# Linear characters of Sylow subgroups of the symmetric group

Stacey Law

University of Cambridge

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- Isaacs–Malle–Navarro (2007): reduction to simple groups;
- Malle–Späth (2016): all $G$, $p = 2$.

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*Let* $p$ *be odd,* $P \in \mathrm{Syl}_p(G)$*. Suppose* $P = N_G(P)$*. There is a bijection*

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- $\mathrm{Irr}_{p'}(P) = \mathrm{Lin}(P) := \{\text{linear characters of } P\}$

- Navarro–Tiep–Vallejo false for $p = 2$, e.g. $G = S_5$.
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*For $n \in \mathbb{N}$, let $P_n \in \mathrm{Syl}_2(S_n)$. When $n = 2^k$, there is a bijection*

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*Moreover, for all $n \in \mathbb{N}$ and all $\chi \in \mathrm{Irr}(S_n)$, $\chi\big\downarrow_{P_n}$ has a linear constituent.*

- If $\chi(1) > 1$, then $\chi\big\downarrow_{P_n}$ has a *unique* linear constituent if and only if $n = 2^k$ and $\chi(1)$ is odd.

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- So for <u>all</u> $n$, $p$ and <u>all</u> $\chi \in \mathrm{Irr}(S_n)$, $\chi\downarrow_{P_n}$ has a linear constituent.

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**Question**

*Given $\chi \in \mathrm{Irr}(S_n)$, what are the linear constituents of $\chi\big\downarrow_{P_n}$?*

## Aim

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## Theorem (Giannelli, L.)

*Let* $n \in \mathbb{N}_{>10}$, *p odd. Let* $\chi^\lambda \in \mathrm{Irr}(S_n)$. *Then* $\mathbb{1}_{P_n} \nmid \chi^\lambda\big\downarrow_{P_n}$ *if and only if* $n = p^k$, $k \in \mathbb{N}$ *and* $\lambda \in \{(p^k-1,1),(2,1^{p^k-2})\}$.

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- So *almost all* $\chi\big\downarrow_{P_n}$ contain the trivial $\mathbb{1}_{P_n}$ as a constituent.

◇ **Corollary**: determined $\mathrm{Irr}(\mathcal{H})$ for Hecke algebra $\mathcal{H}$ associated with permutation character $\mathbb{1}_{P_n}\big\uparrow^{S_n}$.
$[\mathcal{H} = \mathcal{H}(S_n, P_n, \mathbb{1}_{P_n}) = e\mathbb{C}S_n e,\ \ e = \frac{1}{|P_n|}\sum_{h\in P_n} h]$

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- **Question:** is there a combinatorial description of the multiplicity map
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## Theorem (L.)

*Determined precisely which $\chi \in \mathrm{Irr}(S_n)$ satisfy $\langle \chi\big\downarrow_{P_n}, \mathbb{1}_{P_n}\rangle = 1$, and hence determined the linear characters of $\mathcal{H}$, for $p$ odd.*

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- $p = 2$ much more chaotic:
  e.g. $p = 2$, $\mathbb{1}_{P_n} \nmid sign\big\downarrow_{P_n}$ vs. $p$ odd, $\mathbb{1}_{P_n} \mid \chi\big\downarrow_{P_n}\ \forall \chi \in \mathrm{Irr}(S_n)$ if $n \neq p^k$.

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- Let $\mathcal{B}_n(t) = \{\chi^\lambda \in \mathrm{Irr}(S_n)\ :\ \lambda_1, l(\lambda) \leq t\}$.

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*To determine* $\Omega(\phi) := \{\chi \in \mathrm{Irr}(S_n) : \chi \mid \phi\uparrow^{S_n}\}$*, for every n, p and* $\phi \in \mathrm{Lin}(P_n)$.

- Know $\Omega(\mathbb{1})\ \forall\ n,$ odd $p$: almost all of $\mathrm{Irr}(S_n)$.
- **Proposition:** $p$ odd, $\Omega_n := \bigcap_{\phi\in\mathrm{Lin}(P_n)} \Omega(\phi)$. Then $\lim_{n\to\infty} \frac{|\Omega_n|}{|\mathrm{Irr}(S_n)|} = 1$.
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## Theorem (Giannelli, L., Long)

*Let* $n \in \mathbb{N}$, *p odd* $\geq 5$ *and* $\phi \in \mathrm{Lin}(P_n)$. *We determine* $m(\phi)$ *and* $M(\phi)$, *where*

$$m(\phi) = \max\{t : \mathcal{B}_n(t) \subseteq \Omega(\phi)\} \quad \text{and} \quad M(\phi) = \min\{t : \Omega(\phi) \subseteq \mathcal{B}_n(t)\}.$$

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## Proposition

*When* $n = p^k$*,* $\Omega(\phi) = \Omega(\psi) \implies \phi\uparrow^{S_n} = \psi\uparrow^{S_n}$*.*