Categorification of $\mathbb{Z}$-modular data associated to complex reflection groups

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(M4) for all $i, j, k \in I$,

$$N_{i,j}^k := \sum_{l \in I} \frac{S_{l,i} S_{l,j} \overline{S_{l,k}}}{S_{l,i_0}} \in \mathbb{N} \text{ (resp. } \mathbb{Z}\text{).}$$

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The integers $(N_{i,j}^{k})_{i,j,k\in I}$ are the structure constants of a free associative $\mathbb{Z}$-algebra of rank $|I|$, called fusion algebra associated with the modular data:

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A categorical answer

Using monoidal categories with extra structures.

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braiding + pivotal structure

$\rightsquigarrow$ twist $\theta_X : X \to X$ such that $\theta_{X \otimes Y} = \theta_X \otimes \theta_Y \circ c_{Y,X} \circ c_{X,Y}$.

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Definition

$\mathcal{C}$ is non-degenerate if the symmetric center is generated by $\mathbf{1}$.

Non-degenerate categories

Theorem (Turaev)

Suppose that $\mathcal{C}$ is non-degenerate.

- $I = \mathrm{Irr}(\mathcal{C})$,
- $S_{X,Y} = \lambda \operatorname{tr}^{+}_{X\otimes Y}(c_{Y,X} \circ c_{X,Y})$,
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Question

What is a categorical version of $\mathbb{Z}$-modular data?

Slightly degenerate categories

Suppose that the symmetric center of $\mathcal{C}$ is equal to sVect, the braided category of finite dimensional superspaces: $\mathcal{C}$ is *slightly degenerate*.

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The matrices S and T defined before are

$$S = \begin{pmatrix} \mathbf{S} & -\mathbf{S} \\ -\mathbf{S} & \mathbf{S} \end{pmatrix} \qquad T = \begin{pmatrix} \mathbf{T} & 0 \\ 0 & \mathbf{T} \end{pmatrix}.$$

$\mathbb{Z}$-modular data from slightly degenerate categories

Tensorization by ε induces an action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{Irr}(\mathcal{C})$ without fixed points

$\rightsquigarrow$ choose J a system of representatives such that $\mathbf{1} \in J$.

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Theorem (L.)

Suppose that $\mathcal{C}$ is slightly degenerate and that ε is of dimension -1.

- $I = J$,
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The fusion algebra is $\mathrm{Gr}(\mathcal{C})/([\varepsilon] + [\mathbf{1}])$, the structure constants are

$$sN^{Z}_{X,Y} = N^{Z}_{X,Y} - N^{\varepsilon\otimes Z}_{X,Y}.$$

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A $\mathbb{N}$-modular datum $(\mathcal{F}, S(\mathcal{F}), T(\mathcal{F}), \rho_{\mathrm{sp}})$ is associated with each family $\mathcal{F}$ of unipotent characters of G^F.

A hope? Spetses

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- Malle: $G(d, 1, n)$ and $G(d, d, n)$,
- Broué-Malle-Michel: some exceptional complex reflection groups.

The group $G(d,1,n)$

$W = G(d,1,n)$. There is an extension of Lusztig's combinatorics of symbols in type B describing unipotent characters of $G(d,1,n)$

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Theorem (Malle,Cuntz)

Let $\mathcal{F}$ be a family of unipotent characters of $G(d, 1, n)$, $S(\mathcal{F})$ its Fourier matrix and $T(\mathcal{F})$ the diagonal matrix of eigenvalues of the Frobenius. There is $f_{\mathrm{sp}} \in \mathcal{F}$ such that $(\mathcal{F}, S(\mathcal{F}), T(\mathcal{F}), f_{\mathrm{sp}})$ is a $\mathbb{Z}$-modular datum.

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Question

Is there a categorification of these data?

A family of $G(d, 1, n(n+1)/2)$

n, d be an integers with $d \geq n+1$. There is a family $\mathcal{F}$ with $d^{n-1}\binom{d}{n+1}$ unipotent characters.

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Now we generalize this result for any n, in the framework of slightly degenerate categories.

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$+$ other relations (quantum Serre relations, ...)

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$+$ other relations (quantum Serre relations, ...)

$z_i = K_iL_i^{-1}$ is central and $\mathcal{D}_q/(z_i - 1)_{1\leq i\leq n} \simeq \mathcal{U}_q(\mathfrak{sl}_{n+1})$.

Representations of $\mathcal{D}_q$

- $\Phi \subset V$ the root system of type A_n,
- $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ a set of simple roots,
- $\langle \cdot, \cdot \rangle$ a symmetric bilinear form such that $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in \Phi$.

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- $\Delta = \{\alpha_1, \dots, \alpha_n\}$ a set of simple roots,
- $\langle \cdot, \cdot \rangle$ a symmetric bilinear form such that $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in \Phi$.
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Fact

For each $M \in \mathcal{C}_q$, $z_i = K_i L_i^{-1}$ acts by a power of q^2.

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Proposition

The category $\mathcal{C}_q$ is semisimple and there is a bijection

$$\{(\lambda, \mu) \in P \times P \mid \lambda + \mu \in 2P^+\} \xrightarrow{\sim} \left\{ \begin{matrix} \textit{isomorphism classes of} \\ \textit{simple objects in } \mathcal{C}_q \end{matrix} \right\}$$

$$(\lambda, \mu) \longmapsto L(\lambda, \mu)$$

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Working with q a root of unity will truncate this category.

A version of $\mathcal{D}_q$ at a root of unity

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The algebra $\mathcal{D}_\xi$ is still a Hopf algebra and has a quasi-R-matrix.

Representation theory of $\mathcal{D}_\xi$

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Remark

The category $\mathcal{C}_\xi$ is not semisimple.

Tilting modules

For each $(\lambda, \mu) \in P \times P$ with $\lambda + \mu \in 2P^+$, there is a $\mathcal{D}_q^{\mathrm{res}}(\mathfrak{sl}_{n+1})$ form $L^{\mathrm{res}}(\lambda, \mu)$ of $L(\lambda, \mu)$

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Theorem (Andersen)

Tensor product of tilting modules is a tilting module.

Semisimplification

$\mathcal{C}$ a pivotal monoidal category (+ other technical assumptions)

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Proposition

Isomorphism classes of simple objects in $\mathbb{Z}(\mathcal{T}_\xi)$ are in bijection with pairs $(\lambda, \mu) \in P \times P$ with $\lambda + \mu \in 2C$ and $\mu \in Q$ where

$$C = \{\eta \in P^+ \mid \langle \eta, \theta_0 \rangle \leq d - (n+1)\},$$

θ_0 being the longest root of Φ.

Relation with the modular data of $G(d,1,n(n+1)/2)$

$\mathbb{Z}(\mathcal{T}_\xi)$ has a lot of invertible objects which are in the symmetric center. If I is such an object of quantum dimension 1, add an isomorphism between X and $X \otimes I$ for any X. Denote by $\mathbb{Z}(\mathcal{T}_\xi) \rtimes \mathcal{S}$ the category with these additional isomorphisms.

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In any cases, the $\mathbb{Z}$-modular datum defined by $\mathbb{Z}(\mathcal{T}_\xi) \rtimes \mathcal{S}$ coincide with the modular datum associated with the family $\mathcal{F}$ of $G\left(d, 1, \frac{n(n+1)}{2}\right)$ defined before.

A conjecture of Cuntz

Corollary

If we replace $N_{f,g}^{h}$ *in the definition of the fusion algebra* $A_{\mathcal{F}}$ *by* $|N_{f,g}^{h}|$*, we obtain an associative algebra* $A_{\mathcal{F}}^{\mathrm{abs}}$*.*

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If n is odd, we have a commutative diagram

$$\begin{array}{ccc} & \mathrm{Gr}(\mathcal{C}) & \\ \swarrow & & \searrow \\ A_{\mathcal{F}}^{\mathrm{abs}} \simeq \mathrm{Gr}(\mathcal{C})/([\mathbf{1}] - [\varepsilon]) & & \mathrm{Gr}(\mathcal{C})/([\mathbf{1}] + [\varepsilon]) \simeq A_{\mathcal{F}}. \end{array}$$

with $\mathcal{C} = \mathbb{Z}(\mathcal{T}_\xi) \rtimes \mathcal{S}$.

Thank you for your attention!