Irreducible components of the global nilpotent cone

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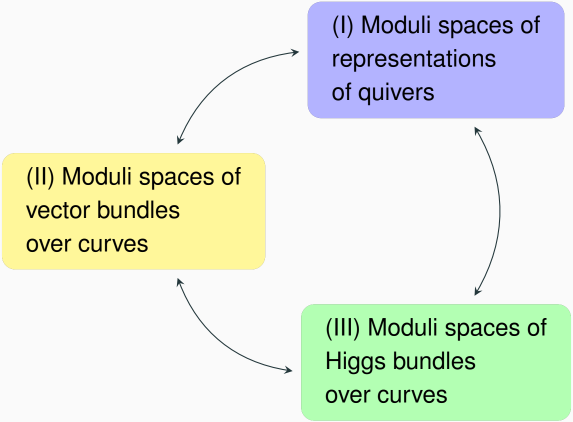
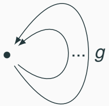

(I) Moduli spaces of representations of quivers
(II) Moduli spaces of vector bundles over curves
(III) Moduli spaces of Higgs bundles over curves

(I) Representations of quivers

Quiver: oriented graph.

Relevant when studying a curve of genus g: $S_g =$ 

↯Wild!↯

$\Bbbk$-representation of dimension r of S_g: g-tuple of $r \times r$ matrices with coefficients in $\Bbbk$.

Definition

$A_{S_g,r}(\mathbb{F}_q)$: number of isoclasses of r-dimensional $\mathbb{F}_q$-representations of S_g which are indecomposable over $\overline{\mathbb{F}}_q$.

Theorem (Kac '82)

$A_{S_g,r}(\mathbb{F}_q) \in \mathbb{Z}[q]$.

Conjecture (Kac '82, †Hausel, Letellier, Rodriguez-Villegas '12)

$A_{S_g,r}(\mathbb{F}_q) \in \mathbb{N}[q]$.

Theorem (Hua '00)

$$Exp\left(\frac{1}{q-1}\sum_r A_{S_g,r}(q)z^r\right) = \sum_\lambda \left\{ q^{(g-1)\sum_k \lambda_k^2} \prod_k [\infty, \lambda_k - \lambda_{k+1}]_{q^{-1}} \right\} z^{|\lambda|}$$

where $[\infty, n]_t = \prod_{p=1}^n (1-t^p)^{-1}$.

If $r = 3$:

$$A_{S_g,3}(q) = \frac{q^{9g-3} - (1+q+q^2)q^{5g-3} - (1+q)q^{3g-2}}{(q^2-1)(q^3-1)}.$$

(II) Vector bundles over curves

X: smooth, projective curve of genus g over $\mathbb{F}_q$.

$Vec_{r,d}(X)$: moduli space of vector bundles $\mathcal{F} \to X$ of rank r and degree d.

☞ smooth, of dimension $(g-1)r^2$.

Theorem (Schiffmann '14)

The number $A_{g,r,d}(X)$ of isoclasses of $\mathcal{F} \in Vec_{r,d}(X)$ which are indecomposable over $\overline{\mathbb{F}}_q$ is polynomial in the Weil numbers of X.

+ implicit formula for $A_{g,r,d}$, similar to Hua's but more complicated.

{Weil numbers $(\sigma_1, \ldots, \sigma_{2g})$ of X:

eigenvalues of the Frobenius $\circlearrowright H^1(\overline{X}, \overline{\mathbb{Q}}_l)$, l prime $\nmid q$;

satisfy $\bar{\sigma}_{2i-1} = \sigma_{2i}$, $\sigma_{2i-1}\sigma_{2i} = q$, $i = 1 \ldots g$}

(III) Higgs bundles

X: curve of genus g over $\mathbb{C}$.

Ω: canonical bundle of X.

Fact

$T^* Vec_{r,d}(X) \simeq Higgs_{r,d}(X)$ is the moduli space of Higgs pairs $(\mathcal{F}, \theta)$ over X, consisting of $\mathcal{F} \in Vec_{r,d}(X)$ and a morphism $\theta : \mathcal{F} \to \mathcal{F} \otimes \Omega$.

Definition

A pair $(\mathcal{F}, \theta)$ is stable if

$$\forall \mathcal{G} \subset \mathcal{F}, \qquad \theta(\mathcal{G}) \subseteq \mathcal{G} \otimes \Omega \Rightarrow \mu(\mathcal{G}) < \mu(\mathcal{F})$$

with respect to the slope $\mu = deg/rk$.

☞ $Higgs^{st}_{r,d}(X)$ smooth

(III) Higgs bundles

Theorem (Schiffmann '14)

The Poincaré polynomial of the moduli space $Higgs^{st}_{r,d}(X)$ of stable Higgs pairs is given by a specialization of the polynomial $A_{g,r,d}$.

Thanks to a simplification by Mellit ('17) of Schiffmann's formula, one then gets $A_{g,r,d}(0) = A_{S_g,r}(1)$.

Fact

Define the (stable) global nilpotent cone $\Lambda^{st}_{r,d}(X) \subset Higgs^{st}_{r,d}(X)$ by requiring θ to be nilpotent. It is a Lagrangian subvariety (Laumon '88). Then

$$\#Irr(\Lambda^{st}_{r,d}(X)) = A_{g,r,d}(0).$$

Notations

$[\mathcal{F}] = (rk(\mathcal{F}), deg(\mathcal{F}))$

$\mathcal{F}(k\Omega) = \mathcal{F} \otimes \Omega^{\otimes k}$

$l = deg(\Omega) = 2g - 2$

Then $[\mathcal{F}(k\Omega)] = (r, d + klr)$.

Accordingly, if $\alpha = (r, d)$, we set $\alpha(p) := (r, d + pr)$, so that if $[\mathcal{F}] = \alpha$ and $k \in \mathbb{Z}$,

$$[\mathcal{F}(k\Omega)] = \alpha(kl).$$

Jordan type

Forget about stability for now and define the global nilpotent cone $\Lambda_{r,d}(X) \subset \mathit{Higgs}_{r,d}(X)$ by requiring $(\mathcal{F}, \theta)$ to be nilpotent, *i.e.*

$$\theta^k := \mathcal{F} \xrightarrow{\theta} \mathcal{F}(\Omega) \xrightarrow{\theta \otimes \mathrm{id}_\Omega} \mathcal{F}(2\Omega) \longrightarrow \dots \longrightarrow \mathcal{F}(k\Omega)$$

$= 0$ for $k >> 0$.

Set $\mathcal{F}_k = \operatorname{Im} \theta^k(-k\Omega) \subset \mathcal{F}$ and s the index of θ, which induces:

$$\mathcal{F}_0 \twoheadrightarrow \mathcal{F}_1(\Omega) \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{F}_s(s\Omega) = \{0\}$$

$$\{0\} = \mathcal{F}_s \subset \mathcal{F}_{s-1} \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{F}$$

hence if $\mathcal{F}'_k = \mathcal{F}_k / \mathcal{F}_{k+1}$:

$$\mathcal{F}'_0 \twoheadrightarrow \mathcal{F}'_1(\Omega) \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{F}'_s(s\Omega) = \{0\}$$

Definition

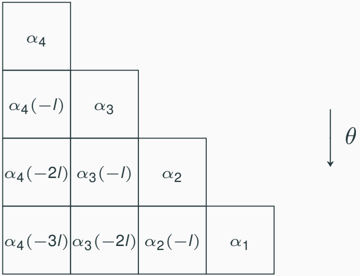
Jordan type of $(\mathcal{F}, \theta)$ is the collection $\underline{\alpha} = (\alpha_k)_{k=1}^s$ where

$$\alpha_k = \left[\ker\{\mathcal{F}'_{k-1}((k-1)\Omega) \to \mathcal{F}'_k(k\Omega)\}\right].$$

Jordan type: diagrammatic approach

Write $\alpha_k = (r_k, d_k)$, so that $\alpha_k(-tl) = (r_k, d_k - tlr_k)$.

If $s = 4$:



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If $s = 4$:

$$(r, d) = [\mathcal{F}] = \sum_{\square} \begin{array}{|c|cccc} \cline{1-1} \alpha_4 & \multicolumn{4}{c}{} \\ \cline{1-2} \alpha_4(-l) & \multicolumn{1}{c|}{\alpha_3} & \multicolumn{3}{c}{} \\ \cline{1-3} \alpha_4(-2l) & \multicolumn{1}{|c|}{\alpha_3(-l)} & \multicolumn{1}{c|}{\alpha_2} & \multicolumn{2}{c}{} \\ \cline{1-4} \alpha_4(-3l) & \multicolumn{1}{|c|}{\alpha_3(-2l)} & \multicolumn{1}{c|}{\alpha_2(-l)} & \multicolumn{1}{c|}{\alpha_1} & \\ \cline{1-4} \end{array} \qquad \Big\downarrow \theta$$

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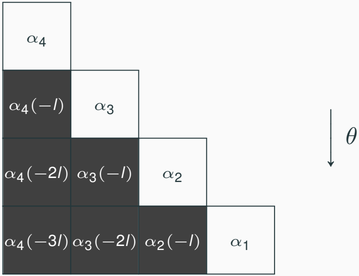
Denote by $|\underline{\alpha}| = (r, d)$ the weight of $\underline{\alpha}$.

Jordan type: diagrammatic approach, twisted images

Write $\alpha_k = (r_k, d_k)$, so that $\alpha_k(-tl) = (r_k, d_k - tlr_k)$.

If $s = 4$:

$$[\mathcal{F}_1] = \sum_{\blacksquare}$$



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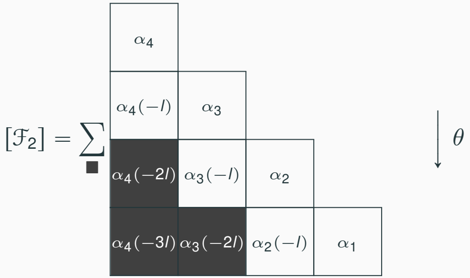

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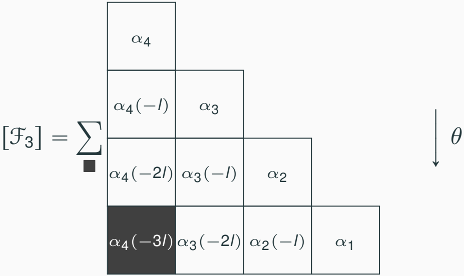

Jordan type: diagrammatic approach, kernels

Write $\alpha_k = (r_k, d_k)$, so that $\alpha_k(-tl) = (r_k, d_k - tlr_k)$.

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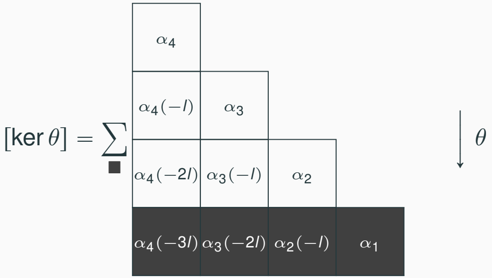
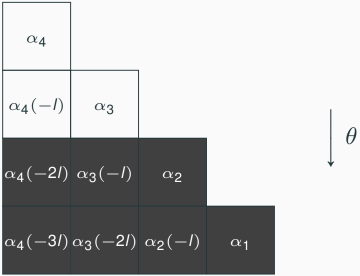


Jordan type: diagrammatic approach, kernels

Write $\alpha_k = (r_k, d_k)$, so that $\alpha_k(-tl) = (r_k, d_k - tlr_k)$.

If $s = 4$:

$[\ker \theta^2] = \sum_{\blacksquare}$



Jordan type: diagrammatic approach, kernels

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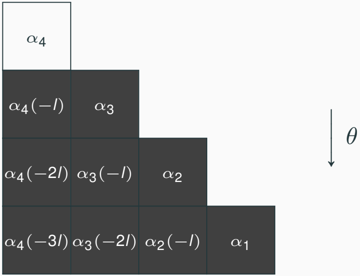
If $s = 4$:

$$[\ker \theta^3] = \sum_{\blacksquare}$$



Jordan stratification

Theorem (B '17)

The irreducible components of $\Lambda_{r,d}(X)$ are given by the closures of the strata

$$\Lambda_{\underline{\alpha}}(X) := \{(\mathcal{F}, \theta) \text{ of Jordan type } \underline{\alpha}\}.$$

These are indexed by finite sequences $\underline{\alpha} = (r_k, d_k)_{1 \leqslant k \leqslant s}$ of $\mathbb{N} \times \mathbb{Z}$ satisfying

$$|\underline{\alpha}| = (r, d)$$

and

$$d_k \geqslant 0 \text{ if } r_k = 0.$$

Remark: $|\underline{\alpha}| = (r, d) \Leftrightarrow \begin{cases} \sum k r_k = r \\ \sum k d_k = d + (g-1) \sum k(k-1) r_k. \end{cases}$

Stable locus

Recall that a pair is stable $(\mathcal{F}, \theta)$ if

$$\forall \mathcal{G} \subset \mathcal{F}, \qquad \theta(\mathcal{G}) \subseteq \mathcal{G} \otimes \Omega \Rightarrow \mu(\mathcal{G}) < \mu(\mathcal{F})$$

with respect to the slope $\mu = deg/rk$.

☞ open condition, so

$$Irr(\Lambda^{st}_{r,d}(X)) \subset Irr(\Lambda_{r,d}(X))$$

and

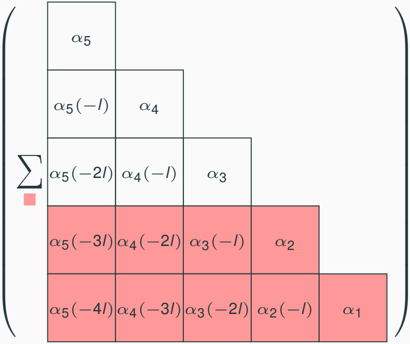
$$\overline{\Lambda_{\underline{\alpha}}(X)} \in Irr(\Lambda^{st}_{r,d}(X)) \Leftrightarrow \Lambda_{\underline{\alpha}}(X) \cap Higgs^{st}_{r,d}(X) \neq \emptyset.$$

Stability: diagrammatic approach

Call canonical the Higgs subsheaves obtained as sums and intersections of $\mathcal{F}_k$'s and $\ker \theta^k$'s.

For instance (s=5)

$\mu(\ker \theta^2 \qquad) = \mu$



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For instance (s=5)

$$\mu(\mathcal{F}_2) = \mu\left(\sum_{\square} \cdots \right)$$

α_5				
$\alpha_5(-l)$	α_4			
$\alpha_5(-2l)$	$\alpha_4(-l)$	α_3		
$\alpha_5(-3l)$	$\alpha_4(-2l)$	$\alpha_3(-l)$	α_2	
$\alpha_5(-4l)$	$\alpha_4(-3l)$	$\alpha_3(-2l)$	$\alpha_2(-l)$	α_1

Stability: diagrammatic approach

Call canonical the Higgs subsheaves obtained as sums and intersections of $\mathcal{F}_k$'s and $\ker\theta^k$'s.

For instance (s=5)

$$\mu(\ker\theta^2 \cap \mathcal{F}_2 \qquad) = \mu\left(\sum_{\blacksquare} \begin{array}{|c|c|c|c|c|} \hline \alpha_5 & & & & \\ \hline \alpha_5(-l) & \alpha_4 & & & \\ \hline \alpha_5(-2l) & \alpha_4(-l) & \alpha_3 & & \\ \hline \alpha_5(-3l) & \alpha_4(-2l) & \alpha_3(-l) & \alpha_2 & \\ \hline \alpha_5(-4l) & \alpha_4(-3l) & \alpha_3(-2l) & \alpha_2(-l) & \alpha_1 \\ \hline \end{array}\right)$$

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For instance (s=5)

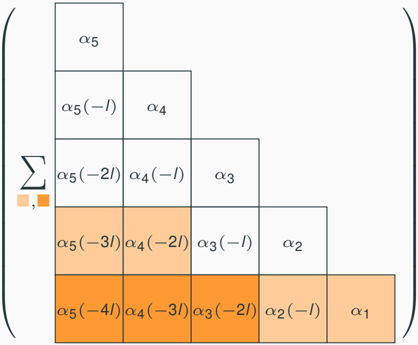
$$\mu(\qquad \ker\theta) = \mu\left(\sum_{\blacksquare} \begin{array}{|c|c|c|c|c|} \hline \alpha_5 & & & & \\ \hline \alpha_5(-l) & \alpha_4 & & & \\ \hline \alpha_5(-2l) & \alpha_4(-l) & \alpha_3 & & \\ \hline \alpha_5(-3l) & \alpha_4(-2l) & \alpha_3(-l) & \alpha_2 & \\ \hline \alpha_5(-4l) & \alpha_4(-3l) & \alpha_3(-2l) & \alpha_2(-l) & \alpha_1 \\ \hline \end{array}\right)$$

Stability: diagrammatic approach

Call canonical the Higgs subsheaves obtained as sums and intersections of $\mathcal{F}_k$'s and $\ker \theta^k$'s.

For instance (s=5)

$$\mu(\ker \theta^2 \cap \mathcal{F}_2 + \ker \theta) = \mu\left(\sum_{\blacksquare,\blacksquare} \cdots \right)$$



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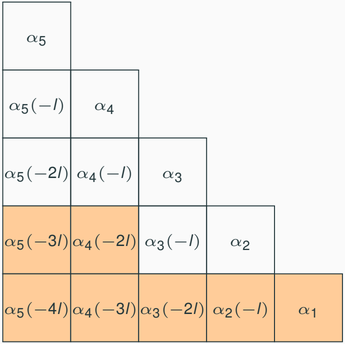


$$\left. \right) = \frac{(2d_5 - 7lr_5) + (2d_4 - 5lr_4) + (d_3 - 2lr_3) + (d_2 - lr_2) + d_1}{2r_5 + 2r_4 + r_3 + r_2 + r_1}$$

Theorem (B '17)

The stratum $\overline{\Lambda_{\underline{\alpha}}(X)}$ subsists in the stable locus if for a given pair $(\mathcal{F}, \theta)$ of Joran type $\underline{\alpha}$, the canonical subsheaves have a slope $\leqslant d/r$.

☞ Get an *optimal* set of conditions that only depends on $\underline{\alpha}$, hence strictly *numerical*.

☞ More precisely, we get the integer points inside an union of polytopes.

☞ Relies on an analogous result of Bradlow, Garcia-Prada, Gothen and Heinloth ('17), obtained for moduli spaces of chains $\mathcal{E}_s \to \ldots \to \mathcal{E}_1$, which can be understood as Higgs pairs fixed by the $\mathbb{C}^*$-action $t.(\mathcal{F}, \theta) = (\mathcal{F}, t\theta)$.

Example: $r = 2, d = 1$

(r_k) partition of $r = 2$ implies $s = 1, 2$.

$s = 1$ ($\theta = 0$): one component corresponding to $\underline{\alpha} = \alpha_1 = (2, 1)$. There *exist* stable sheaves $\mathcal{F}$ of class $(2, 1)$.

$s = 2$ ($\theta^2 = 0$): components indexed by $(\alpha_1, \alpha_2) = ((0, d_1), (1, d_2))$ such that

$$(2,1) = \sum\nolimits_{\square} \begin{array}{|c|c} \hline \alpha_2 & \\ \hline \alpha_2(-l) & \multicolumn{1}{c|}{\alpha_1} \\ \hline \end{array} \Leftrightarrow d_1 = 1 - 2d_2 + l \geqslant 0 \text{ (with } l = 2g - 2\text{);}$$

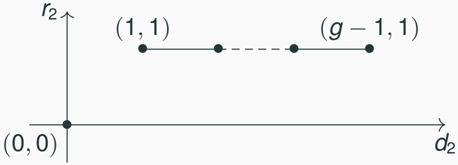
$$\mu\left(\sum\nolimits_{\blacksquare} \begin{array}{|c|c} \hline \alpha_2 & \\ \hline \alpha_2(-l) & \multicolumn{1}{c|}{\alpha_1} \\ \hline \end{array}\right) = d_2 - l < 1/2;$$

$$\mu\left(\sum\nolimits_{\blacksquare} \begin{array}{|c|c} \hline \alpha_2 & \\ \hline \alpha_2(-l) & \multicolumn{1}{c|}{\alpha_1} \\ \hline \end{array}\right) = d_2 + d_1 - l = 1 - d_2 < 1/2 \text{ (implies the previous one).}$$

We simply get $1 \leqslant d_2 \leqslant g - 1$.

Example: $r = 2, d = 1$

We get the union of a 0-dimensional polytope and a 1-dimensional polytope.



In particular

$$\#Irr(\Lambda^{st}_{2,1}(X)) = g = A_{S_g,2}(1)$$

since $A_{S_g,2}(q) = q^{2g-1}\dfrac{q^{2g}-1}{q^2-1}$.

Sketch of proof

Idea: build an injection

$$\{\underline{\alpha} \text{ s.t. } \mu(\text{can. sheaves}) < d/r\} \hookrightarrow \{\text{stable types of chains}\}.$$

For $\underline{\alpha}$ on the LHS, and any given $(\mathcal{F}, \theta)$ of type $\underline{\alpha}$, need to build a filtration $\mathcal{F}_\bullet = (\{0\} = \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_s = \mathcal{F})$ by canonical subsheaves such that θ induces a chain

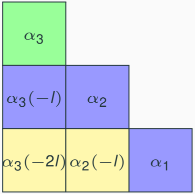
$$\frac{\mathcal{F}_s}{\mathcal{F}_{s-1}} \to \frac{\mathcal{F}_{s-1}}{\mathcal{F}_{s-2}}(\Omega) \to \ldots \to \frac{\mathcal{F}_2}{\mathcal{F}_1}((s-2)\Omega) \to \mathcal{F}_1((s-1)\Omega)$$

of stable type.

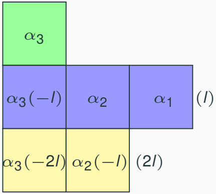
Example (s=3)

$\underline{\alpha} \mapsto [\mathcal{E}_\bullet] = \{\alpha_3 \to (\alpha_3(-l) + \alpha_2 + \alpha_1)(l) \to (\alpha_3(-2l) + \alpha_2(-l))(2l)\}$, or diagrammatically

$\underline{\alpha} =$



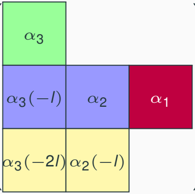
$\mapsto$



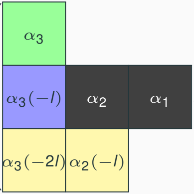
$$= \begin{bmatrix} \mathcal{F} \\ \downarrow \theta \\ (\ker \theta^2 / \ker \theta \cap \mathcal{F}_1)(\Omega) \\ \downarrow \theta \\ (\ker \theta \cap \mathcal{F}_1)(2\Omega) \end{bmatrix}$$

if

$\mu\Bigg($



$\Bigg) < d/r\ ,\ \mu\Bigg($



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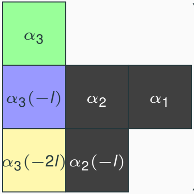

$\Bigg) \geqslant d/r$

i.e. if $\mu(\alpha_1) < d/r \leqslant \min\{\mu(\alpha_2 + \alpha_1), \mu(\alpha_2 + \alpha_2(-l) + \alpha_1)\}$.