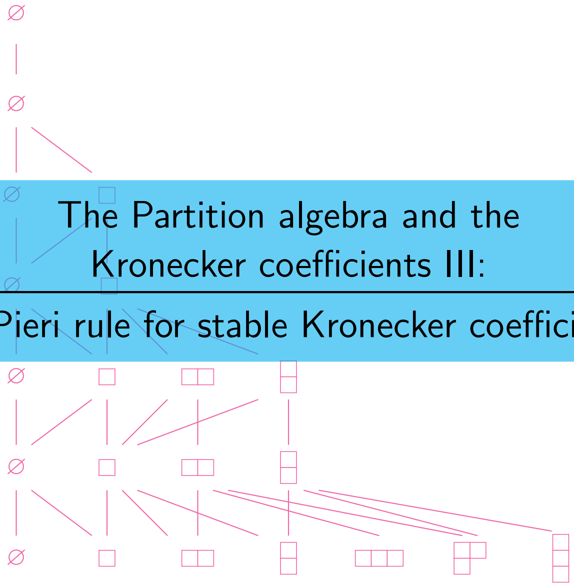
The Partition algebra and the Kronecker coefficients III:

Co-Pieri rule for stable Kronecker coefficients



Recap

The stable Kronecker coefficients equal dimensions of homomorphism spaces for path-theoretic $P_s(n)$-modules:

$$\overline{g}(\lambda, \mu, \nu) = \dim_{\mathbb{C}}(\mathsf{Hom}_{P_s(n)}(\Delta_s(\mu), \Delta_s(\nu \setminus \lambda)))$$

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- Plus we benefit from the **extra** $P_s(n)$-structure.
- We can define local operators on paths.
- And hence calculate $\overline{g}(\lambda, \mu, \nu)$ via combinatorial resolutions.

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$$c(\lambda, \mu, \nu) = \dim_{\mathbb{C}}(\mathsf{Hom}_{\mathbb{C}\mathfrak{S}_s}(\Delta_s(\mu), \Delta_s(\nu \setminus \lambda)))$$

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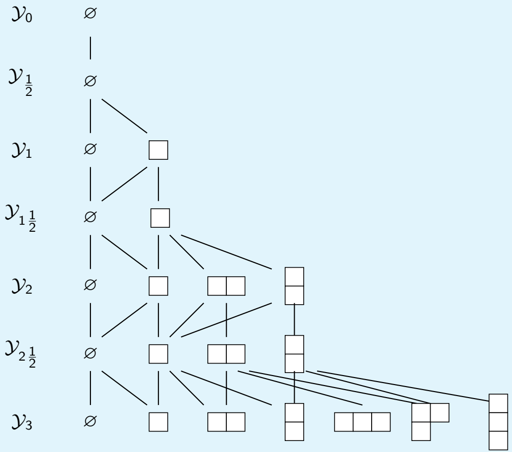
- And so we generalise one half of the Littlewood–Richardson rule to the $P_s(n)$-branching graph.
- And hence solve one half of the stable Kronecker problem.

Definition



Let $\lambda \in \mathcal{Y}_{r-s}$ and $\nu \in \mathcal{Y}_r$.

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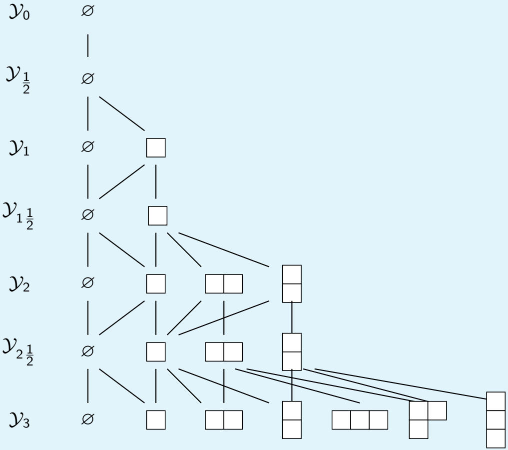
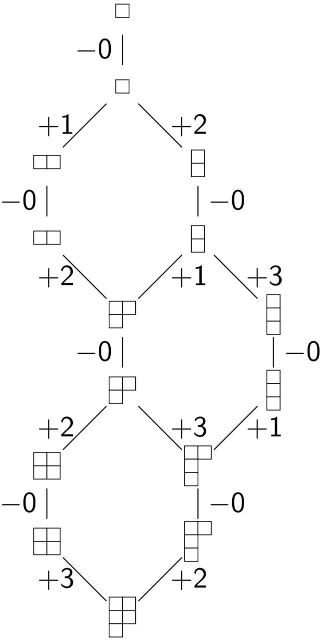


Let $\lambda \in \mathcal{Y}_{r-s}$ and $\nu \in \mathcal{Y}_r$. A skew Kronecker tableau of shape $\nu \setminus \lambda$ and degree s is a path t of the form

$$\lambda = \mathsf{t}(0) \to \mathsf{t}(\tfrac{1}{2}) \to \mathsf{t}(1) \to \cdots \to \mathsf{t}(s - \tfrac{1}{2}) \to \mathsf{t}(s) = \nu.$$



	1
2	3
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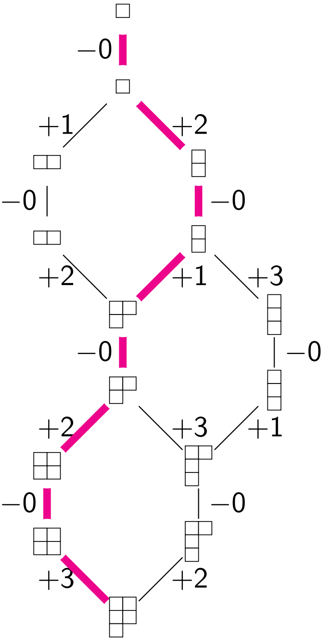

−0
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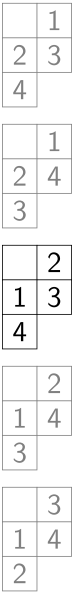

1
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2
1 4
3
3
1 4
2

Let $\mathsf{t} \in \mathsf{Std}_s(\nu \setminus \lambda)$. For $1 \le k \le s$ consider the steps

$$\cdots\ \mathsf{t}(k-1) \xrightarrow{-\varepsilon_t} \mathsf{t}(k-\tfrac{1}{2}) \xrightarrow{+\varepsilon_u} \mathsf{t}(k) \xrightarrow{-\varepsilon_v} \mathsf{t}(k+\tfrac{1}{2}) \xrightarrow{+\varepsilon_w} \mathsf{t}(k+1) \cdots$$

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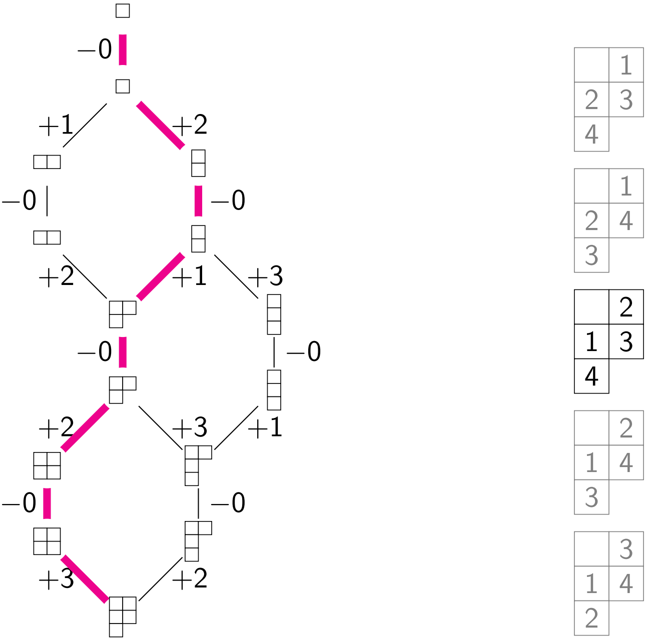
We say that (λ, ν, s) is a co-Pieri triple if $\mathsf{t}_{k \leftrightarrow k+1}$ exists for all $1 \leq k \leq s$ and $\mathsf{t} \in \mathrm{Std}_s(\nu \setminus \lambda)$.

$((2^2, 1), (1), 4)$ is not a co-Pieri triple. Let $k = 2$ and t as follows

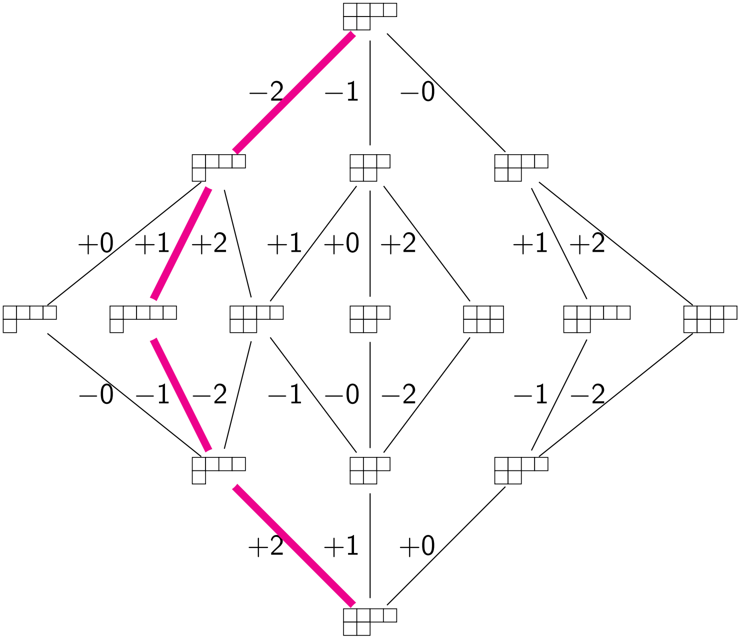


Action on skew modules

Let (λ, ν, s) be a co-Pieri triple. Given $\mathsf{t} \in \mathsf{Std}_s(\nu \setminus \lambda)$, we have that

$$s_k(\mathsf{t}) = \mathsf{t}_{k \leftrightarrow k+1}$$

The action of s_1 on the co-Pieri triple $((4,2),(4,2),2)$.



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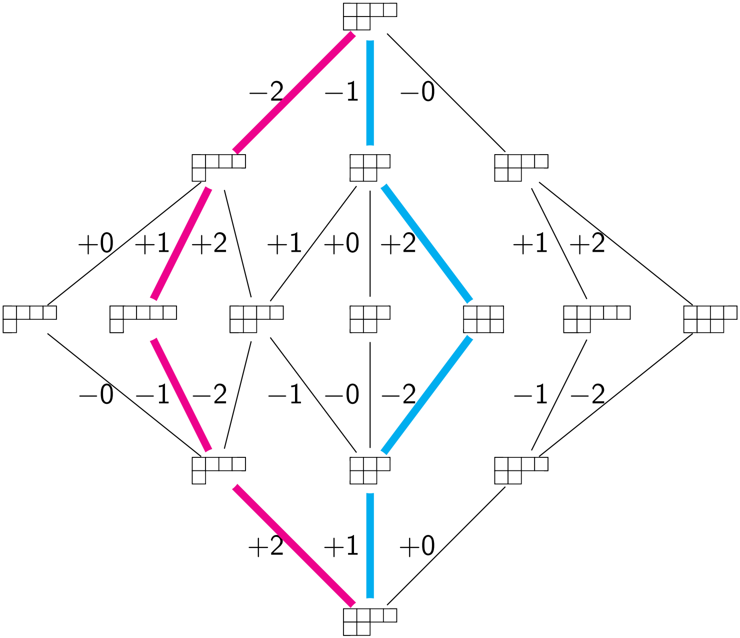

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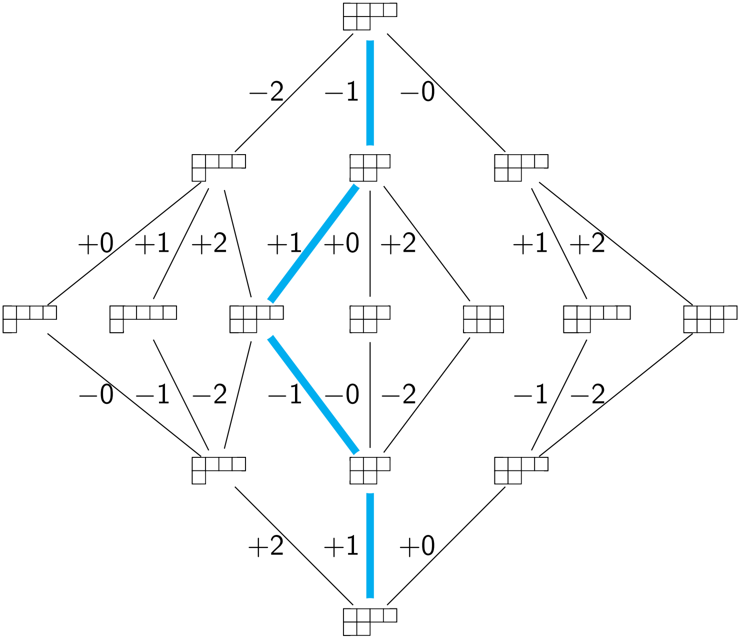

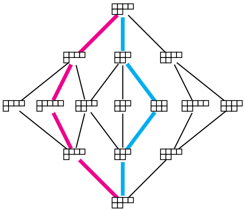

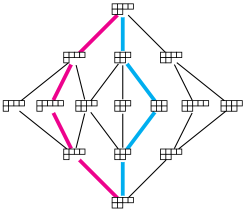

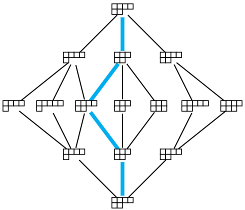

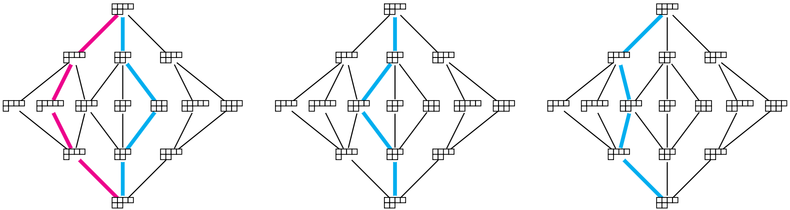

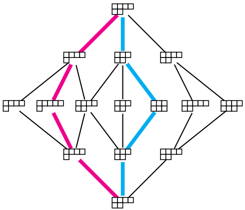
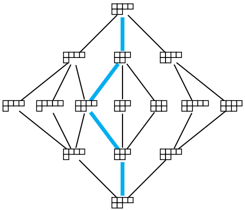
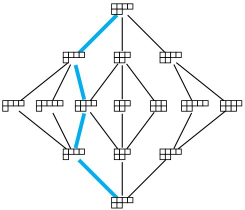
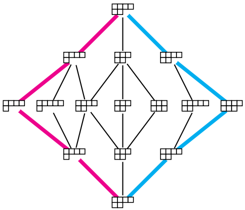

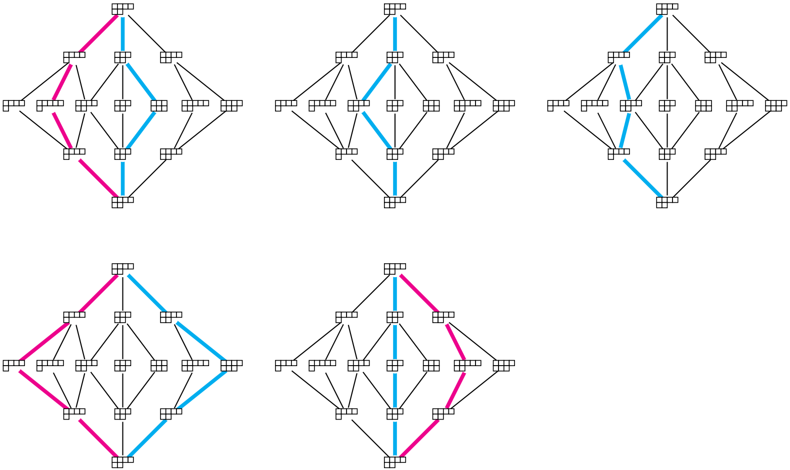

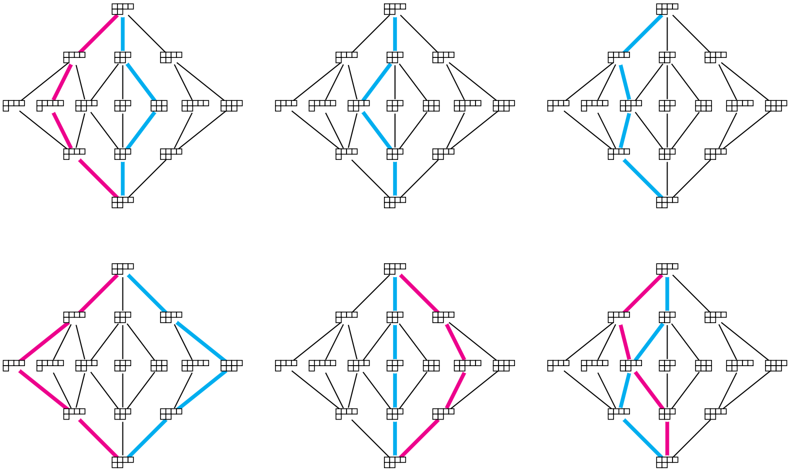

And so

$$\overline{g}((4,2),(4,2),\mu) = \begin{cases} 6 & \text{for } \nu = (2) \\ 4 & \text{for } \nu = (1^2). \end{cases}$$

Section 1

Semistandard Kronecker tableaux

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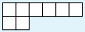
Some semistandard tableaux.



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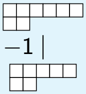

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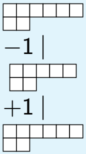

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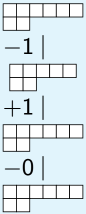

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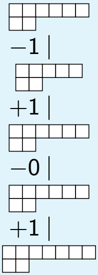

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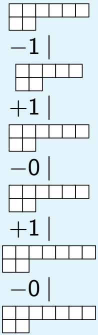

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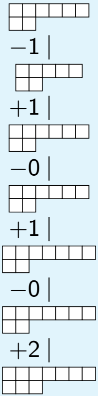

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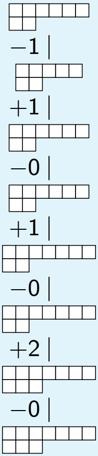

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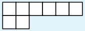

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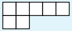

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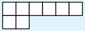

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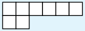

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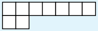

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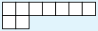
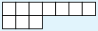


$+2$



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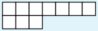

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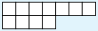

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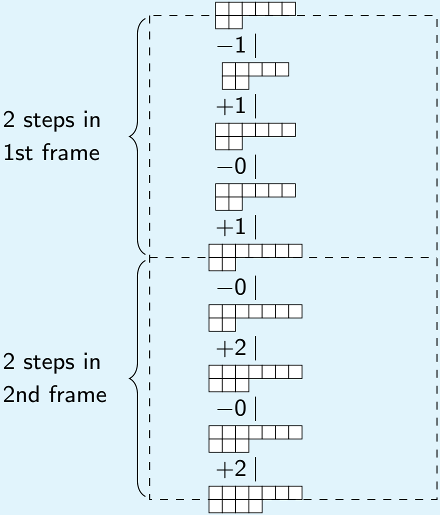

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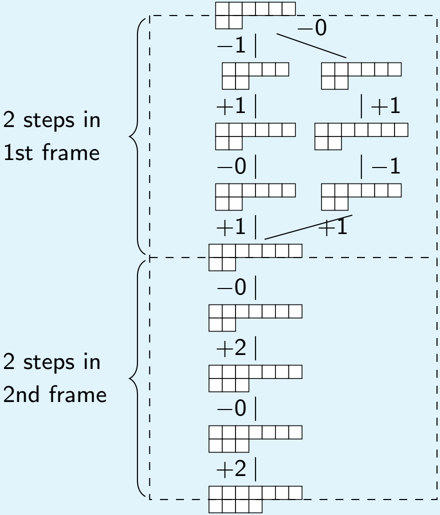

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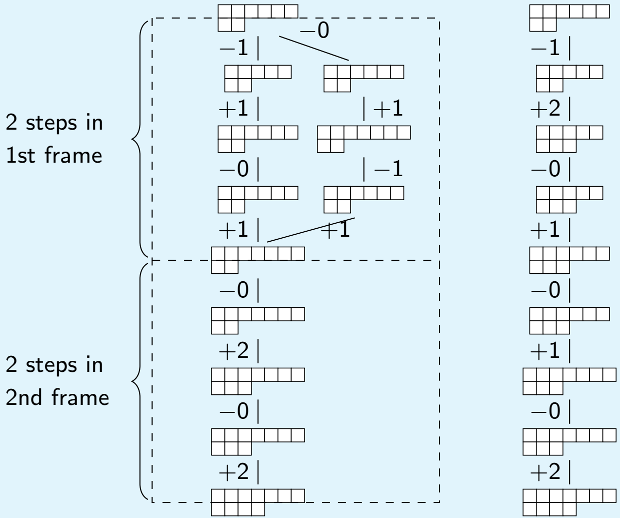

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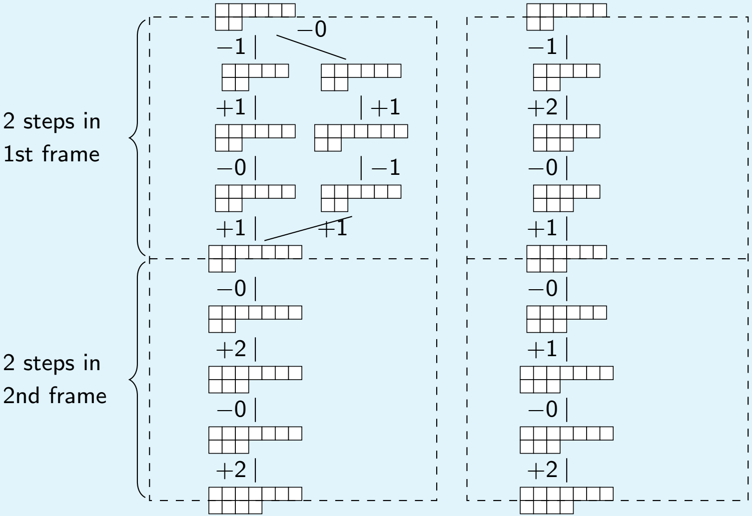

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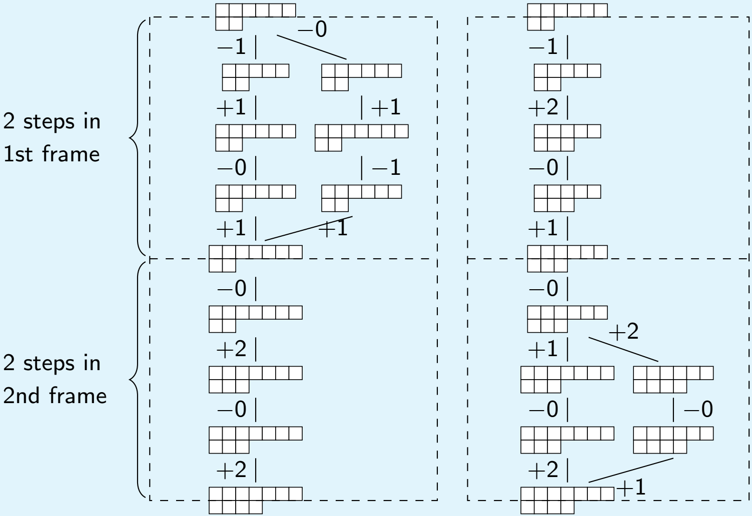

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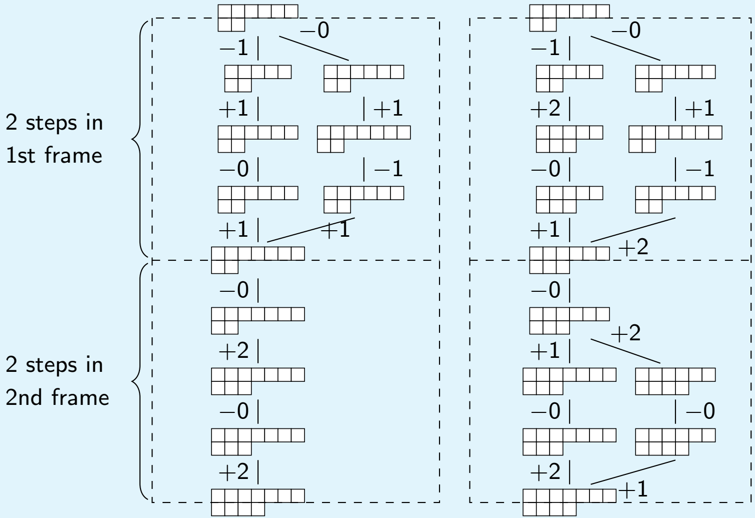

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The $P_s(n)$-module

$$\mathrm{Hom}_{P_s(n)}(\mathrm{ind}_{P_{\mu_1}(n)\times P_{\mu_2}(n)\dots}^{P_s(n)}(\mathbb{C}), \Delta_s(\nu \setminus \lambda))$$

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where φ_{T} is determined by

$$\varphi_{\mathsf{T}}(\mathsf{t}^{\mu}) = \sum_{\mathsf{t}\in\mathsf{T}} \mathsf{t}.$$

Section 2

The lattice permutation condition

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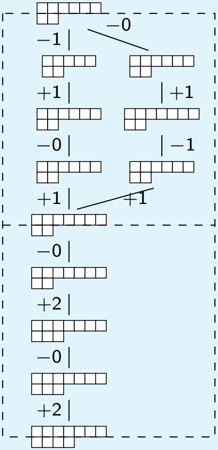
for $p > q$ and $u < v$. We can refine this to a total ordering.

Some latticed tableaux.



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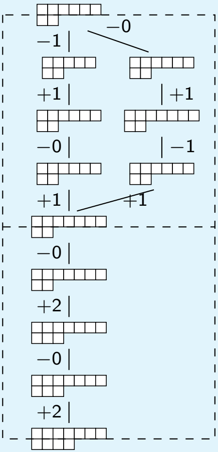

$$\left[\begin{array}{c|c|cc} d(1) & a(1) & a(2) & a(2) \\ 1 & 1 & 2 & 2 \end{array}\right]$$

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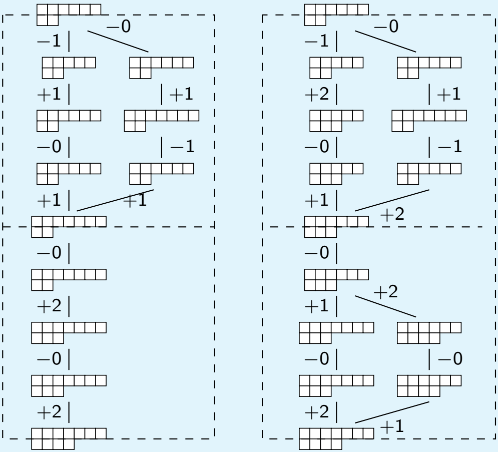

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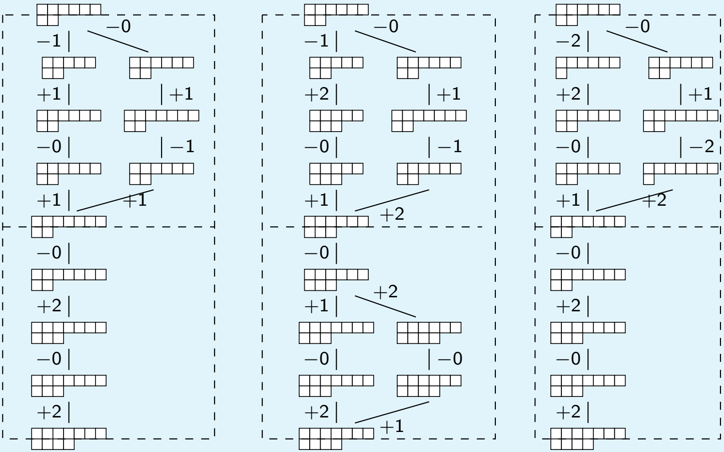

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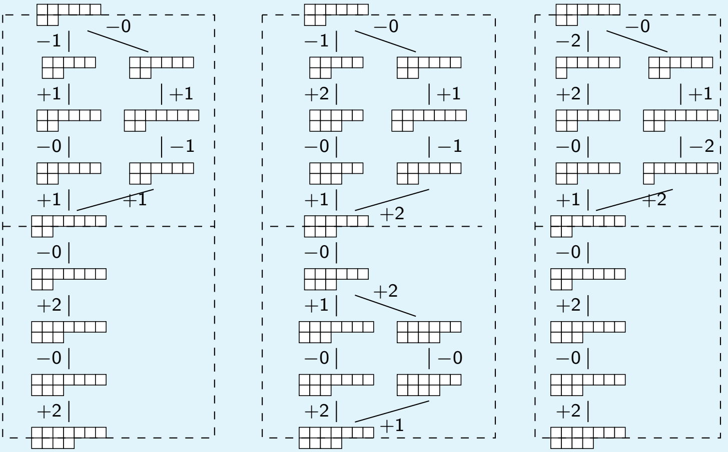

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Theorem* [B., De Visscher, Enyang]

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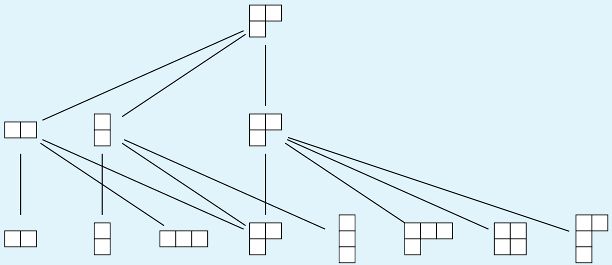
- $\lambda = \nu = (d\ell, d(\ell-1), \dots, 2d, d)$ for any $\ell, d \geq 1$, $|\mu| \leq d$.
- the two skew partitions $\lambda \setminus (\lambda \setminus \nu)$ and $\nu \setminus (\lambda \cap \nu)$ have no two boxes in the same column and

$$|\mu| = \max\{|\lambda \setminus (\lambda \cap \nu)|, |\nu \ominus (\lambda \cap \nu)|\}.$$

Example

Recall from the first lecture the following graph.



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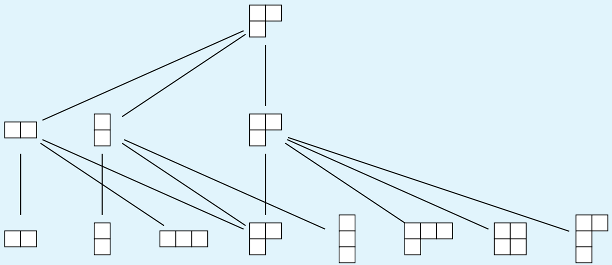

- $\overline{g}((2^2,1),(1),\nu)) = g((n-5,2^2,1),(n-|\nu|,\nu),(n-1))$ for $n \geq 7$ is equal to the number of paths from λ to ν.

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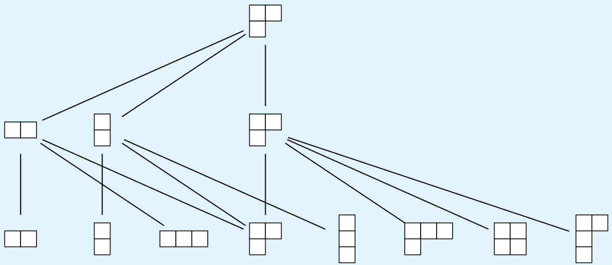
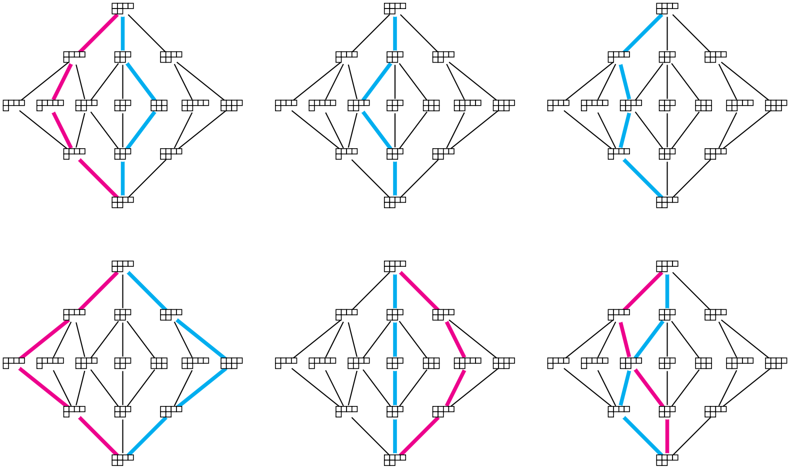


- $\overline{g}((2^2,1),(1),\nu)) = g((n-5,2^2,1),(n-|\nu|,\nu),(n-1))$ for $n \geq 7$ is equal to the number of paths from λ to ν.
- Since $s = 1$ there are no symmetric group generators here, and so all these paths satisfy the semistandard and lattice permutation conditions trivially.

Recall our earlier example of a co-Pieri triple



And so

$$\overline{g}((4,2),(4,2),\mu) = \begin{cases} 6 & \text{for } \nu = (2) \\ 4 & \text{for } \nu = (1^2). \end{cases}$$

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Let $\lambda = (4, 2) = \nu$ as in our earlier example. The semistandard tableaux of weight (2) are

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and all steps occur in the first frame. All steps in the first frame are good and so

$$\overline{g}(\lambda, \nu, (2)) = 6$$

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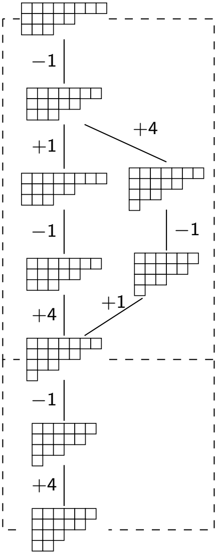
and so only one satisfies the lattice permutation property.



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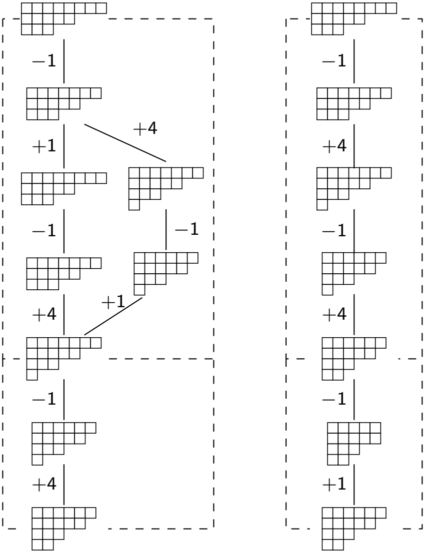

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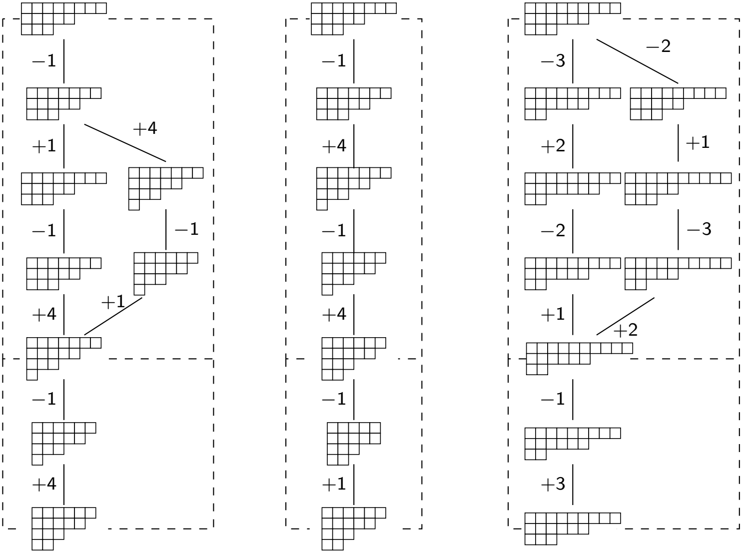
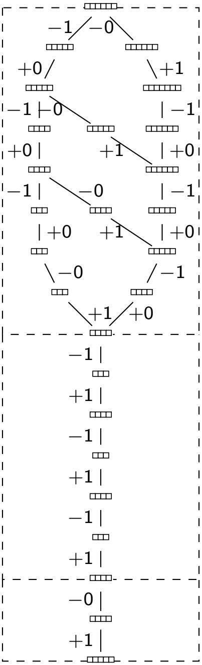


$$\begin{bmatrix} d(1) & m{\downarrow}(1,4) & m{\downarrow}(1,4)) \\ 1 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} d(1) & m{\downarrow}(1,4) & m{\downarrow}(1,4)) \\ 2 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} m{\uparrow}(3,2) & m{\uparrow}(2,1) & m{\downarrow}(1,3)) \\ 1 & 1 & 2 \end{bmatrix}$$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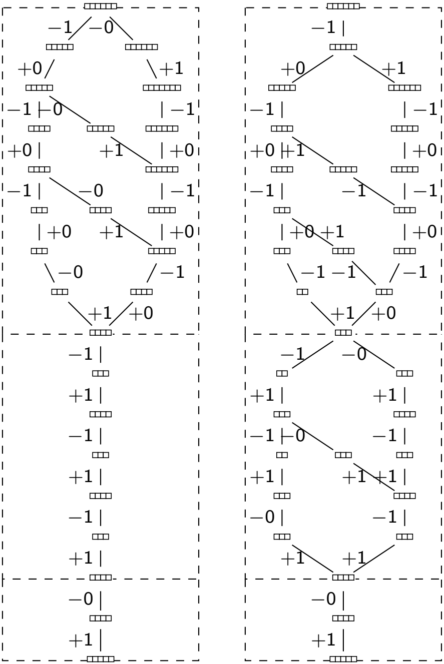

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−0 −1 −1 −1 −1
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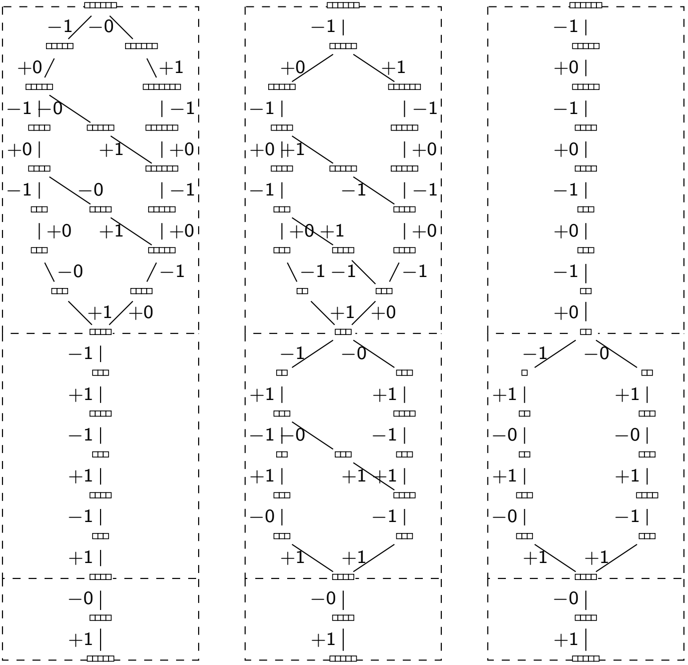
+1 +1 +1 +1
−1 −0 −1
+1 +1 +1
−0 −0
+1 +1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Example

Let $\lambda = (7)$ and $\nu = (6)$ and $\mu = (4,3,1)$. The three elements of $\mathsf{S} \in \mathrm{Latt}_8(\nu \setminus \lambda, \mu)$ from previous slide have $\mathrm{read}(\mathsf{S})$ equal to one of the following

$$\left(\begin{array}{ccc|ccc|cc} r(1) & r(1) & r(1) & d(1) & d(1) & d(1) & a(1) & a(1) \\ 1 & 1 & 1 & 2 & 2 & 2 & 3 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc|cc} r(1) & r(1) & r(1) & d(1) & d(1) & d(1) & a(1) & a(1) \\ 1 & 1 & 1 & 2 & 2 & 1 & 3 & 2 \end{array}\right)$$

$$\left(\begin{array}{cccc|c|ccc} r(1) & r(1) & r(1) & r(1) & d(1) & a(1) & a(1) & a(1) \\ 1 & 1 & 1 & 1 & 2 & 3 & 2 & 2 \end{array}\right)$$

Therefore

$$g((n-7,7),(n-6,6),(n-8,4,3,1)) = 3$$

for $n \geq 15$.

THE END!

We now explain the $*$ which occurred on some definitions and theorems. The partition algebra module

$$\mathrm{Hom}_{P_s(n)}(\Delta_s(\mu), \Delta_s(\nu \setminus \lambda))$$

doesn't just see the Kronecker coefficients $g(\lambda, \nu, \mu)$ for $\mu \vdash s$. It also sees those for μ a partition of $s-1$, $s-2$, etc. This can be taken care of by identifying tableaux

$$\mathrm{Std}_s^0(\nu \setminus \lambda) \subset \mathrm{Std}_s(\nu \setminus \lambda)$$

$$\mathrm{SStd}_s^0(\nu \setminus \lambda) \subseteq \mathrm{SStd}_s(\nu \setminus \lambda) \quad \mathrm{Latt}_s^0(\nu \setminus \lambda) \subseteq \mathrm{Latt}_s(\nu \setminus \lambda)$$

which discard the "offending tableaux" in a way made precise in [B., De Visscher, Enyang]. However, it has a technical flavour which makes for a boring talk. Notice that in the pictures which claim to give "all tableaux" of a given shape, we actually don't include all tableaux. For example, no-where in the talk does the obvious tableau

$$-\varepsilon_0 + \varepsilon_0 - \varepsilon_0 + \varepsilon_0 \dots$$

appear. We only picture $\mathrm{Std}_s^0(\nu \setminus \lambda)$.