Balanced system of cell representations in affine Hecke algebras and Lusztig conjectures

Jérémie Guilhot (University of Tours)

joint work with James Parkinson (University of Sydney)

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Note: no affine Weyl group such that $w_0 \neq -\mathrm{Id}$ were harmed during the making of this talk

Coxeter Group, weight functions and Hecke algebras

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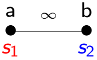
  Example: $\tilde{A}_1$: 

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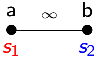
  Example: $\tilde{A}_1$: 

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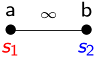
  Example: $\tilde{A}_1$: 

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$$T_w T_s = \begin{cases} T_{ws} & \text{if } \ell(ws) > \ell(w) \\ T_{ws} + (\mathrm{q}^{L(s)} - \mathrm{q}^{-L(s)}) T_w & \text{if } \ell(ws) < \ell(w) \end{cases}$$

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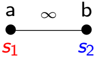
  Example: $\tilde{A}_1$: 

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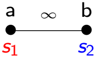
  Example: $\tilde{A}_1$: 

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$\rightsquigarrow$ we have$(T_s - q^{L(s)})(T_s + q^{-L(s)}) = 0$

Kazhdan-Lusztig basis

The $\bar{\ }$-involution $q \to q^{-1}$ on R extends to $\mathcal{H}$:

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Example: If $s \in S$ with $L(s) = a$ we have $C_s = T_s + \mathrm{q}^{-a}$. Indeed

$$\overline{C}_s = \overline{T}_s + \overline{\mathrm{q}^{-a}} = T_s^{-1} + \mathrm{q}^a = (T_s - (\mathrm{q}^a - \mathrm{q}^{-a})) + \mathrm{q}^a = C_s$$

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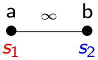
Example: back to $\tilde{A}_1$: 

Group of reflection in 1-dim space

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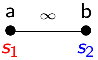
Example: back to $\tilde{A}_1$: 

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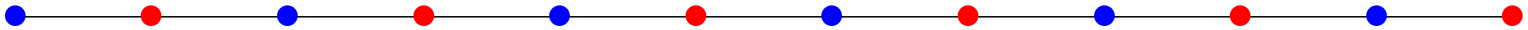
$a > b$ 

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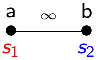
Example: back to $\tilde{A}_1$: 

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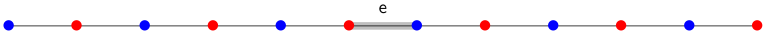

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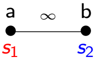
Example: back to $\tilde{A}_1$: 

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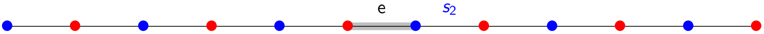
$a > b$ 

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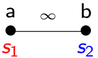
Example: back to $\tilde{A}_1$: 

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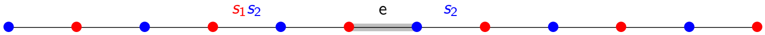
$a > b$ 

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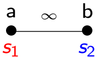
Example: back to $\tilde{A}_1$: 

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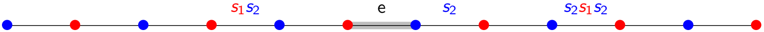
$a > b$ 

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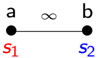
Example: back to $\tilde{A}_1$: 

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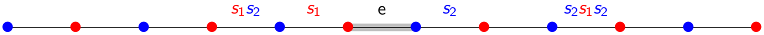
$a > b$ 

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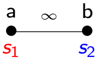
Example: back to $\tilde{A}_1$: 

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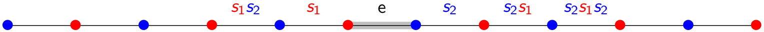
$a > b$ 

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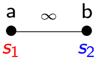
Example: back to $\tilde{A}_1$: 

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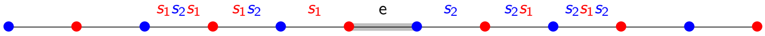
$a > b$ 

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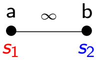
Example: back to $\tilde{A}_1$: 

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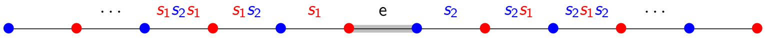
$a > b$ 

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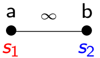
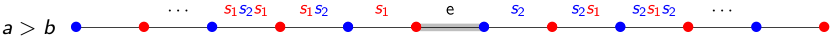
Example: back to $\tilde{A}_1$: 

Group of reflection in 1-dim space



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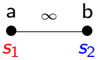
Example: back to $\tilde{A}_1$: 

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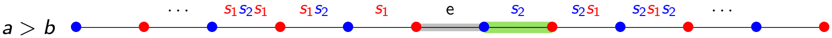

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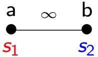
Example: back to $\tilde{A}_1$: 

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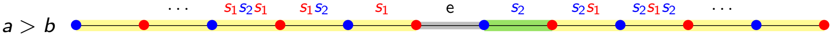

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Careful! One may have $h_{x,y,z} \neq 0$ with $\gamma_{x,y,z^{-1}} = 0$

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P9. If $z' \leq_{\mathcal{L}} z$ and $\mathbf{a}(z') = \mathbf{a}(z)$, then $z' \sim_{\mathcal{L}} z$

P14. For each $z \in W$ we have $z \sim_{\mathcal{LR}} z^{-1}$.

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Theorem. (G.,Parkinson 2018)

Lusztig conjectures **P1–P15** holds in affine Weyl groups of rank 2 for any choices of parameters.

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Theorem. If such a system exists then $\mathbf{a}_\Gamma = \mathbf{a}(\Gamma)$ for all $\Gamma \in \Lambda$

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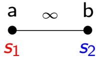

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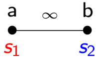

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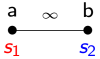

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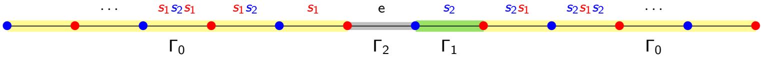

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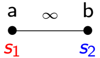

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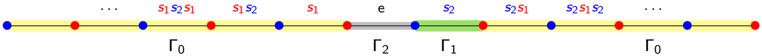
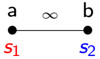


Consider the 4 one dimensional representations of $\mathcal{H}(\tilde{A}_1)$ where $\tilde{A}_1$:



$$\rho_{\emptyset} : \quad T_{s_1} \longrightarrow -\mathsf{q}^{-a}, \quad T_{s_2} \longrightarrow -\mathsf{q}^{-b}$$

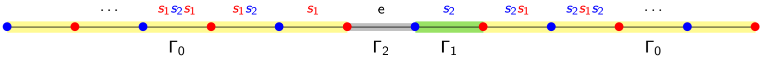
$$\rho_{\{2\}} : \quad T_{s_1} \longrightarrow -\mathsf{q}^{-a}, \quad T_{s_2} \longrightarrow \mathsf{q}^{b}$$

$$\rho_{\{1\}} : \quad T_{s_1} \longrightarrow \mathsf{q}^{a}, \quad T_{s_2} \longrightarrow -\mathsf{q}^{-b}$$

$$\rho_{\{1,2\}} : \quad T_{s_1} \longrightarrow \mathsf{q}^{a}, \quad T_{s_2} \longrightarrow \mathsf{q}^{b}$$

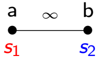
Recall that

$a > b$



We have $\deg(\rho_{\emptyset}(T_w)) < 0$ for all $w \neq \mathsf{e}$ and $\deg(\rho_{\emptyset}(T_{\mathsf{e}})) = 0$:

Consider the 4 one dimensional representations of $\mathcal{H}(\tilde{A}_1)$ where $\tilde{A}_1$:

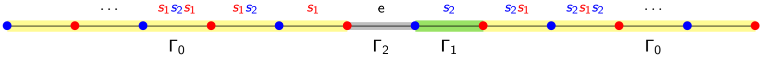


$$\rho_\emptyset : \begin{array}{ccc} T_{s_1} & \longrightarrow & -\mathsf{q}^{-a} \\ T_{s_2} & \longrightarrow & -\mathsf{q}^{-b} \end{array} \qquad \rho_{\{2\}} : \begin{array}{ccc} T_{s_1} & \longrightarrow & -\mathsf{q}^{-a} \\ T_{s_2} & \longrightarrow & \mathsf{q}^{b} \end{array}$$

$$\rho_{\{1\}} : \begin{array}{ccc} T_{s_1} & \longrightarrow & \mathsf{q}^{a} \\ T_{s_2} & \longrightarrow & -\mathsf{q}^{-b} \end{array} \qquad \rho_{\{1,2\}} : \begin{array}{ccc} T_{s_1} & \longrightarrow & \mathsf{q}^{a} \\ T_{s_2} & \longrightarrow & \mathsf{q}^{b} \end{array}$$

Recall that

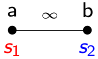
$a > b$



We have $\deg(\rho_\emptyset(T_w)) < 0$ for all $w \neq \mathrm{e}$ and $\deg(\rho_\emptyset(T_\mathrm{e})) = 0$:

$\rho_\emptyset$ is bounded by 0 and Γ_2-balanced

Consider the 4 one dimensional representations of $\mathcal{H}(\tilde{A}_1)$ where $\tilde{A}_1$:



$$\rho_{\emptyset}: \quad T_{s_1} \longrightarrow -\mathsf{q}^{-a}, \quad T_{s_2} \longrightarrow -\mathsf{q}^{-b}$$

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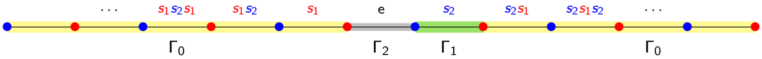
Recall that

$a > b$

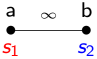


We have $\deg(\rho_{\emptyset}(T_w)) < 0$ for all $w \neq \mathsf{e}$ and $\deg(\rho_{\emptyset}(T_{\mathsf{e}})) = 0$:

$\rho_{\emptyset}$ is bounded by 0 and Γ_2-balanced

Next we look at $\max\{\deg_{\mathsf{q}}\left(\rho_{\{2\}}(T_w)\right)\}$ (in the case $a - b > 0$)

Consider the 4 one dimensional representations of $\mathcal{H}(\tilde{A}_1)$ where $\tilde{A}_1$:



$$\rho_{\emptyset}: \quad T_{s_1} \longrightarrow -\mathsf{q}^{-a}, \quad T_{s_2} \longrightarrow -\mathsf{q}^{-b}$$

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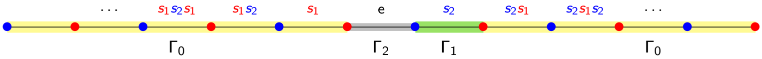
$$\rho_{\{1,2\}}: \quad T_{s_1} \longrightarrow \mathsf{q}^{a}, \quad T_{s_2} \longrightarrow \mathsf{q}^{b}$$

Recall that

$a > b$



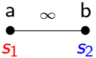
We have $\deg(\rho_{\emptyset}(T_w)) < 0$ for all $w \neq e$ and $\deg(\rho_{\emptyset}(T_e)) = 0$:

$\rho_{\emptyset}$ is bounded by 0 and Γ_2-balanced

Next we look at $\max\{\deg_{\mathsf{q}}\left(\rho_{\{2\}}(T_w)\right)\}$ (in the case $a - b > 0$)

$$(s_1 s_2)^n \qquad (s_2 s_1)^n \qquad (s_1 s_2)^n s_1 \qquad (s_2 s_1)^n s_2$$

Consider the 4 one dimensional representations of $\mathcal{H}(\tilde{A}_1)$ where $\tilde{A}_1$:



$$\rho_\emptyset : \quad T_{s_1} \longrightarrow -\mathsf{q}^{-a}, \quad T_{s_2} \longrightarrow -\mathsf{q}^{-b}$$

$$\rho_{\{2\}} : \quad T_{s_1} \longrightarrow -\mathsf{q}^{-a}, \quad T_{s_2} \longrightarrow \mathsf{q}^{b}$$

$$\rho_{\{1\}} : \quad T_{s_1} \longrightarrow \mathsf{q}^{a}, \quad T_{s_2} \longrightarrow -\mathsf{q}^{-b}$$

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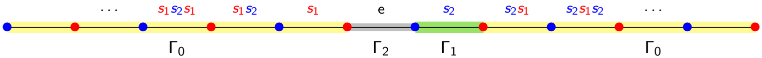
Recall that

$a > b$



We have $\deg(\rho_\emptyset(T_w)) < 0$ for all $w \neq \mathsf{e}$ and $\deg(\rho_\emptyset(T_\mathsf{e})) = 0$:

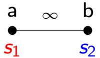
$\rho_\emptyset$ is bounded by 0 and Γ_2-balanced

Next we look at $\max\{\deg_\mathsf{q}(\rho_{\{2\}}(T_w))\}$ (in the case $a - b > 0$)

$$(s_1 s_2)^n \qquad (s_2 s_1)^n \qquad (s_1 s_2)^n s_1 \qquad (s_2 s_1)^n s_2$$

$(s_1 s_2)^n \longrightarrow n(b - a)$

Consider the 4 one dimensional representations of $\mathcal{H}(\tilde{A}_1)$ where $\tilde{A}_1$:



$$\rho_\emptyset : \quad T_{s_1} \longrightarrow -\mathrm{q}^{-a}, \quad T_{s_2} \longrightarrow -\mathrm{q}^{-b}$$

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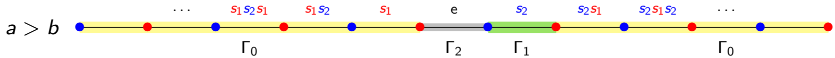
$$\rho_{\{1,2\}} : \quad T_{s_1} \longrightarrow \mathrm{q}^{a}, \quad T_{s_2} \longrightarrow \mathrm{q}^{b}$$

Recall that



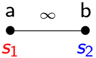
We have $\deg(\rho_\emptyset(T_w)) < 0$ for all $w \neq \mathrm{e}$ and $\deg(\rho_\emptyset(T_\mathrm{e})) = 0$:

$\rho_\emptyset$ is bounded by 0 and Γ_2-balanced

Next we look at $\max\{\deg_\mathrm{q}\left(\rho_{\{2\}}(T_w)\right)\}$ (in the case $a - b > 0$)

$$(s_1 s_2)^n \longrightarrow n(b-a) \qquad (s_2 s_1)^n \longrightarrow n(b-a) \qquad (s_1 s_2)^n s_1 \qquad (s_2 s_1)^n s_2$$

Consider the 4 one dimensional representations of $\mathcal{H}(\tilde{A}_1)$ where $\tilde{A}_1$:



$$\rho_{\emptyset} : \quad T_{s_1} \longrightarrow -\mathsf{q}^{-a}, \quad T_{s_2} \longrightarrow -\mathsf{q}^{-b}$$

$$\rho_{\{2\}} : \quad T_{s_1} \longrightarrow -\mathsf{q}^{-a}, \quad T_{s_2} \longrightarrow \mathsf{q}^{b}$$

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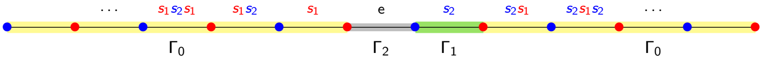
$$\rho_{\{1,2\}} : \quad T_{s_1} \longrightarrow \mathsf{q}^{a}, \quad T_{s_2} \longrightarrow \mathsf{q}^{b}$$

Recall that

$a > b$



We have $\deg(\rho_{\emptyset}(T_w)) < 0$ for all $w \neq \mathrm{e}$ and $\deg(\rho_{\emptyset}(T_{\mathrm{e}})) = 0$:

$\rho_{\emptyset}$ is bounded by 0 and Γ_2-balanced

Next we look at $\max\{\deg_{\mathsf{q}}\left(\rho_{\{2\}}(T_w)\right)\}$ (in the case $a - b > 0$)

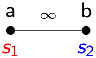
$(s_1 s_2)^n \longrightarrow n(b-a)$

$(s_2 s_1)^n \longrightarrow n(b-a)$

$(s_1 s_2)^n s_1 \longrightarrow -a + n(b-a)$

$(s_2 s_1)^n s_2$

Consider the 4 one dimensional representations of $\mathcal{H}(\tilde{A}_1)$ where $\tilde{A}_1$:



$$\rho_\emptyset : \quad T_{s_1} \longrightarrow -q^{-a}, \quad T_{s_2} \longrightarrow -q^{-b}$$

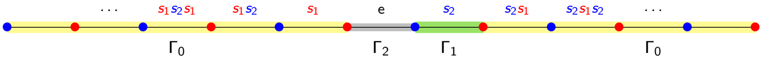
$$\rho_{\{2\}} : \quad T_{s_1} \longrightarrow -q^{-a}, \quad T_{s_2} \longrightarrow q^{b}$$

$$\rho_{\{1\}} : \quad T_{s_1} \longrightarrow q^{a}, \quad T_{s_2} \longrightarrow -q^{-b}$$

$$\rho_{\{1,2\}} : \quad T_{s_1} \longrightarrow q^{a}, \quad T_{s_2} \longrightarrow q^{b}$$

Recall that

$a > b$



We have $\deg(\rho_\emptyset(T_w)) < 0$ for all $w \neq e$ and $\deg(\rho_\emptyset(T_e)) = 0$:

$\rho_\emptyset$ is bounded by 0 and Γ_2-balanced

Next we look at $\max\{\deg_q(\rho_{\{2\}}(T_w))\}$ (in the case $a - b > 0$)

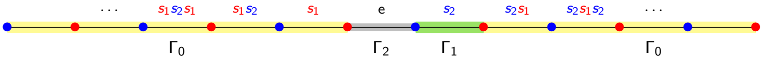
$$(s_1 s_2)^n \longrightarrow n(b-a) \qquad (s_2 s_1)^n \longrightarrow n(b-a) \qquad (s_1 s_2)^n s_1 \longrightarrow -a + n(b-a) \qquad (s_2 s_1)^n s_2 \longrightarrow b + n(b-a)$$

Consider the 4 one dimensional representations of $\mathcal{H}(\tilde{A}_1)$ where $\tilde{A}_1$: $\overset{a}{\underset{s_1}{\bullet}} \overset{\infty}{\text{———}} \overset{b}{\underset{s_2}{\bullet}}$

$$\rho_\emptyset : \quad \begin{array}{ccc} T_{s_1} & \longrightarrow & -\mathsf{q}^{-a} \\ T_{s_2} & \longrightarrow & -\mathsf{q}^{-b} \end{array} \qquad\qquad \rho_{\{2\}} : \quad \begin{array}{ccc} T_{s_1} & \longrightarrow & -\mathsf{q}^{-a} \\ T_{s_2} & \longrightarrow & \mathsf{q}^{b} \end{array}$$

$$\rho_{\{1\}} : \quad \begin{array}{ccc} T_{s_1} & \longrightarrow & \mathsf{q}^{a} \\ T_{s_2} & \longrightarrow & -\mathsf{q}^{-b} \end{array} \qquad\qquad \rho_{\{1,2\}} : \quad \begin{array}{ccc} T_{s_1} & \longrightarrow & \mathsf{q}^{a} \\ T_{s_2} & \longrightarrow & \mathsf{q}^{b} \end{array}$$

Recall that

$a > b$ 

We have $\deg(\rho_\emptyset(T_w)) < 0$ for all $w \neq e$ and $\deg(\rho_\emptyset(T_e)) = 0$:

$\rho_\emptyset$ is bounded by 0 and Γ_2-balanced

Next we look at $\max\{\deg_{\mathsf{q}}\left(\rho_{\{2\}}(T_w)\right)\}$ (in the case $a - b > 0$)

$$\begin{array}{cccc} (s_1s_2)^n & (s_2s_1)^n & (s_1s_2)^n s_1 & (s_2s_1)^n s_2 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ n(b-a) & n(b-a) & -a + n(b-a) & b + n(b-a) \end{array}$$

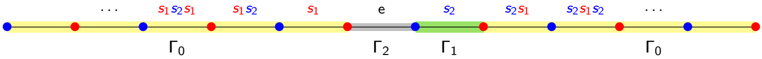
$\rho_{\{2\}}$ is bounded by b

Consider the 4 one dimensional representations of $\mathcal{H}(\tilde{A}_1)$ where $\tilde{A}_1$: $\overset{a}{\underset{s_1}{\bullet}} \overset{\infty}{—} \overset{b}{\underset{s_2}{\bullet}}$

$$\rho_\emptyset : \quad \begin{array}{ccc} T_{s_1} & \longrightarrow & -\mathsf{q}^{-a} \\ T_{s_2} & \longrightarrow & -\mathsf{q}^{-b} \end{array} \qquad \rho_{\{2\}} : \quad \begin{array}{ccc} T_{s_1} & \longrightarrow & -\mathsf{q}^{-a} \\ T_{s_2} & \longrightarrow & \mathsf{q}^{b} \end{array}$$

$$\rho_{\{1\}} : \quad \begin{array}{ccc} T_{s_1} & \longrightarrow & \mathsf{q}^{a} \\ T_{s_2} & \longrightarrow & -\mathsf{q}^{-b} \end{array} \qquad \rho_{\{1,2\}} : \quad \begin{array}{ccc} T_{s_1} & \longrightarrow & \mathsf{q}^{a} \\ T_{s_2} & \longrightarrow & \mathsf{q}^{b} \end{array}$$

Recall that

$a > b$ 

We have $\deg(\rho_\emptyset(T_w)) < 0$ for all $w \neq \mathrm{e}$ and $\deg(\rho_\emptyset(T_\mathrm{e})) = 0$:

$\rho_\emptyset$ is bounded by 0 and Γ_2-balanced

Next we look at $\max\{\deg_\mathsf{q}\left(\rho_{\{2\}}(T_w)\right)\}$ (in the case $a - b > 0$)

$$\begin{array}{cccc} (s_1 s_2)^n & (s_2 s_1)^n & (s_1 s_2)^n s_1 & (s_2 s_1)^n s_2 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ n(b-a) & n(b-a) & -a + n(b-a) & b + n(b-a) \end{array}$$

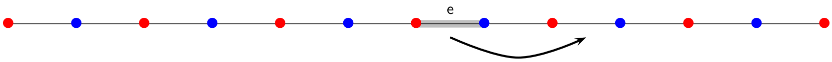
$\rho_{\{2\}}$ is bounded by b and Γ_1-balanced

Translated weight and direction

Translated weight and direction

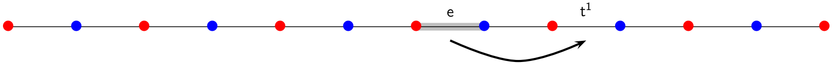
t $= s_2 s_1$ is a translation

Translated weight and direction



$t = s_2 s_1$ is a translation

Translated weight and direction



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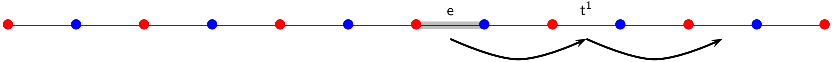

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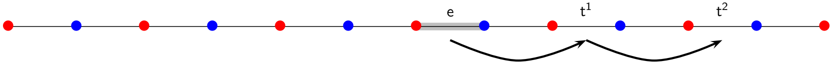
Translated weight and direction



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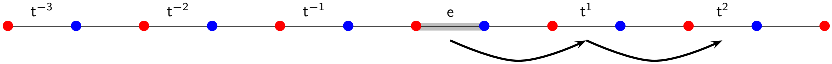

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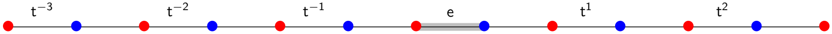
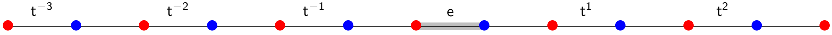


$t = s_2 s_1$ is a translation

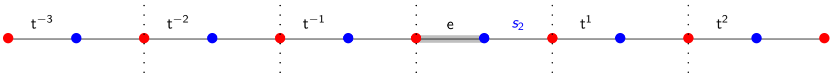
Translated weight and direction



$\mathrm{t} = s_2 s_1$ is a translation

$\mathrm{B} = \{\mathrm{e}, s_2\}$ is a fundamental domain for t

Translated weight and direction



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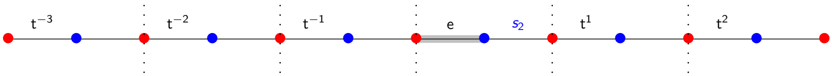
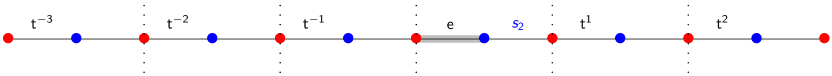


$t = s_2 s_1$ is a translation

$B = \{e, s_2\}$ is a fundamental domain for t

w can be written as $w = t^n u$ where $u \in B, n \in \mathbb{Z}$:

Translated weight and direction



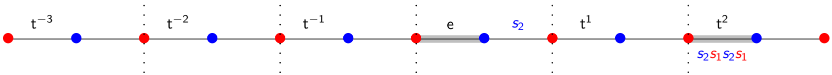
t = $s_2 s_1$ is a translation

B = $\{e, s_2\}$ is a fundamental domain for t

w can be written as $w = t^n u$ where $u \in B$, $n \in \mathbb{Z}$:

$$s_2 s_1 s_2 s_1$$

Translated weight and direction



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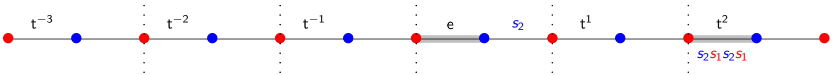
Translated weight and direction



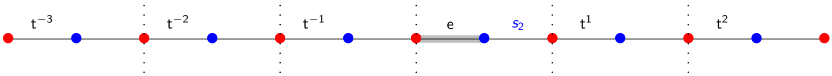
$t = s_2 s_1$ is a translation

$B = \{e, s_2\}$ is a fundamental domain for t

w can be written as $w = t^n u$ where $u \in B, n \in \mathbb{Z}$:

$$s_2 s_1 s_2 s_1 = t^2$$

Translated weight and direction



$\mathrm{t} = s_2 s_1$ is a translation

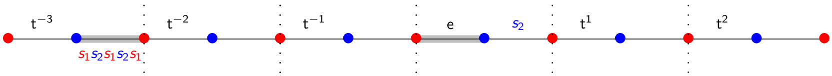
$\mathrm{B} = \{\mathrm{e}, s_2\}$ is a fundamental domain for t

w can be written as $w = t^n u$ where $u \in \mathrm{B}, n \in \mathbb{Z}$:

$$s_2 s_1 s_2 s_1 = \mathrm{t}^2$$

$$s_1 s_2 s_1 s_2 s_1$$

Translated weight and direction



$\mathrm{t} = s_2 s_1$ is a translation

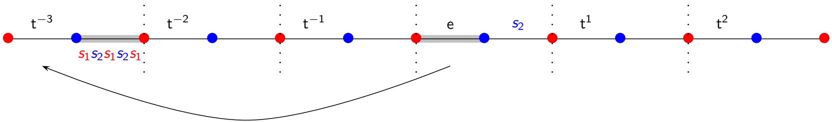
$\mathrm{B} = \{\mathrm{e}, s_2\}$ is a fundamental domain for t

w can be written as $w = t^n u$ where $u \in \mathrm{B}, n \in \mathbb{Z}$:

$$s_2 s_1 s_2 s_1 = \mathrm{t}^2$$

$$s_1 s_2 s_1 s_2 s_1$$

Translated weight and direction



$t = s_2 s_1$ is a translation

$B = \{e, s_2\}$ is a fundamental domain for t

w can be written as $w = t^n u$ where $u \in B$, $n \in \mathbb{Z}$:

$$s_2 s_1 s_2 s_1 = t^2$$

$$s_1 s_2 s_1 s_2 s_1$$

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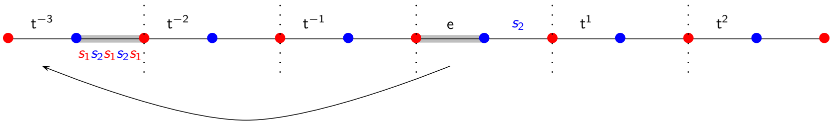

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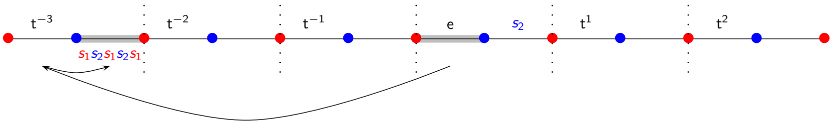

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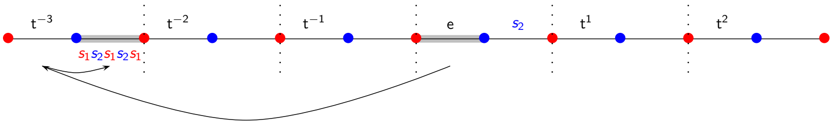

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translated weight of w (in $\mathbb{Z}$) direction of w (in B)

$$w = t^{\mathrm{wt}(w)} \theta(w)$$

Positively folded alcove paths

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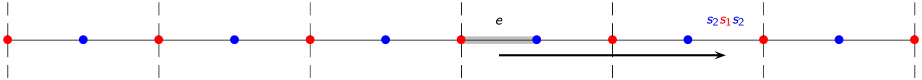
A positively folded alcove path $\vec{p}$ of shape $s_2 s_1 s_2$ starting at e:



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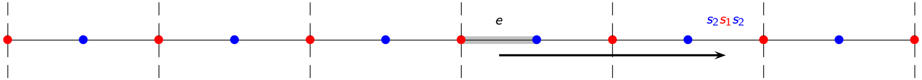

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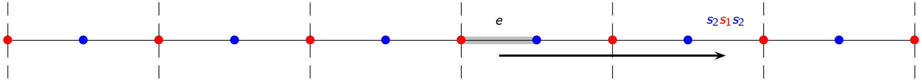

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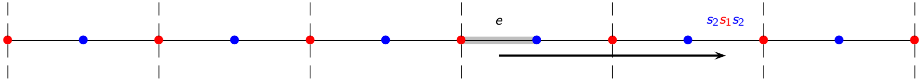

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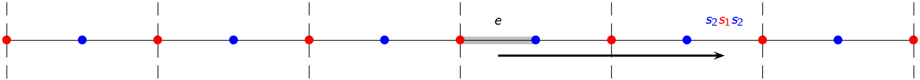

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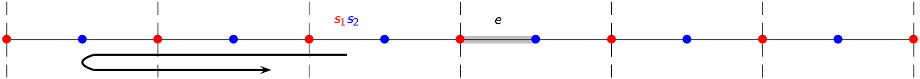

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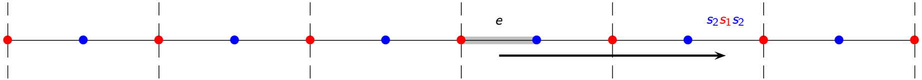

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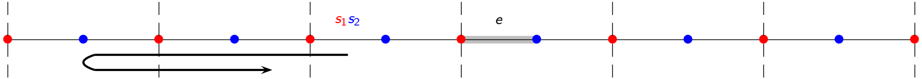

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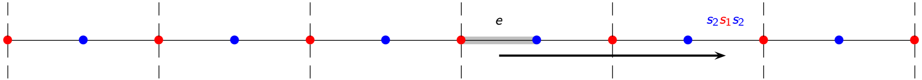

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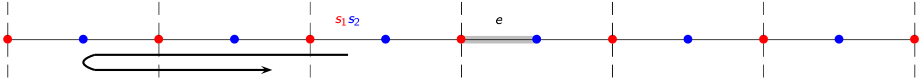

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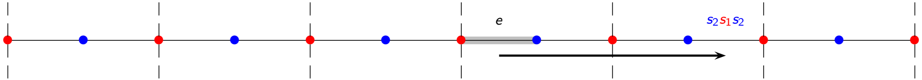

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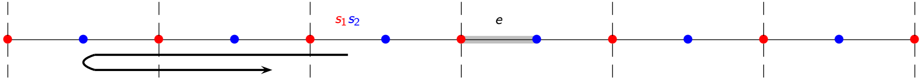

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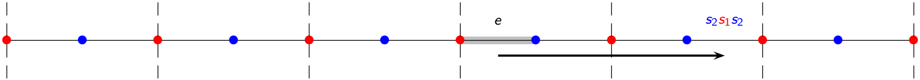

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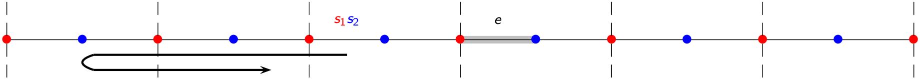

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$$\mathcal{P}(u; \vec{w}) = \{\text{positively alcove path starting at } u\}$$

A two-dimensional Γ_0-balanced representation of $\tilde{A}_1$

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Let π_0 be the 2×2 matrix representation over $R[\xi]$ defined by

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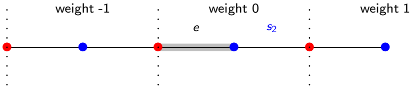

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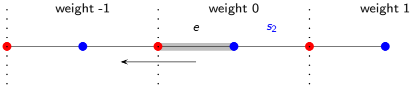

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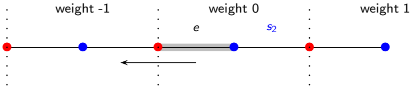

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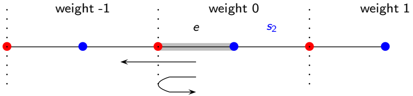

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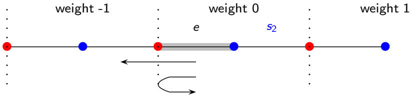

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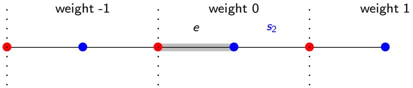

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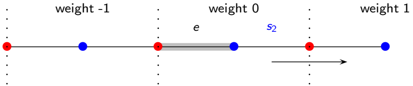

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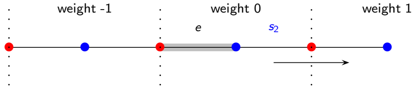

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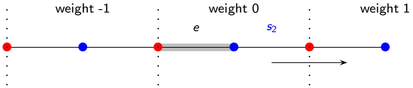

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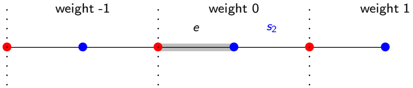

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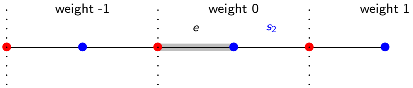

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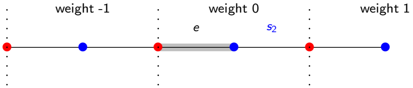

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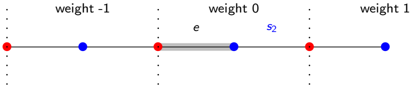


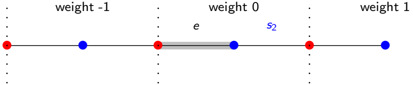
A two-dimensional Γ_0-balanced representation of $\tilde{A}_1$

Let π_0 be the 2×2 matrix representation over $R[\xi]$ defined by

$$[\pi_0(w)]_{\mathrm{u,v}} = \sum_{\vec{p} \in \mathcal{P}(\mathrm{u};\vec{w}), \theta(\vec{p}) = \mathrm{v}} \mathcal{Q}(\vec{p}) \cdot \xi^{\mathrm{wt}(\vec{p})} \text{ where } u, v \in \{\mathrm{e}, s_2\}$$

Example:

$$[\pi_0(s_1)] = \begin{array}{c} \\ \mathrm{e} \\ s_2 \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{e} & s_2 \end{array} \\ \begin{pmatrix} \mathrm{q}^a - \mathrm{q}^{-a} & \xi^{-1} \\ \xi & 0 \end{pmatrix} \end{array}$$



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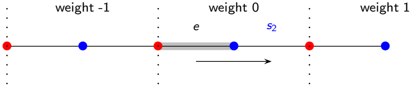

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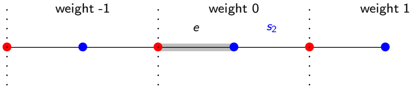

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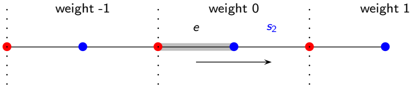

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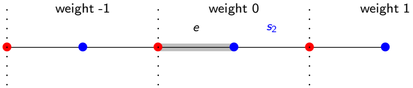

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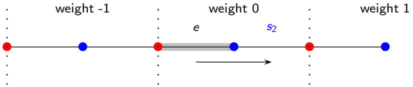

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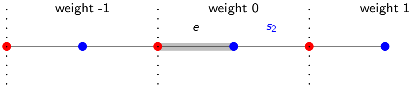

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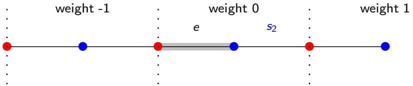

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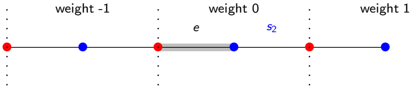

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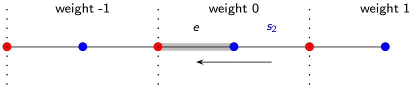

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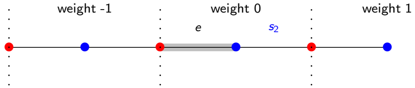

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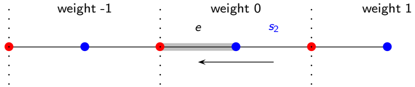

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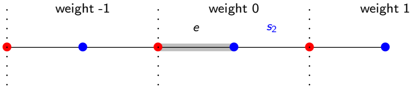

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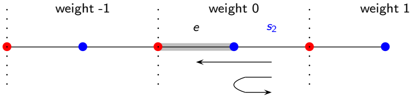

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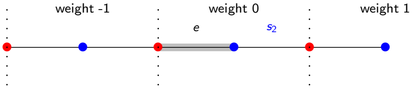

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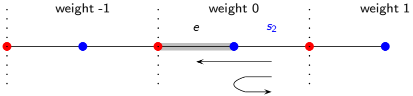

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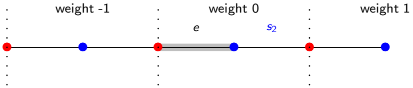

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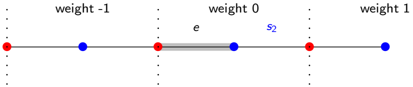

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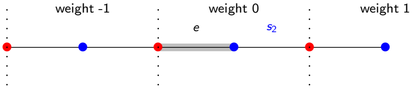

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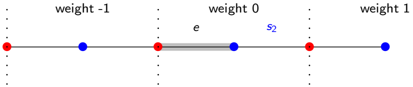

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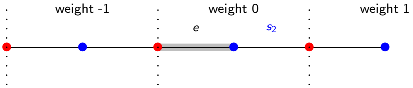

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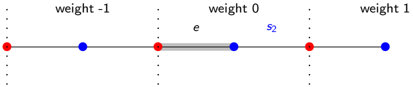

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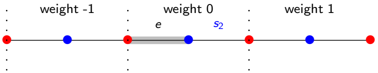

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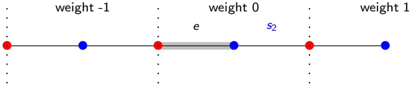

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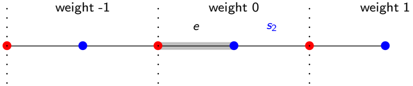

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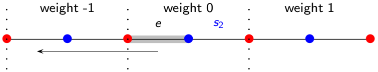

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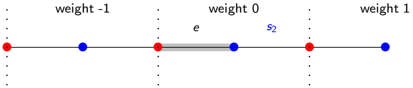

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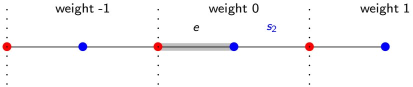

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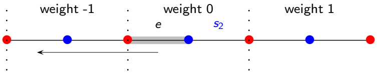

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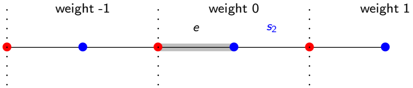

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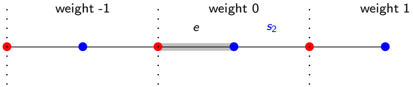

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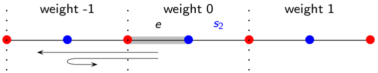

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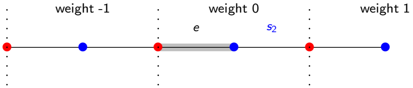

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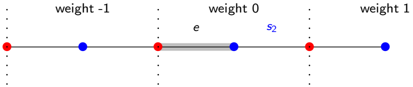

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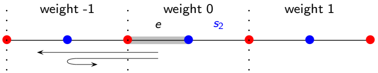

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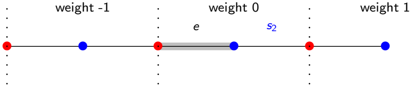

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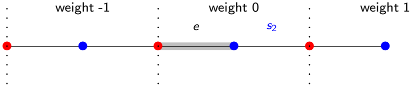

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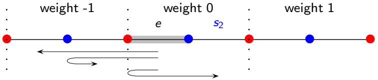

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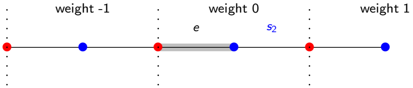

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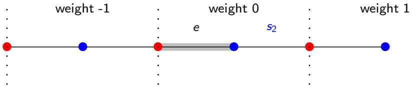

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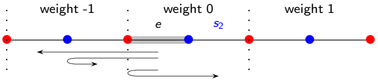

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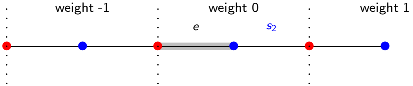

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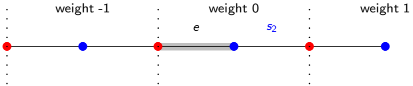

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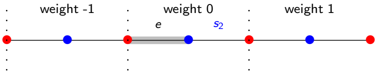

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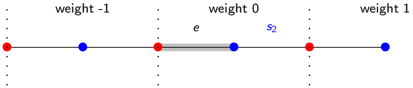

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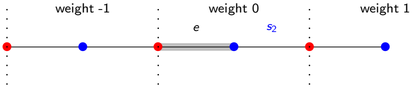

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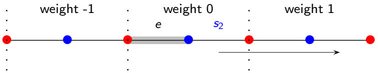

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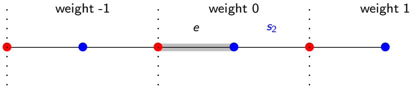

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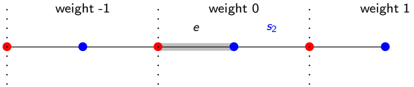

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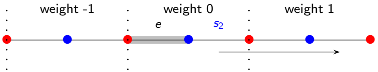

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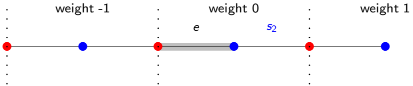

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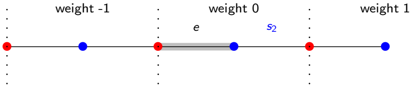

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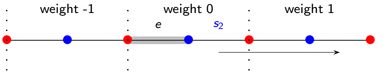

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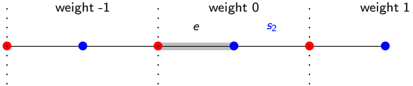

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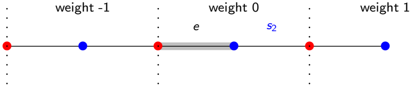

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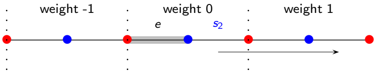
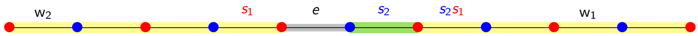


π_0 is a bounded by a and Γ_0-balanced.

Kazhdan-Lusztig cell decomposition

$a > b$



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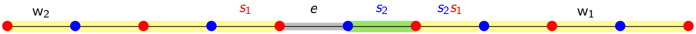

Cell factorisation :

$$\forall w \in \square, w = u^{-1} \cdot s_1 \cdot t^n \cdot v \quad \text{where } u, v \in \mathrm{B} = \{\mathrm{e}, s_2\}$$

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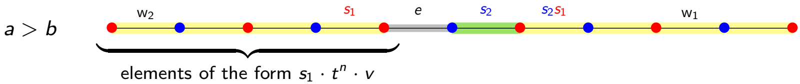

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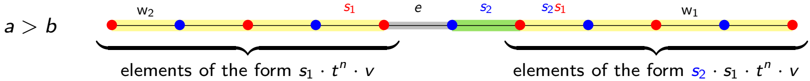

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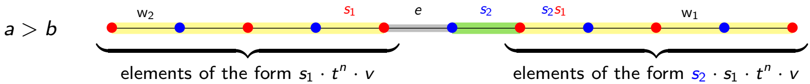

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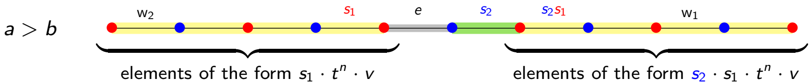

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We have

$$w \sim_{\mathcal{R}} w' \iff \mathrm{u}_w = \mathrm{u}_{w'}$$
$$w \sim_{\mathcal{L}} w' \iff \mathrm{v}_w = \mathrm{v}_{w'}$$

Leading matrices associated to π_0

Let $w = s_1 s_2 s_1 s_2 s_1 = s_1 \cdot \mathrm{t}^2$, we want compute

$$\mathfrak{c}_{\pi_0}(w) = \begin{array}{c} \\ \mathrm{e} \\ s_2 \end{array} \begin{array}{cc} \mathrm{e} & s_2 \\ \multicolumn{2}{c}{\left(\begin{array}{cc} \quad & \quad \\ \quad & \quad \end{array} \right)} \end{array}$$

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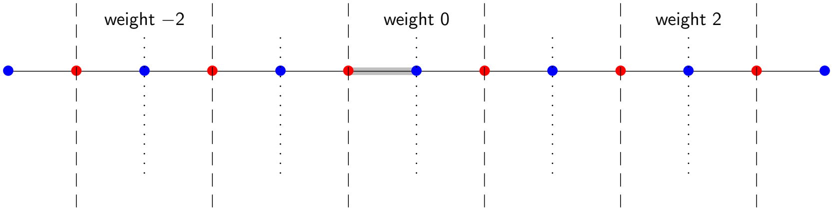
Let $\vec{p} \in \mathcal{P}_{\mathrm{u}}(\mathrm{v}; w)$ be a path that will contribute to $\mathfrak{c}_{\pi_0}(w)$:

Leading matrices associated to π_0

Let $w = s_1 s_2 s_1 s_2 s_1 = s_1 \cdot \mathrm{t}^2$, we want compute

$$\mathfrak{c}_{\pi_0}(w) = \begin{array}{c} \\ \mathrm{e} \\ s_2 \end{array} \begin{array}{cc} \mathrm{e} & s_2 \\ \left(\begin{array}{c} \\ \\ \end{array} \right. & \left. \begin{array}{c} \\ \\ \end{array} \right) \end{array}$$

Let $\vec{p} \in \mathcal{P}_{\mathrm{u}}(\mathrm{v}; w)$ be a path that will contribute to $\mathfrak{c}_{\pi_0}(w)$:



Leading matrices associated to π_0

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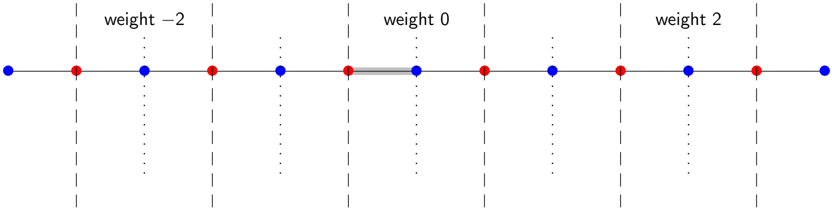

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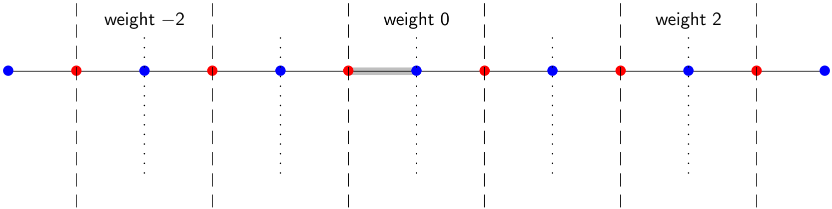

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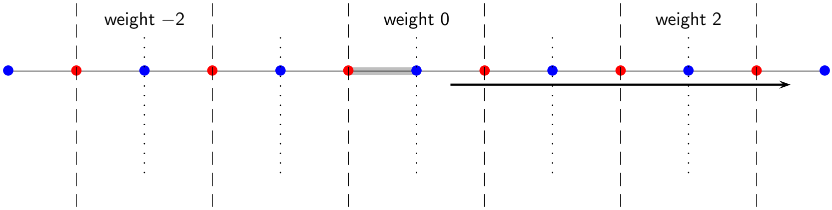

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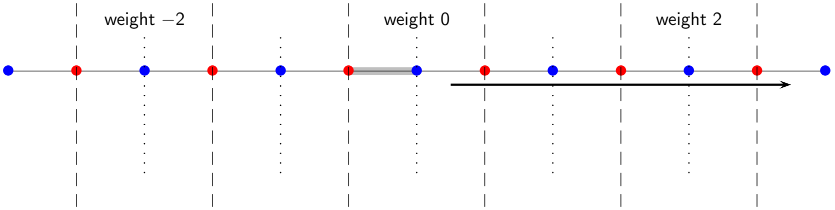

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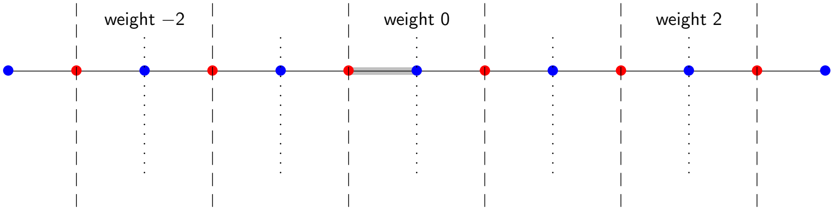

Leading matrices associated to π_0

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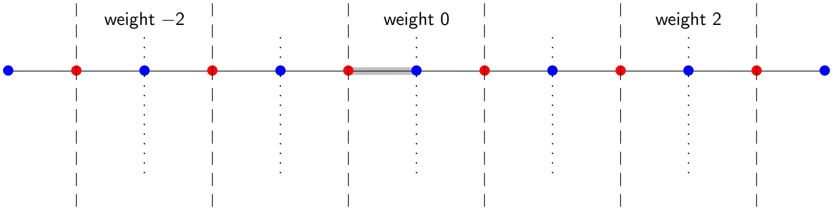

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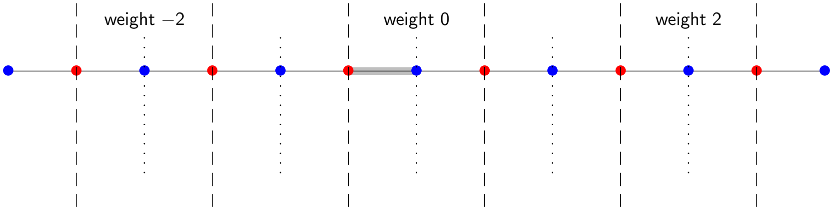

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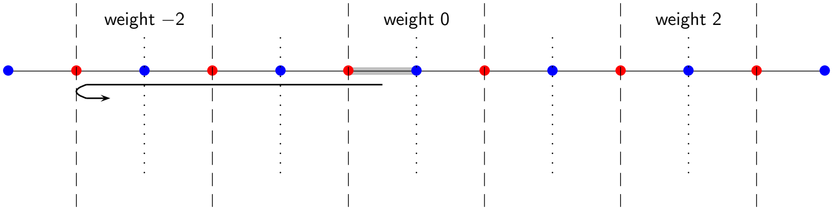

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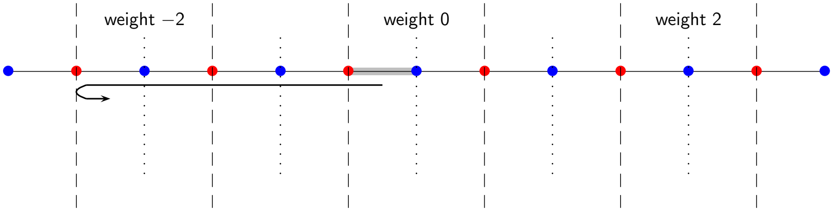

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$$\mathfrak{c}_{\pi_0}(w) = \begin{array}{c} \\ \mathrm{e} \\ s_2 \end{array} \begin{array}{cc} \mathrm{e} & s_2 \\ \left(\xi^{-2} \right. & \\ 0 & \left. 0 \right) \end{array}$$

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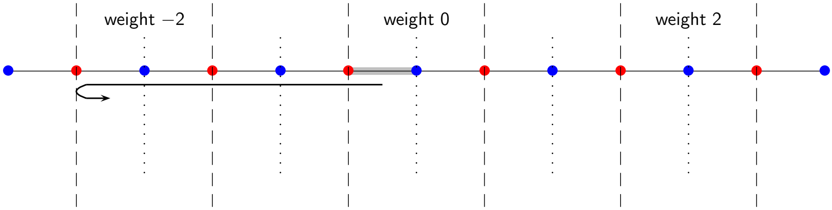

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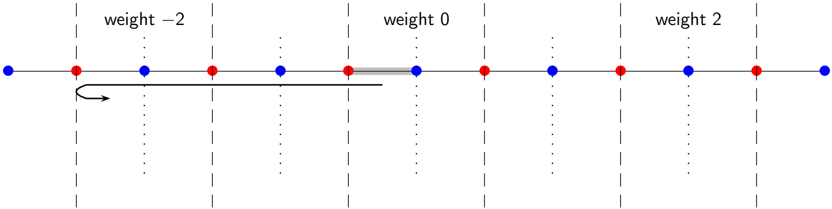

Leading matrices associated to π_0

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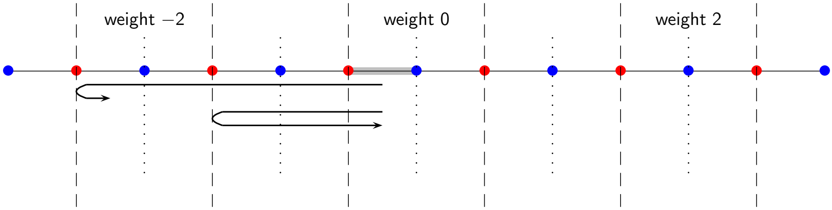

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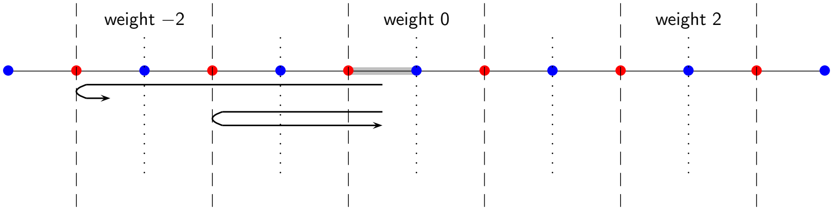

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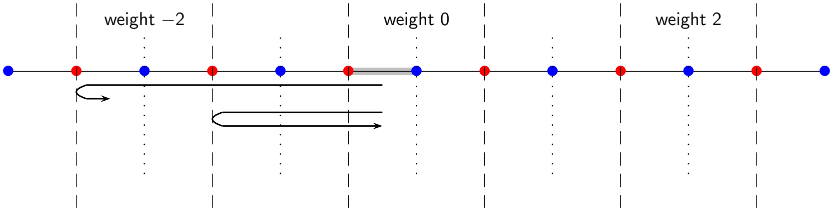

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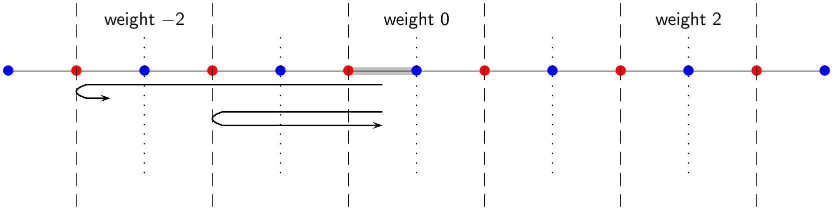

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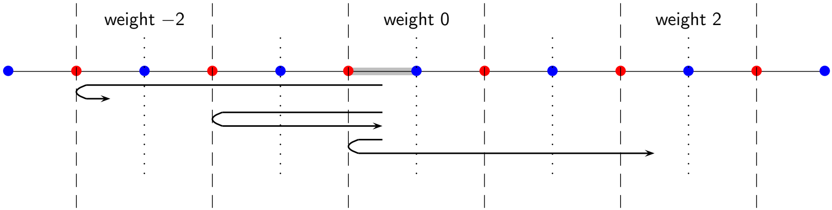

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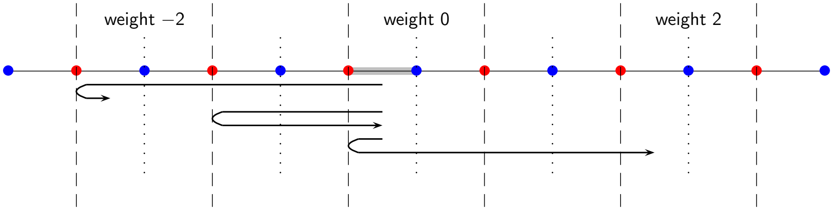

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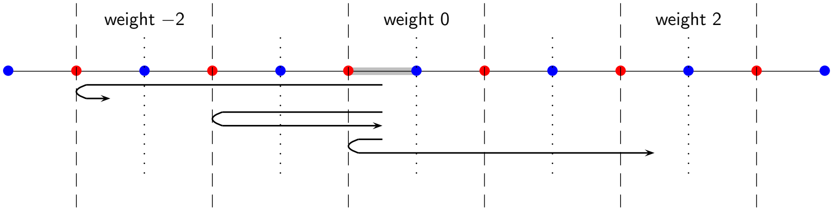

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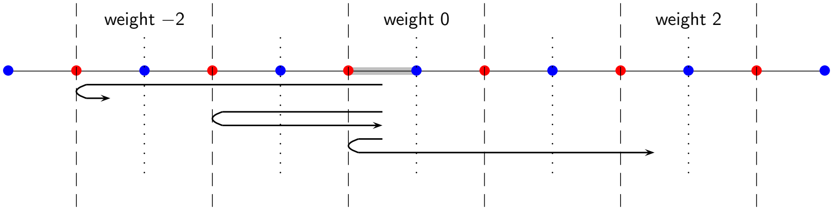

Leading matrices associated with π_0

$$\mathfrak{c}_{\pi_0}(e \cdot s_1 \cdot t^2 \cdot e) \rightsquigarrow \begin{array}{c|cc} & e & s_2 \\ \hline e & \xi^{-2}+1+\xi^2 & 0 \\ s_2 & 0 & 0 \end{array}$$

$$\mathfrak{c}_{\pi_0}(e \cdot s_1 \cdot t^2 \cdot s_2) \rightsquigarrow \begin{array}{c|cc} & e & s_2 \\ \hline e & 0 & \xi^{-2}+1+\xi^2 \\ s_2 & 0 & 0 \end{array}$$

$$\begin{array}{c|cc} & e & s_2 \\ \hline e & 0 & 0 \\ s_2 & \xi^{-2}+1+\xi^2 & 0 \end{array} \leftsquigarrow \mathfrak{c}_{\pi_0}(s_2 \cdot s_1 \cdot t^2 \cdot e)$$

$$\begin{array}{c|cc} & e & s_2 \\ \hline e & 0 & 0 \\ s_2 & 0 & \xi^{-2}+1+\xi^2 \end{array} \leftsquigarrow \mathfrak{c}_{\pi_0}(s_2 \cdot s_1 \cdot t^2 \cdot s_2)$$

Leading matrices associated with π_0

$$\mathfrak{c}_{\pi_0}(e \cdot s_1 \cdot t^2 \cdot e)$$

$$\rightsquigarrow$$

$$\begin{array}{c|cc} & e & s_2 \\ \hline e & \mathfrak{s}_2(\xi) & 0 \\ s_2 & 0 & 0 \end{array}$$

$$\mathfrak{c}_{\pi_0}(e \cdot s_1 \cdot t^2 \cdot s_2)$$

$$\rightsquigarrow$$

$$\begin{array}{c|cc} & e & s_2 \\ \hline e & 0 & \mathfrak{s}_2(\xi) \\ s_2 & 0 & 0 \end{array}$$

$$\begin{array}{c|cc} & e & s_2 \\ \hline e & 0 & 0 \\ s_2 & \mathfrak{s}_2(\xi) & 0 \end{array}$$

$$\rightsquigarrow$$

$$\mathfrak{c}_{\pi_0}(s_2 \cdot s_1 \cdot t^2 \cdot e)$$

$$\begin{array}{c|cc} & e & s_2 \\ \hline e & 0 & 0 \\ s_2 & 0 & \mathfrak{s}_2(\xi) \end{array}$$

$$\rightsquigarrow$$

$$\mathfrak{c}_{\pi_0}(s_2 \cdot s_1 \cdot t^2 \cdot s_2)$$

Leading matrices associated with π_0

$$\mathfrak{c}_{\pi_0}(\mathrm{e} \cdot s_1 \cdot \mathrm{t}^n \cdot \mathrm{e}) \rightsquigarrow \begin{array}{c|cc} & \mathrm{e} & s_2 \\ \hline \mathrm{e} & \mathfrak{s}_n(\xi) & 0 \\ s_2 & 0 & 0 \end{array}$$

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Leading matrices associated with π_0

$$\mathfrak{c}_{\pi_0}(\mathrm{e} \cdot s_1 \cdot \mathrm{t}^n \cdot \mathrm{e}) \rightsquigarrow \begin{array}{c|cc} & \mathrm{e} & s_2 \\ \hline \mathrm{e} & \mathfrak{s}_n(\xi) & 0 \\ s_2 & 0 & 0 \end{array}$$

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$$\mathfrak{c}_{\pi_0}(\mathrm{u}^{-1} \cdot s_1 \cdot t^n \cdot \mathrm{v}) = \mathfrak{s}_n(\xi) E_{\mathrm{u},\mathrm{v}}$$

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