Distinction of the Steinberg representation and a conjecture of Prasad and Takloo-Bighash

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Introduction

- F: local non-archimedean local field, char(F)$\neq 2$
- E/F: quadratic extension and $N_{E/F}$ its norm map
- D: central division F-algebra of dimension d^2

  For $n \geq 1$, $\mathcal{M}_n(D)$ is a central simple F-algebra of dimension n^2d^2. E is embedded in $\mathcal{M}_n(D)$ as an F-subalgebra if and only if nd is even.
- For nd even, $C_{\mathcal{M}_n(D)}(E)$: centralizer of E in $\mathcal{M}_n(D)$
  - If d is even, $C_{\mathcal{M}_n(D)}(E) = \mathcal{M}_n(C_D(E))$.
  - If d is odd and n is even, $C_{\mathcal{M}_n(D)}(E) = \mathcal{M}_{\frac{n}{2}}(D \otimes_F E)$.

Introduction

Definition

For G a locally profinite group, a representation $\pi : G \to GL_{\mathbb{C}}(V)$ is said to be ***smooth*** *if $Stab_G(v) = \{g \in G | \pi(g) \cdot v = v\}$ is open for all v in V.*

All representations are considered smooth on complex vector spaces.

- $G = GL(n, D)$ and $N_{rd,F}$ its reduced norm
- $H = (C_{\mathcal{M}_n(D)}(E))^\times$
- For χ a character of F^*, we set $\tilde{\chi} = \chi \circ N_{rd,F}$.

Definition

For χ a character of F^*, we denote $St(\chi) = St(n, \chi)$ the ***Steinberg representation*** *of G:*

$$St(\chi) = ind_{P_\emptyset}^G(\tilde{\chi}) / \sum_P ind_P^G(\tilde{\chi})$$

where $P_\emptyset$ denotes the minimal standard parabolic subgroup of G and the standard parabolic subgroups P in the sum correspond to a partition of n with all elements equal to 1 except one of them which is equal to 2.

Definition

A representation (π, V) *of* G *is said to be* H-***distinguished*** *if* $Hom_H(\pi, \mathbb{1}) \neq \{0\}$ *or equivalently if there exists a non-zero linear form* L *on* V *such that*

$$L(\pi(h) \cdot v) = L(v) \, \forall v \in V, \, \forall h \in H$$

Questions:

According to χ, when is the Steinberg representation $St(\chi)$ H-distinguished ?

Does it satisfy the conjecture of Prasad and Takloo-Bighash ?

- As F is of characteristic not 2, there exists $\delta \in E$ with $\delta^2 \in F$ such that $E = F[\delta]$.
- As E is quadratic and d (the index of D) is even, E can be seen as a subfield of D.
- $C_D(E)$, the centralizer of E in D, is a central division E-algebra of dimension $\frac{d^2}{4}$ and $C_D(E) = D^{int(\delta)}$.
- $C_{\mathcal{M}_n(D)}(E) = \mathcal{M}_n(C_D(E)) = \mathcal{M}_n(D^{int(\delta)})$.
- We set $\sigma = int(\delta)$ which is an involution of $G = GL(n, D)$. Then, $H = (C_{\mathcal{M}_n(D)}(E))^\times = G^\sigma$.

Let P be a standard parabolic subgroup of G, associated to a partition $\bar{n} = (n_1, \ldots, n_r)$ of n.

$$P = \begin{pmatrix} A_1 & & & & \\ & A_2 & & \text{X} & \\ & & \ddots & & \\ & 0 & & A_{r-1} & \\ & & & & A_r \end{pmatrix}$$

with $A_1 \in GL(n_1, D), \ldots, A_r \in GL(n_r, D)$.

Proposition

We have a finite (and explicit) set of representatives of the double cosets $P\backslash G/H$.

We set $u_1 = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$ the representative of the open double coset $P_\emptyset u_1 H$.

Theorem

Suppose that $St(\chi)$ is H-distinguished. Then, $ind_{P_\emptyset}^G(\tilde{\chi})$ is H-distinguished, $\chi^2 = 1$ if n is even and $\chi \circ N_{E/F} = 1$ otherwise. Moreover, only the open orbit $P_\emptyset u_1 H$ supports a H-equivariant linear form and dim($Hom_H(St(\chi, 1))$)$= 1$.

Some elements of the proof:

- Short exact sequence:
  $0 \longrightarrow \sum_P ind_P^G(\tilde{\chi}) \longrightarrow ind_{P_\emptyset}^G(\tilde{\chi}) \longrightarrow St(\chi) \longrightarrow 0$
- We have a filtration of H-modules:

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{r-1} \subset V_r = ind_{P_\emptyset}^G(\tilde{\chi})$$

with $V_s = \mathcal{C}_c^\infty(P_\emptyset \backslash \bigsqcup_{i=1}^{s} P_\emptyset u_i H, \tilde{\chi})$ such that $\mathrm{Hom}_H(V_{i+1}/V_i, \mathbb{C}) = 0$ for $i \geq 1$ and $\dim(\mathrm{Hom}_H(V_1, \mathbb{C})) = 1$.

Theorem (C.)

- *If n is odd, $St(\chi)$ is H-distinguished if and only if $\chi \circ N_{E/F} = 1$.*
- *If n is even, $St(\chi)$ is H-distinguished if and only if $\chi^2 = 1$ and $\chi \circ N_{E/F} \neq 1$.*

- $n = 2m$
- $D \otimes_F E$ is a central division E-algebra of dimension d^2. Then, $D \otimes_F E$ is 2-dimensional over D so there exists $\delta \in (D \otimes_F E) \setminus D$ such that $D \otimes_F E = D \bigoplus \delta D$ as a right-D-vector space.
- Let $(e_1, \ldots, e_m)$ be the canonical basis of $(D \otimes_F E)^m$. Then, $(D \otimes_F E)^m$ identifies with D^{2m} via the basis $\mathcal{B} = (e_1 \ldots, e_m, \delta e_m, \ldots, \delta e_1)$ of D^{2m}.

- An application $u \in \mathrm{End}((D \otimes_F E)^m)_D$ is right-$D \otimes_F E$-linear if and only if u commutes with the multiplication by δ whose matrix in $\mathcal{B}$ is

$$\begin{pmatrix} & & & & & \Delta \\ & & & & \iddots & \\ & & & \Delta & & \\ & & 1 & & & \\ & \iddots & & & & \\ 1 & & & & & \end{pmatrix} =: U_\Delta \qquad \text{with } \Delta = \delta^2$$

so $\mathcal{M}_m(D \otimes_F E) = \mathcal{M}_n(D)^{int(U_\Delta)}$.

- $$E \longrightarrow \mathcal{M}_n(D)$$
  $$x \longmapsto \begin{pmatrix} 1\otimes x & & \\ & \ddots & \\ & & 1\otimes x \end{pmatrix} \in \mathcal{M}_m(D\otimes_F E) \subset \mathcal{M}_{2m}(D)$$

- $g \in C_{\mathcal{M}_n(D)}(E)$ if and only if $g \in \mathcal{M}_m(D\otimes_F E)$ so
  $H = (C_{\mathcal{M}_n(D)}(E))^{\times} = GL(m, D\otimes_F E) = G^{\sigma}$ with $\sigma = int_{U_\Delta}$.

Theorem (C.)

$St(\chi)$ is H-distinguished if and only if $\chi^2 = 1$ and $\chi \circ N_{E/F} \neq 1$.

Definition

*Let G be a locally profinite group, (π, V) a smooth irreducible representation of G with unitary central character and $(\check{\pi}, \check{V})$ its contragredient. π is said to be **square integrable** if there exists $v \in V$, $\check{v} \in \check{V}$ such that $g \mapsto | < \check{v}, \pi(g) \cdot v > |^2$ is a function of $Z \backslash G$ which is integrable on $Z \backslash G$.*

The Jacquet-Langlands correspondence

For D a central division F-algebra of dimension d^2, there is a bijection:

$$\left\{ \begin{array}{c} \text{equivalent classes} \\ \text{of square integrable} \\ \text{representations of } GL(n, D) \end{array} \right\} \xrightarrow{JL} \left\{ \begin{array}{c} \text{equivalent classes} \\ \text{of square integrable} \\ \text{representations of } GL(nd, F) \end{array} \right\}$$

The local Langlands correspondence

There is a bijection

$$\left\{ \begin{array}{c} \text{equivalent classes of} \\ \text{irreducible representations} \\ \text{of } GL(n,F) \end{array} \right\} \xrightarrow{CLL} \left\{ \begin{array}{c} \text{equivalent classes of } n\text{-dimensional} \\ \text{semisimple representations} \\ \text{of the Weil-Deligne group of } F \;\; : \\ W_F' = W_F \times SL(2,\mathbb{C}) \end{array} \right\}$$

The image of an irreducible smooth representation π of $GL(n,F)$ is called the **Langlands parameter** of π.

Definition

An irreducible smooth representation π *of* $GL(n, D)$ *is said to be* ***symplectic*** *if the Langlands parameter* Ψ *of* $JL(\pi)$ *satisfies: there exists* $< \cdot, \cdot >$ *a nondegenerate alternating bilinear form on* $\mathbb{C}^n$ *such that* $\forall w \in W'_F, \forall v, v' \in \mathbb{C}^n, < \Psi(w) \cdot v, \Psi(w) \cdot v' >=< v, v' >$.

- BC_E: the base change to E

Conjecture of Prasad and Takloo-Bighash, 2011

Let π be a square integrable representation of $G = GL(n, D)$. Then, π is H-distinguished if and only if it is symplectic and $\epsilon(\frac{1}{2}, BC_E(\pi)) = (-1)^n$.

- $Sp(n)$: the unique irreducible representation of $SL(2,\mathbb{C})$ of dimension n
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$St(n, \chi)$

rep. of $GL(n, D)$

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Theorem (C.)

$St(n,\chi)$ is H-distinguished if and only if it is symplectic and $\epsilon(\frac{1}{2}, BC_E(St(n,\chi))) = (-1)^n$.