ and

$$\mathcal{F}^{\mathrm{gl}_n}((g,h)\cdot f) = (h,g)\cdot \mathcal{F}^{\mathrm{gl}_n}(f).$$

  I.e. $\mathcal{F}^{\mathrm{gl}_n} : \mathcal{C}(\mathrm{gl}_n^F) \to \mathcal{C}(\mathrm{gl}_n^F)^\iota$ isomorphism of $\mathrm{GL}_n^F \times \mathrm{GL}_n^F$-mod.
- Deligne-Lusztig theory :

$$\mathrm{Irr}(G^F) = \coprod_{(s)} \mathcal{E}_G(s),$$

  where (s) runs over $F^\flat$-stable semisimple conjugacy classes of $G^\flat$.
  $\mathcal{E}_G(s)$: Lusztig series.

Spectral aspect of Fourier transform $\mathcal{F}^{\mathrm{gl}_n}$

- We have

$$\mathcal{C}(\mathrm{GL}_n^F) \simeq \bigoplus_{\pi \in \widehat{\mathrm{GL}_n^F}} V_\pi \boxtimes V_\pi^\vee.$$

Spectral aspect of Fourier transform $\mathcal{F}^{\mathrm{gl}_n}$

- We have

$$\mathcal{C}(\mathrm{GL}_n^F) \simeq \bigoplus_{\pi \in \widehat{\mathrm{GL}_n^F}} V_\pi \boxtimes V_\pi^\vee.$$

where for $x \boxtimes x^\vee \in V_\pi \boxtimes V_\pi^\vee$, we get

$$\mathrm{GL}_n^F \to \overline{\mathbb{Q}}_\ell,\ h \mapsto x^\vee(\rho_\pi(h)(x)).$$

Spectral aspect of Fourier transform $\mathcal{F}^{\mathrm{gl}_n}$

- We have

$$\mathcal{C}(\mathrm{GL}_n^F) \simeq \bigoplus_{\pi \in \widehat{\mathrm{GL}_n^F}} V_\pi \boxtimes V_\pi^\vee.$$

where for $x \boxtimes x^\vee \in V_\pi \boxtimes V_\pi^\vee$, we get

$$\mathrm{GL}_n^F \to \overline{\mathbb{Q}}_\ell,\ h \mapsto x^\vee(\rho_\pi(h)(x)).$$

- $$\mathcal{C}(\mathrm{GL}_n^F) = \bigoplus_{(s)} \mathcal{C}(\mathrm{GL}_n^F)_{(s)}.$$

where (s) runs over the F-stable semisimple conjugacy classes of GL_r and

$$\mathcal{C}(\mathrm{GL}_n^F)_{(s)} = \bigoplus_{\pi \in \mathcal{E}_{\mathrm{GL}_n}(s)} V_\pi \boxtimes V_\pi^\vee.$$

Spectral aspect of Fourier transform $\mathcal{F}^{\mathrm{gl}_n}$

- $$\mathcal{C}(\mathrm{gl}_n^F) = \bigoplus_{(s)} \mathcal{C}(\mathrm{gl}_n^F)_{(s)},$$

  where (s) runs over F-stable semisimple conjugacy classes of GL_n.

Spectral aspect of Fourier transform $\mathcal{F}^{\mathrm{gl}_n}$

- $$\mathcal{C}(\mathrm{gl}_n^F) = \bigoplus_{(s)} \mathcal{C}(\mathrm{gl}_n^F)_{(s)},$$

  where (s) runs over F-stable semisimple conjugacy classes of GL_n.
- Moreover $$\mathcal{F}^{\mathrm{gl}_n} : \mathcal{C}(\mathrm{gl}_n)_{(s)} \to \mathcal{C}(\mathrm{gl}_n)_{(s^{-1})}.$$

Spectral aspect of Fourier transform $\mathcal{F}^{\mathrm{gl}_n}$

- $$\mathcal{C}(\mathrm{gl}_n^F) = \bigoplus_{(s)} \mathcal{C}(\mathrm{gl}_n^F)_{(s)},$$

  where (s) runs over F-stable semisimple conjugacy classes of GL_n.
- Moreover $\mathcal{F}^{\mathrm{gl}_n} : \mathcal{C}(\mathrm{gl}_n)_{(s)} \to \mathcal{C}(\mathrm{gl}_n)_{(s^{-1})}$.
- If the eigenvalues of (s) are all $\neq 1$, then $\mathcal{C}(\mathrm{gl}_n)_{(s)} = \mathcal{C}(\mathrm{GL}_n)_{(s)}$.

Spectral aspect of Fourier transform $\mathcal{F}^{\mathrm{gl}_n}$

- **Jordan decomposition for Fourier :**

- **Jordan decomposition for Fourier :**

$$\mathcal{C}(\mathrm{gl}_n)_{(s)} = R^{\mathrm{GL}_n \times \mathrm{GL}_n}_{L_{r'} \times L_{r'}} \left(\mathcal{C}(\mathrm{GL}^F_{r'})_{(s')} \boxtimes \mathcal{C}(\mathrm{gl}^F_{n-r'})_{(1)} \right),$$

with $r' = \#\{\text{eigenvalues of } s \neq 1\}$ and $L_{r'} = \mathrm{GL}_{r'} \times \mathrm{GL}_{n-r'}$.

- **Jordan decomposition for Fourier :**

$$\mathcal{C}(\mathrm{gl}_n)_{(s)} = R_{L_{r'}\times L_{r'}}^{\mathrm{GL}_n\times\mathrm{GL}_n}\left(\mathcal{C}(\mathrm{GL}_{r'}^F)_{(s')}\boxtimes\mathcal{C}(\mathrm{gl}_{n-r'}^F)_{(1)}\right),$$

with $r' = \#\{\text{eigenvalues of } s \neq 1\}$ and $L_{r'} = \mathrm{GL}_{r'} \times \mathrm{GL}_{n-r'}$.

Fourier on both sides coincide.

- **Jordan decomposition for Fourier :**

$$\mathcal{C}(\mathrm{gl}_n)_{(s)} = R^{\mathrm{GL}_n \times \mathrm{GL}_n}_{L_{r'} \times L_{r'}} \left(\mathcal{C}(\mathrm{GL}^F_{r'})_{(s')} \boxtimes \mathcal{C}(\mathrm{gl}^F_{n-r'})_{(1)} \right),$$

  with $r' = \#\{\text{eigenvalues of } s \neq 1\}$ and $L_{r'} = \mathrm{GL}_{r'} \times \mathrm{GL}_{n-r'}$.
  Fourier on both sides coincide.

- We are reduced to understand Fourier on $\mathcal{C}(\mathrm{gl}^F_n)_{(1)}$.

- **Jordan decomposition for Fourier :**

$$\mathcal{C}(\mathrm{gl}_n)_{(s)} = R^{\mathrm{GL}_n \times \mathrm{GL}_n}_{L_{r'} \times L_{r'}} \left(\mathcal{C}(\mathrm{GL}^F_{r'})_{(s')} \boxtimes \mathcal{C}(\mathrm{gl}^F_{n-r'})_{(1)} \right),$$

with $r' = \#\{\text{eigenvalues of } s \neq 1\}$ and $L_{r'} = \mathrm{GL}_{r'} \times \mathrm{GL}_{n-r'}$.
Fourier on both sides coincide.

- We are reduced to understand Fourier on $\mathcal{C}(\mathrm{gl}^F_n)_{(1)}$.
- **Example :** $n = 2$

$\mathcal{C}(\mathrm{GL}^F_{\mathbf{2}})$	$\mathbf{1} \boxtimes \mathbf{1}$	$\mathrm{St} \boxtimes \mathrm{St}$			$\{V^{\mathrm{GL}_{\mathbf{2}}}_{\alpha,\mathbf{1}}\}_{\alpha \neq \mathbf{1}}$	$\cdots$
$\mathcal{C}(\mathrm{gl}^F_{\mathbf{2},\mathbf{1}})$	$\mathbf{1} \boxtimes \mathbf{1}$	$\mathrm{St} \boxtimes \mathrm{St}$	$\mathbf{1} \boxtimes \mathrm{St}$	$\mathrm{St} \boxtimes \mathbf{1}$	$\{V^{\mathrm{GL}_{\mathbf{2}}}_{\alpha,\mathbf{1}}\}_{\alpha \neq \mathbf{1}}$	
$\mathcal{C}(\mathrm{gl}^F_{\mathbf{2},\mathbf{0}})$	$\mathbf{1} \boxtimes \mathbf{1}$					

- **Fourier matrices when $n = 2$**

$$\begin{array}{c} \\ (1 \boxtimes 1)_2 \\ (1 \boxtimes 1)_1 \\ (1 \boxtimes 1)_0 \end{array} \begin{array}{c} \begin{array}{ccc} (1 \boxtimes 1)_2 & (1 \boxtimes 1)_1 & (1 \boxtimes 1)_0 \end{array} \\ \begin{pmatrix} q & -(q+1) & 1 \\ -q(q-1) & q^2 - q - 1 & 1 \\ q(q-1)^2(q+1) & q^3 + q^2 - q - 1 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ (\mathrm{St} \boxtimes \mathrm{St})_2 \\ (\mathrm{St} \boxtimes \mathrm{St})_1 \end{array} \begin{array}{c} \begin{array}{cc} (\mathrm{St} \boxtimes \mathrm{St})_2 & (\mathrm{St} \boxtimes \mathrm{St})_1 \end{array} \\ \begin{pmatrix} q & -q \\ q(q-1) & q(q+1) \end{pmatrix} \end{array}$$

Spectral aspect of Fourier transform $\mathcal{F}^{\mathfrak{gl}_n}$

- **Fourier matrices when $n = 2$**

$$
\begin{array}{c|ccc}
 & (1\boxtimes 1)_2 & (1\boxtimes 1)_1 & (1\boxtimes 1)_0 \\
(1\boxtimes 1)_2 & q & -(q+1) & 1 \\
(1\boxtimes 1)_1 & -q(q-1) & q^2-q-1 & 1 \\
(1\boxtimes 1)_0 & q(q-1)^2(q+1) & q^3+q^2-q-1 & 1
\end{array}
$$

$$
\begin{array}{c|cc}
 & (\mathrm{St}\boxtimes \mathrm{St})_2 & (\mathrm{St}\boxtimes \mathrm{St})_1 \\
(\mathrm{St}\boxtimes \mathrm{St})_2 & q & -q \\
(\mathrm{St}\boxtimes \mathrm{St})_1 & q(q-1) & q(q+1)
\end{array}
$$

- In the $n = 2$ case $\mathcal{F}^{\mathfrak{gl}_n}$ can be reconstructed from these two matrices and $(1\boxtimes \mathrm{St}) \to (\mathrm{St}\boxtimes 1)$, $x \mapsto -q^2x$.

Spectral aspect of Fourier transform $\mathcal{F}^{\mathrm{gl}_n}$

- **Fourier matrices when $n = 2$**

$$\begin{array}{c} (1\boxtimes 1)_2 \\ (1\boxtimes 1)_1 \\ (1\boxtimes 1)_0 \end{array} \overset{\begin{array}{ccc} (1\boxtimes 1)_2 & (1\boxtimes 1)_1 & (1\boxtimes 1)_0 \end{array}}{\begin{pmatrix} q & -(q+1) & 1 \\ -q(q-1) & q^2-q-1 & 1 \\ q(q-1)^2(q+1) & q^3+q^2-q-1 & 1 \end{pmatrix}}$$

$$\begin{array}{c} (\mathrm{St}\boxtimes \mathrm{St})_2 \\ (\mathrm{St}\boxtimes \mathrm{St})_1 \end{array} \overset{\begin{array}{cc} (\mathrm{St}\boxtimes \mathrm{St})_2 & (\mathrm{St}\boxtimes \mathrm{St})_1 \end{array}}{\begin{pmatrix} q & -q \\ q(q-1) & q(q+1) \end{pmatrix}}$$

- In the $n = 2$ case $\mathcal{F}^{\mathrm{gl}_n}$ can be reconstructed from these two matrices and $(1 \boxtimes \mathrm{St}) \to (\mathrm{St} \boxtimes 1)$, $x \mapsto -q^2 x$.

Spectral aspect of Fourier transform $\mathcal{F}^{\mathrm{gl}_n}$

- **Fourier matrices when $n = 2$**

$$\begin{array}{c|ccc} & (1\boxtimes 1)_2 & (1\boxtimes 1)_1 & (1\boxtimes 1)_0 \\ (1\boxtimes 1)_2 & q & -(q+1) & 1 \\ (1\boxtimes 1)_1 & -q(q-1) & q^2-q-1 & 1 \\ (1\boxtimes 1)_0 & q(q-1)^2(q+1) & q^3+q^2-q-1 & 1 \end{array}$$

$$\begin{array}{c|cc} & (\mathrm{St}\boxtimes \mathrm{St})_2 & (\mathrm{St}\boxtimes \mathrm{St})_1 \\ (\mathrm{St}\boxtimes \mathrm{St})_2 & q & -q \\ (\mathrm{St}\boxtimes \mathrm{St})_1 & q(q-1) & q(q+1) \end{array}$$

- In the $n = 2$ case $\mathcal{F}^{\mathrm{gl}_n}$ can be reconstructed from these two matrices and $(1 \boxtimes \mathrm{St}) \to (\mathrm{St} \boxtimes 1)$, $x \mapsto -q^2 x$.

Back to our problem

- Given $\rho^\flat : N_{G^\flat}(T^\flat) \to \mathrm{GL}_n$, we would like to transfert $(\mathrm{GL}_n \subset \mathrm{gl}_n, \mathcal{F}^{\mathrm{gl}_n})$ to $(G \subset \mathcal{X}_\rho, \mathcal{F}^{\mathcal{X}_\rho})$.

Back to our problem

- Given $\rho^\flat : N_{G^\flat}(T^\flat) \to \mathrm{GL}_n$, we would like to transfert $(\mathrm{GL}_n \subset \mathrm{gl}_n, \mathcal{F}^{\mathrm{gl}_n})$ to $(G \subset \mathcal{X}_\rho, \mathcal{F}^{\mathcal{X}_\rho})$.

- $\mathcal{F}^{\mathrm{GL}_n}$:

$$
\begin{array}{ccc}
\mathcal{C}(\mathrm{gl}_n^F) & \xrightarrow{\mathcal{F}^{\mathrm{gl}_n}} & \mathcal{C}(\mathrm{gl}_n^F) \\
\big\uparrow & & \big\downarrow \text{Restriction} \\
\mathcal{C}(\mathrm{GL}_n^F) & \xrightarrow{\mathcal{F}^{\mathrm{GL}_n}} & \mathcal{C}(\mathrm{GL}_n^F)
\end{array}
$$

Back to our problem

- Given $\rho^\flat : N_{G^\flat}(T^\flat) \to \mathrm{GL}_n$, we would like to transfert $(\mathrm{GL}_n \subset \mathrm{gl}_n, \mathcal{F}^{\mathrm{gl}_n})$ to $(G \subset \mathcal{X}_\rho, \mathcal{F}^{\mathcal{X}_\rho})$.

- $\mathcal{F}^{\mathrm{GL}_n}$:

$$\begin{array}{ccc}
\mathcal{C}(\mathrm{gl}_n^F) & \xrightarrow{\quad \mathcal{F}^{\mathrm{gl}_n} \quad} & \mathcal{C}(\mathrm{gl}_n^F) \\
\Big\uparrow & & \Big\downarrow \text{Restriction} \\
\mathcal{C}(\mathrm{GL}_n^F) & \xrightarrow{\quad \mathcal{F}^{\mathrm{GL}_n} \quad} & \mathcal{C}(\mathrm{GL}_n^F)
\end{array}$$

- We first transfert $\mathcal{F}^{\mathrm{GL}_n}$ to $\mathcal{F}_\rho^G : \mathcal{C}(G^F) \to \mathcal{C}(G^F)$.

The map $\mathfrak{t}_\rho$

- Deligne-Lusztig theory :

$$\mathrm{Irr}(G^F) = \coprod_{(s)} \mathcal{E}_G(s),$$

where (s) runs over $F^\flat$-stable semisimple conjugacy classes of $G^\flat$.
$\mathcal{E}_G(s)$: Lusztig series.

The map $\mathfrak{t}_\rho$

- Deligne-Lusztig theory :

$$\mathrm{Irr}(G^F) = \coprod_{(s)} \mathcal{E}_G(s),$$

where (s) runs over $F^\flat$-stable semisimple conjugacy classes of $G^\flat$.
$\mathcal{E}_G(s)$: Lusztig series.

- $\rho^\flat : N_{G^\flat}(T^\flat) \to H^\flat$ (H arbitrary connected reductive group) induces a map from $F^\flat$-stable semisimple conjugacy classes of $G^\flat$ to $F^\flat$-stable semisimple conjugacy classes of $H^\flat$.

The map $\mathfrak{t}_\rho$

- Deligne-Lusztig theory :

$$\mathrm{Irr}(G^F) = \coprod_{(s)} \mathcal{E}_G(s),$$

  where (s) runs over $F^\flat$-stable semisimple conjugacy classes of $G^\flat$.
  $\mathcal{E}_G(s)$: Lusztig series.

- $\rho^\flat : N_{G^\flat}(T^\flat) \to H^\flat$ (H arbitrary connected reductive group) induces a map from $F^\flat$-stable semisimple conjugacy classes of $G^\flat$ to $F^\flat$-stable semisimple conjugacy classes of $H^\flat$.

  $\rightsquigarrow \mathfrak{t}_\rho : \{\text{Lusztig series of } G^F\} \to \{\text{Lusztig series of } H^F\}$ given by

$$\mathfrak{t}_\rho(\mathcal{E}_G(s)) = \mathcal{E}_H(\rho^\flat(s)).$$

Cases where $\mathfrak{t}_\rho$ is given by a functor

- If $\rho^\flat : G^\flat \to H^\flat$ is a *normal* morphism, then there exists

$$\rho : H \to G$$

and $\mathfrak{t}_\rho$ is given by the functor $\rho^* : \mathrm{Rep}(G^F) \to \mathrm{Rep}(H^F)$.

Cases where $\mathfrak{t}_\rho$ is given by a functor

- If $\rho^\flat : G^\flat \to H^\flat$ is a *normal* morphism, then there exists

$$\rho : H \to G$$

and $\mathfrak{t}_\rho$ is given by the functor $\rho^* : \mathrm{Rep}(G^F) \to \mathrm{Rep}(H^F)$.

Examples :

$\rho^\flat : \mathrm{SL}_n \subset \mathrm{GL}_n$, then $\rho : \mathrm{GL}_n \to \mathrm{PGL}_n$.
$\rho^\flat : \mathrm{GL}_1 \hookrightarrow \mathrm{GL}_n$, $\lambda \mapsto \lambda.\mathrm{I}_n$, then $\rho = \det : \mathrm{GL}_n \to \mathrm{GL}_1$.

Cases where $\mathfrak{t}_\rho$ is given by a functor

- If $\rho^\flat : G^\flat \to H^\flat$ is a *normal* morphism, then there exists

  $$\rho : H \to G$$

  and $\mathfrak{t}_\rho$ is given by the functor $\rho^* : \mathrm{Rep}(G^F) \to \mathrm{Rep}(H^F)$.

  Examples :

  $\rho^\flat : \mathrm{SL}_n \subset \mathrm{GL}_n$, then $\rho : \mathrm{GL}_n \to \mathrm{PGL}_n$.
  $\rho^\flat : \mathrm{GL}_1 \hookrightarrow \mathrm{GL}_n$, $\lambda \mapsto \lambda.\mathrm{I}_n$, then $\rho = \det : \mathrm{GL}_n \to \mathrm{GL}_1$.

- If $\rho^\flat : L^\flat \hookrightarrow H^\flat$ is the inclusion of a Levi, then $\mathfrak{t}_\rho$ is given by the Lusztig induction functor

  $$R_L^H : \mathrm{Rep}(L^F) \to \mathrm{Rep}(H^F).$$

Transfert of restricted Fourier transform

- Fact :

  $\mathcal{F}^{\mathrm{GL}_n} : \mathcal{C}(\mathrm{GL}_n^F) \to \mathcal{C}(\mathrm{GL}_n^F)^\iota$ isomorphism of $\mathrm{GL}_n^F \times \mathrm{GL}_n^F$-mod.

Transfert of restricted Fourier transform

- Fact :

  $\mathcal{F}^{\mathrm{GL}_n} : \mathcal{C}(\mathrm{GL}_n^F) \to \mathcal{C}(\mathrm{GL}_n^F)^\iota$ isomorphism of $\mathrm{GL}_n^F \times \mathrm{GL}_n^F$-mod.
- Recall

$$\mathcal{C}(\mathrm{GL}_n^F) \simeq \bigoplus_{\pi \in \widehat{\mathrm{GL}_n^F}} V_\pi \boxtimes V_\pi^\vee.$$

- Fact :

  $\mathcal{F}^{\mathrm{GL}_n} : \mathcal{C}(\mathrm{GL}_n^F) \to \mathcal{C}(\mathrm{GL}_n^F)^{\iota}$ isomorphism of $\mathrm{GL}_n^F \times \mathrm{GL}_n^F$-mod.
- Recall
$$\mathcal{C}(\mathrm{GL}_n^F) \simeq \bigoplus_{\pi \in \widehat{\mathrm{GL}_n^F}} V_\pi \boxtimes V_\pi^\vee .$$
- We get a *gamma* function $\gamma^{\mathrm{GL}_n} : \widehat{\mathrm{GL}_n^F} \to \overline{\mathbb{Q}}_\ell^\times$ such that
$$\mathcal{F}^{\mathrm{GL}_n}(x \boxtimes x^\vee) = \gamma^{\mathrm{GL}_n}(\pi)\,(x^\vee \boxtimes x)$$
for all $x \boxtimes x^\vee \in V_\pi \boxtimes V_\pi^\vee$.

Transfert of restricted Fourier transform

- Fact : γ^{GL_n} constant on Lusztig series of GL_n^F.

Transfert of restricted Fourier transform

- Fact : γ^{GL_n} constant on Lusztig series of GL_n^F.
- Recall that $\rho^\flat : N_{G^\flat}(T^\flat) \to \mathrm{GL}_n$ defines

$$\mathfrak{t}_\rho : \{\text{Lusztig series of } G^F\} \to \{\text{Lusztig series of } \mathrm{GL}_n^F\}$$

Transfert of restricted Fourier transform

- Fact : γ^{GL_n} constant on Lusztig series of GL_n^F.
- Recall that $\rho^\flat : N_{G^\flat}(T^\flat) \to \mathrm{GL}_n$ defines

$$\mathfrak{t}_\rho : \{\text{Lusztig series of } G^F\} \to \{\text{Lusztig series of } \mathrm{GL}_n^F\}$$

Hence we get

$$\gamma_\rho^G : \widehat{G^F} \to \overline{\mathbb{Q}}_\ell$$

given by $\gamma^{\mathrm{GL}_n} \circ \mathfrak{t}_\rho$.

Transfert of restricted Fourier transform

- Fact : γ^{GL_n} constant on Lusztig series of GL_n^F.
- Recall that $\rho^\flat : N_{G^\flat}(T^\flat) \to \mathrm{GL}_n$ defines

$$\mathfrak{t}_\rho : \{\text{Lusztig series of } G^F\} \to \{\text{Lusztig series of } \mathrm{GL}_n^F\}$$

  Hence we get

$$\gamma_\rho^G : \widehat{G^F} \to \overline{\mathbb{Q}}_\ell$$

  given by $\gamma^{\mathrm{GL}_n} \circ \mathfrak{t}_\rho$.
- $\rightsquigarrow$ isomorphism $\mathcal{F}_\rho^G : \mathcal{C}(G^F) \to \mathcal{C}(G^F)^\iota$ of $G^F \times G^F$-modules s.t.

$$\mathcal{F}_\rho^G(x \boxtimes x^\vee) = \gamma_\rho^G(\pi)\,(x^\vee \boxtimes x),$$

  for all $x \boxtimes x^\vee \in V_\pi \boxtimes V_\pi^\vee \subset \mathcal{C}(G^F)$.

  (Construction of $\mathcal{F}_\rho^G$ due to Braverman-Kazhdan.)

- **Problem** : compute explicitely $\phi_\rho^G \in \mathcal{C}_c(G^F)$ such that

$$\mathcal{F}_\rho^G(f)(y) = \sum_{x \in G^F} \phi_\rho^G(xy) f(y).$$

Explicit construction of exotic restricted Fourier kernel

- **Problem** : compute explicitely $\phi_\rho^G \in \mathcal{C}_c(G^F)$ such that

$$\mathcal{F}_\rho^G(f)(y) = \sum_{x \in G^F} \phi_\rho^G(xy) f(y).$$

- Note that abstractly, for all $g \in G^F$

$$\phi_\rho^G(g) = \sum_{\pi \in \widehat{G^F}} \gamma_\rho^G(\pi)\pi(1)\overline{\pi(g)}.$$

Explicit construction of exotic restricted Fourier kernel

- **Problem** : compute explicitely $\phi_\rho^G \in \mathcal{C}_c(G^F)$ such that

$$\mathcal{F}_\rho^G(f)(y) = \sum_{x \in G^F} \phi_\rho^G(xy) f(y).$$

- Note that abstractly, for all $g \in G^F$

$$\phi_\rho^G(g) = \sum_{\pi \in \widehat{G^F}} \gamma_\rho^G(\pi)\pi(1)\overline{\pi(g)}.$$

- If the image of $\rho^\flat : G^\flat \to \mathrm{GL}_n$ is normal in some F-stable Levi L of GL_n, then $\rho : L \to G$ and

$$\phi_\rho^G = \rho_!(\psi \circ \mathrm{Tr}) \text{ up to some explicit scalar}$$

Braverman-Kazhdan conjecture

- Recall $\rho^\flat : N_{G^\flat}(T^\flat) \to \mathrm{GL}_n$, put $L := C_{\mathrm{GL}_n}(\rho^\flat(T^\flat))$ Levi subgroup of GL_n.

Braverman-Kazhdan conjecture

- Recall $\rho^\flat : N_{G^\flat}(T^\flat) \to \mathrm{GL}_n$, put $L := C_{\mathrm{GL}_n}(\rho^\flat(T^\flat))$ Levi subgroup of GL_n.

  Then $\rho^\flat : T^\flat \to L$ normal and so we have

$$\rho : L \to T.$$

Braverman-Kazhdan conjecture

- Recall $\rho^\flat : N_{G^\flat}(T^\flat) \to \mathrm{GL}_n$, put $L := C_{\mathrm{GL}_n}(\rho^\flat(T^\flat))$ Levi subgroup of GL_n.

  Then $\rho^\flat : T^\flat \to L$ normal and so we have

$$\rho : L \to T.$$

- Artin-Schreier sheaf : $h : \mathbb{A}^1 \to \mathbb{A}^1$, $x \mapsto x^q - x$ Galois covering with Galois group $\mathbb{F}_q$.

$$\mathcal{L}_\psi := \text{subsheaf of } h_*(\overline{\mathbb{Q}}_\ell) \text{ on which } \mathbb{F}_q \text{ acts as } \psi^{-1}.$$

  Consider $\Phi^L := \mathrm{Tr}^*(\mathcal{L}_\psi)$ with $\mathrm{Tr} : L \to \mathbb{A}_1$, and put

$$\Phi^T_\rho := \rho_! \Phi^L[\dim L](\dim V_L) \in \mathcal{D}^b_c(T)^F.$$

- **Proposition** [Braverman-Kazhdan, Cheng-Ngô, Laumon-L.] If $\rho : L \to T$ is surjective + positivity assumptions on weights of $\rho^\flat$, then Φ^T_ρ is an irreducible perverse smooth ℓ-adic sheaf on T.

- **Proposition** [Braverman-Kazhdan, Cheng-Ngô, Laumon-L.] If $\rho : L \to T$ is surjective + positivity assumptions on weights of $\rho^\flat$, then Φ_ρ^T is an irreducible perverse smooth ℓ-adic sheaf on T.
- We have geometric induction :

$$\mathrm{Ind}_T^G : \mathcal{D}_c^b(T) \to \mathcal{D}_c^b(G).$$

given by $(pr_2)_! \circ (pr_1)^*[\dim G - \dim T]$ where

$$T \longleftarrow \{(t, x, g) \in T \times G \times G/B \mid g^{-1}xg \in tU\} \longrightarrow G \ .$$

Braverman-Kazhdan conjecture

- **Proposition** [Braverman-Kazhdan, Cheng-Ngô, Laumon-L.] If $\rho : L \to T$ is surjective + positivity assumptions on weights of $\rho^\flat$, then Φ_ρ^T is an irreducible perverse smooth ℓ-adic sheaf on T.
- We have geometric induction :

$$\mathrm{Ind}_T^G : \mathcal{D}_c^b(T) \to \mathcal{D}_c^b(G).$$

given by $(pr_2)_! \circ (pr_1)^*[\dim G - \dim T]$ where

$$T \longleftarrow \{(t, x, g) \in T \times G \times G/B \,|\, g^{-1}xg \in tU\} \longrightarrow G \ .$$

- **Theorem** [Lusztig] Under above assumptions, $\mathrm{Ind}_T^G(\Phi_\rho^T)$ is a semisimple perverse sheaf on which $W := W_G(T)$ acts.

Braverman-Kazhdan conjecture

- **Conjecture** [Braverman-Kazhdan, 2002] The characteristic function of $\mathrm{Ind}_T^G(\Phi_\rho^T)^W$ equals ϕ_ρ^G.

Braverman-Kazhdan conjecture

- **Conjecture** [Braverman-Kazhdan, 2002] The characteristic function of $\mathrm{Ind}_T^G(\Phi_\rho^T)^W$ equals ϕ_ρ^G.

  (Proved by BK if $\rho^\flat : G^\flat = \mathrm{GL}_m \to \mathrm{GL}_n$).

Braverman-Kazhdan conjecture

- **Conjecture** [Braverman-Kazhdan, 2002] The characteristic function of $\mathrm{Ind}_T^G(\Phi_\rho^T)^W$ equals ϕ_ρ^G.

  (Proved by BK if $\rho^\flat : G^\flat = \mathrm{GL}_m \to \mathrm{GL}_n$).

- **Theorem** [Laumon-L., 2018] BK conjecture is true.

- **Conjecture** [Braverman-Kazhdan, 2002] The characteristic function of $\mathrm{Ind}_T^G(\Phi_\rho^T)^W$ equals ϕ_ρ^G.
  (Proved by BK if $\rho^\flat : G^\flat = \mathrm{GL}_m \to \mathrm{GL}_n$).
- **Theorem** [Laumon-L., 2018] BK conjecture is true.
  In fact we prove (without any assumptions on ρ) that

$$\phi_\rho^G = \frac{1}{|W|} \sum_{w \in W} R_{T_w}^G(\phi_\rho^{T_w}),$$

  where $R_{T_w}^G$ is Deligne-Lusztig induction and $\phi_\rho^{T_w}$ is push forward of ϕ^{L_v} for some $v \in N_{\mathrm{GL}_n}(L)$ defined from w.

Extending $\mathcal{F}_\rho^G$ to involutive Fourier ?

- Reduces to extending $\mathcal{F}_\rho^G : \mathcal{C}(G^F)_{(1)} \to \mathcal{C}(G^F)_{(1)}$ (by Jordan decomp.).

Extending $\mathcal{F}_\rho^G$ to involutive Fourier ?

- Reduces to extending $\mathcal{F}_\rho^G : \mathcal{C}(G^F)_{(1)} \to \mathcal{C}(G^F)_{(1)}$ (by Jordan decomp.).
- **Theorem** [Laumon, L.] Explicit construction for all representations $\rho^\flat : N_{G^\flat}(T^\flat) \to \mathrm{GL}_n$ with $G = \mathrm{GL}_2$ or GL_3.

Extending $\mathcal{F}_\rho^G$ to involutive Fourier?

- Reduces to extending $\mathcal{F}_\rho^G : \mathcal{C}(G^F)_{(1)} \to \mathcal{C}(G^F)_{(1)}$ (by Jordan decomp.).
- **Theorem** [Laumon, L.] Explicit construction for all representations $\rho^\flat : N_{G^\flat}(T^\flat) \to \mathrm{GL}_n$ with $G = \mathrm{GL}_2$ or GL_3.
- Assume that the image of $\rho^\flat : G^\flat \to \mathrm{GL}_n$ is normal in some Levi L, then we have $\rho : L \to G$.

Proposition [Laumon, L.]

$$\mathcal{X}_\rho = [(\mathrm{Lie}(L) \times G)/L] \; (= [\mathrm{Lie}(L)/\mathrm{Ker}(\rho)] \text{ if } \rho \text{ surjective}).$$

where the action is given by $(x, g) \cdot h = (xh, \rho(h)^{-1} g)$.

Extending $\mathcal{F}_\rho^G$ to involutive Fourier ?

- Diag. embedding $\rho^\flat : \mathrm{GL}_n \hookrightarrow \mathrm{GL}_n \times \mathrm{GL}_n \subset \mathrm{GL}_{2n}$. Then

$$m : \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_n, (x, y) \mapsto xy.$$

extends $\rho : \mathrm{T}_n \times \mathrm{T}_n \to \mathrm{T}_n$.

Extending $\mathcal{F}_\rho^G$ to involutive Fourier ?

- Diag. embedding $\rho^\flat : \mathrm{GL}_n \hookrightarrow \mathrm{GL}_n \times \mathrm{GL}_n \subset \mathrm{GL}_{2n}$. Then

$$m : \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_n, (x, y) \mapsto xy.$$

extends $\rho : \mathrm{T}_n \times \mathrm{T}_n \to \mathrm{T}_n$.

Theorem [Laumon, L.]

$$\mathcal{X}_\rho = [\mathrm{gl}_n \times \mathrm{gl}_n / \mathrm{GL}_n],$$

with action $(x, y) \cdot g = (xg, g^{-1}y)$.

Extending $\mathcal{F}_\rho^G$ to involutive Fourier ?

- Diag. embedding $\rho^\flat : \mathrm{GL}_n \hookrightarrow \mathrm{GL}_n \times \mathrm{GL}_n \subset \mathrm{GL}_{2n}$. Then

$$m : \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_n, (x, y) \mapsto xy.$$

extends $\rho : \mathrm{T}_n \times \mathrm{T}_n \to \mathrm{T}_n$.

Theorem [Laumon, L.]

$$\mathcal{X}_\rho = [\mathrm{gl}_n \times \mathrm{gl}_n/\mathrm{GL}_n],$$

with action $(x, y) \cdot g = (xg, g^{-1}y)$.

Fourier kernel :

$$\phi^{\mathcal{X}_\rho}([x', x'']) = \sum_{z \in \mathrm{GL}_n^F} \psi(\mathrm{Tr}(x'z^{-1} + zx'')).$$

Extending $\mathcal{F}_\rho^G$ to involutive Fourier?

- $\rho^\flat : \mathrm{PGL}_2 \to \mathrm{GL}_3$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{ad - bc} \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2cd \\ c^2 & cd & d^2 \end{pmatrix}.$$

Extending $\mathcal{F}_\rho^G$ to involutive Fourier ?

- $\rho^\flat : \mathrm{PGL}_2 \to \mathrm{GL}_3$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{ad-bc} \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2cd \\ c^2 & cd & d^2 \end{pmatrix}.$$

At the Level of tori we have

$$\rho^\flat : \overline{T}_2 \to T_3, \ \ (a,b) \mapsto (a/b, 1, b/a)$$

which dualizes to

$$\rho : T_3 \to T_2' \simeq \{(t,\delta) \in T_2 \times \mathbb{A}^1 \mid \det(t) = \delta^2\}/\mathrm{GL}_1,$$

given by $(a,b,c) \mapsto (a/c, c/a) \mapsto [a^2, c^2]$. This induces a bijective morphism

$$[\mathrm{Lie}(T_3)/\mathrm{Ker}(\rho)] \to [\{(t,\delta) \in \mathrm{Lie}(T_2) \times \mathbb{A}^1 \mid \det(t) = \delta^2\}/\mathrm{GL}_1].$$

Theorem [Laumon, L.]

$$\mathcal{X}_\rho = [\{(x,\delta) \in \mathrm{gl}_2 \times \mathrm{gl}_1 \mid \det(x) = \delta^2\}/\mathrm{GL}_1].$$

Extending $\mathcal{F}_\rho^G$ to involutive Fourier ?

Theorem [Laumon, L.]

$$\mathcal{X}_\rho = [\{(x,\delta) \in \mathrm{gl}_2 \times \mathrm{gl}_1 \mid \det(x) = \delta^2\}/\mathrm{GL}_1].$$

The Fourier kernel is given by

$$\begin{aligned} \phi^{\mathcal{X}_\rho}([x,\delta]) &= \sum_{s \in \mathrm{GL}_1} \psi(s(\mathrm{Tr}(x+2\delta))) \\ &= \begin{cases} q-1 & \text{if } \mathrm{Tr}(x+2\delta) = 0 \\ -1 & \text{otherwise.} \end{cases}. \end{aligned}$$